Performance Limitations in Stabilization and Tracking

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Objective and Background

- Find simple explicit relationship between the plant characteristics and the best achievable performance in certain stabilization and tracking problems.

- A recent book:

Minimum Energy Regulation

- The control energy:
  \[ E = \int_0^\infty u^2(t) dt. \]
- The minimum energy required to stabilize the system
  \[ E^* = \inf_{C \text{ is stabilizing}} E. \]
• **Theorem:** Let $P$ have anti-stable poles $p_1, \ldots, p_\mu$. Then

$$E^* = 2 \sum_{i=1}^{\mu} p_i.$$ 

• Two possible ways to prove this theorem:

  – “Expensive control”: Let $P = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$. Then

$$E^* = \lim_{\epsilon \to 0} \min_u \int_0^\infty \left[ \epsilon^2 y^2(t) + u^2(t) \right] dt, \quad x(0) = B.$$ 

  **Observation:** (Kwakernaak and Sivan, 1972) In the state feedback case, the total shift of eigenvalues when closing the loop is $2 \sum_{i=1}^{\mu} p_i$, i.e.,

Minimum energy = modal shift.

  – $\mathcal{H}_2$ model-matching: Let $P(s) = N(s)M^{-1}(s)$ be a corpime factorization. Then

$$E^* = \min_{Q \in \mathcal{H}_\infty} \| I - M(s)Q(s) \|^2_2.$$
• $\sum_{i=1}^{\mu} p_i$ serves as an instability index.

• It also appears in the Bode sensitivity integral (Freudenberg and Looze, 1985):

- $L(s)$ has relative degree at least 2.
- $L(s)$ has anti-stable poles $p_1, \ldots, p_\mu$.
- $S(s) = \frac{1}{1+L(s)} \in \mathcal{H}_\infty$.
- Then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \ln |S(j\omega)| d\omega = \sum_{i=1}^{\mu} p_i.$$
• It has an “entropy” interpretation.
  
  – Consider a linear system
    \[
    \dot{x}(t) = Ax(t), \quad x(t) \in \mathbb{R}^n.
    \]
    The solution is
    \[
    x(t) = e^{At}x_0.
    \]
  
  – Define a new norm in \( \mathbb{R}^n \) as
    \[
    \|x_0\|_T = \sup_{t \in [0,T]} \|e^{At}x_0\|.
    \]
  
  – Let \( B \) be the closed unit ball in the original norm \( \| \cdot \| \). Fill up \( B \) with as many \( \| \cdot \|_T \) balls with radius \( \epsilon \) as possible. Denote the number of such \( \| \cdot \|_T \) balls as \( r_T(\epsilon) \).
  
  – Define
    \[
    r(\epsilon) = \limsup_{T \to \infty} \frac{1}{T} \ln r_T(\epsilon).
    \]
    \( r(\epsilon) \) is a decreasing function of \( \epsilon \).
  
  – Define the topological entropy of the system as
    \[
    h(A) = \lim_{\epsilon \to 0} r(\epsilon).
    \]
Theorem: \( h(A) = \sum_{i=1}^{\mu} p_i \) where \( p_1, \ldots, p_\mu \) are the anti-stable eigenvalues of \( A \).
• What if the disturbance or the states of the plant cannot be measured?

\[ C \quad u \quad P \]

\[ \delta \]

\[ y \]

• We have

\[
\inf_{C \text{ is stabilizing}} \|u(t)\|_2^2 = 2 \sum_{i=1}^{\mu} p_i
\]

if and only if \( P(s) \) is minimum phase.

• Historic notes:
  
  − Qiu and Chen, in *Learning, Control, and Hybrid Systems*, 1998.
Minimum Error Step Tracking

Here

\[ P = \begin{bmatrix} G \\ H \end{bmatrix}. \]

Wish to design \( C \) so that the closed loop system is internally stable and \( z \) tracks a unit step \( \sigma(t) \).

The tracking error:

\[ J = \int_0^\infty [\sigma(t) - z(t)]^2 dt. \]

The minimum tracking error:

\[ J^* = \inf_{C \text{ is stabilizing}} J. \]
**Theorem:** Let $G(s)$ have nonminimum phase zeros $z_1, \ldots, z_\nu$. Then under mild conditions,

$$J^* = 2 \sum_{i=1}^\nu \frac{1}{z_i}.$$ 

**Two ways to prove this theorem:**

- “Cheap control”:

  $$J^* = \lim_{\epsilon \to 0} \min_u \int_0^\infty e^2(t) + e^2[u(t) - u(\infty)]^2 dt$$

- $\mathcal{H}_2$ model-matching: Let $P(s) = N(s)M^{-1}(s)$ be a coprime factorization. Then

  $$J^* = \min_{Q\in\mathcal{H}_\infty} \| [I - N(s)Q(s)]^{-1}_s \|_2^2.$$
\[ \sum_{i=1}^{\nu} \frac{1}{z_i} \] defines a degree of difficulty in tracking a step.

- It also appears in the Bode complementary sensitivity integral (Middleton, 1991)

\[ L(s) \] is at least of type 2.
- \( L(s) \) has nonminimum phase zeros \( z_1, \ldots, z_\nu \).
- \( T(s) = \frac{L(s)}{1+L(s)} \in \mathcal{H}_\infty \).

\[ \ln |T(j\omega)| \frac{d\omega}{\omega^2} = \sum_{i=1}^{\nu} \frac{1}{z_i} \]

- Does it have an entropy interpretation? The instability index of the zero dynamics?
• What if a unity feedback is used?

\[
\begin{align*}
\text{r} & \quad e \quad C \quad u \quad G \quad z \\
\end{align*}
\]

• It holds

\[
\inf_{C \text{ is stabilizing}} \|e(t)\|_2^2 = 2 \sum_{i=1}^{\nu} \frac{1}{z_i}
\]

iff \(G(s)\) is semi-stable.

• Historic notes

Minimum Error Tracking of Sinusoidal Signals

Here

- $\xi$ is the state of exo-system $S$. (Full information)
- $r(t) = a \sin \omega t + b \cos \omega t$.
- $P = \begin{bmatrix} G \\ H \end{bmatrix}$.

- Wish to design $C$ so that the closed loop system is internally stable and $z$ tracks $r$. 

• The averaged tracking error:

\[ J = \mathbb{E}\{ \int_0^\infty \|e(t)\|_2^2 \, dt : \mathbb{E} \begin{bmatrix} a \\ b \end{bmatrix} = 0, \mathbb{E} \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} = I \} \].

• The minimum tracking error

\[ J^* = \inf_{C \text{ is stabilizing}} J. \]
• **Theorem:** Let $G$ have nonminimum phase zeros $z_1, \ldots, z_\nu$. Then

$$J^* = \sum_{i=1}^{\nu} \left( \frac{1}{z_i^* + j\omega} + \frac{1}{z_i - j\omega} \right).$$

(Su, Qiu, Chen, *IEEE TAC* 2003)
• What if the controller can only access the reference signal, not the state of reference generator. (Partial information)

```
\begin{figure}
\begin{center}
\begin{tikzpicture}
\node (r) at (0,0) {$r$};
\node (C) at (1,0) {$C$};
\node (P) at (2,0) {$P$};
\node (u) at (1,1) {$u$};
\node (z) at (2,1) {$z$};
\node (y) at (1,-1) {$y$};
\draw[->] (r) -- (C);
\draw[->] (C) -- (u);
\draw[->] (u) -- (P);
\draw[->] (P) -- (z);
\draw[->] (z) -- (y);
\end{tikzpicture}
\end{center}
\end{figure}
```

• **Theorem:** Let $G$ be rational and have nonminimum phase zeros $z_1, \ldots, z_\nu$.

$$J^* = \sum_{i=1}^{\nu} \left( \frac{1}{z_i^* + j\omega} + \frac{1}{z_i - j\omega} \right) + \frac{1}{\omega} \sin^2 \left[ 2 \sum_{i=1}^{\nu} \angle(z_i - j\omega) \right].$$

• The extra term in the partial information case is due to the cost of estimating the full information optimal control.

(Su, Qiu, Chen, *IEEE TAC* 2005)
Minimum Energy Stabilization under Disturbance

- \(d\) is \(L_2\) norm bounded
  \[\|d\|_2 \leq \epsilon.\]
- The worst case energy
  \[E = \sup_d \int_0^\infty u^2(t)dt.\]
- The best achievable worst case energy
  \[E^* = \inf_{C \text{ is stabilizing}} E.\]
Theorem:

\[ E^* = 2 \sum_{i=1}^{\mu} p_i + \epsilon^2 \inf_{C \text{ is stabilizing}} \|T_{ud}\|_{\infty}^2. \]
Here $\Delta$ is a possible nonlinear time-varying uncertainty with bounded $\mathcal{L}_2$ induced norm:

$$\|\Delta\| \leq \epsilon.$$ 

- The worst case stabilization energy

$$E = \sup_{\Delta} \int_0^\infty u^2(t) dt$$

- The best achievable worst case stabilization energy

$$E^* = \inf_{C \text{ is stabilizing}} E.$$
Theorem: Let

\[ \gamma = \inf_{C \text{ is stabilizing}} \|T_{ud}\|_{\infty}. \]

Then

\[ E^* = \frac{2}{1 - \epsilon^2 \gamma^2} \sum_{i=1}^{\mu} p_i. \]
Minimum Error Tracking under Disturbance

- Here $d$ is $\mathcal{L}_2$ norm bounded:
  \[ \|d\|_2 \leq \epsilon. \]

- The worst case error
  \[ J = \sup_d \int_0^\infty \|\sigma(t) - z(t)\|_2^2 dt \]

- The best achievable worst case error
  \[ J^* = \inf_{C \text{ is stabilizing}} J. \]
• **Theorem:** Let $G$ have nonminimum phase zeros $z_1, \ldots, z_\nu$. Then under mild conditions,

$$J^* = 2 \sum_{i=1}^{\nu} \frac{1}{z_i} + \epsilon^2 \inf_{C \text{ is stabilizing}} \| T_{ed} \|_\infty^2.$$

(Su, Petersen, and Qiu, IFAC Congress, 2005)
Minimum Error Tracking under Uncertainty

Here $\Delta$ is a possibly nonlinear time-varying uncertainly with bounded $\mathcal{L}_2$ induced norm:

$$\|\Delta\| \leq \delta.$$ 

The worst case error

$$J = \sup_{\Delta} \int_0^\infty \|\sigma(t) - z(t)\|_2^2 dt,$$

and the best achievable worst case error

$$J^* = \inf_{C \text{ is stabilizing}} J.$$
• **Theorem:** Let \[ \gamma = \inf_{C \text{ is stabilizing}} \|T_{ed}\|_\infty. \]

Then

\[ J^* = \frac{2}{1 - \epsilon^2 \gamma^2} \sum_{i=1}^{\nu} \frac{1}{z_i}. \]

(Su, Petersen, Qiu, IFAC Congress, 2005)
Conclusions

• Simple and explicit expressions for certain performance limitations have been obtained.

• These limitations reveals inherent system structures and properties.

• Extensions to
  – multivariable systems,
  – discrete time systems,
  – sampled-data systems,
  – system with delays,
  – nonlinear systems,

are available.