Performance Limitations in Decentralized Feedback Control

by

Graham C. Goodwin
with
Mario M. Salgado and Eduardo I. Silva

Conference on Decision and Control Workshop

New Developments in Control Performance Limitation Research:
A Tale in the Network Age

11th December 2005
Seville, Spain
Almost all practical control systems utilize a decentralized architecture.

Reasons
- simplicity of understanding and design
- physical insights (level should be controlled by flow, etc.)
- cabling issues

There is likely to be even greater emphasis on decentralized architectures in future networked control systems since channel inter-connections are reduced.
Core question: “What performance penalty arises from use of a decentralized architecture?”

Despite practical importance - not much known about this problem.

In this talk we will outline some recent results that give (some) insights into this problem.

Interestingly the results depend, inter-alia, on the Relative Gain Array (RGA).

Gives insight into this standard tool for assessing input-output pairings.
1. Preliminaries
2. Review of known Results
3. Time Domain Limits in Decentralized Control
   (3a) RGA
   (3b) Time Domain Limits that apply to Decentralized Architectures
   (3c) Time Domain Limits that apply generally
4. Interpretation of Results
5. Bounds on Transient Terms
6. Examples
7. Discussion and Open Problems
Also let

\[
\mathbf{G}_0(s) = \text{diag}\{G_{11}(s), G_{22}(s), \cdots, G_{pp}(s)\}
\]  

(1)

diagonal entries of plant transfer function.

\\
\text{Figure: Multivariable control loop}
Define the additive error transfer function as

\[ G_\varepsilon(s) = G(s) - G_0(s). \]  \hfill (2)

**Assumption 1** \( G(0) \) and \( G_0(0) \) are nonsingular.

**Assumption 2** The controller \( C_d(s) \) belongs to the class \( C \) of controllers, which is defined as the class of all diagonal, proper and stabilizing controllers for \( G_0(s) \).
Lemma 1 (Decentralized closed loop sensitivity)

For any stable plant $G(s)$ we have the following:

(i) The class $C$ of controllers can be parameterized as

$$C_d(s) = [I - Q_d(s)G_o(s)]^{-1}Q_d(s) \quad (3)$$

where $Q_d(s)$ is any stable proper diagonal transfer function.

(ii) The achieved sensitivity function $S(s)$ when $C_d(s)$, as in (3), is utilized in the feedback loop of figure 1 is

$$S(s) = S_o(s)S_{\Delta}(s) \quad (4)$$

where

$$S_o(s) = I - G_o(s)Q_d(s) \quad (5)$$

$$S_{\Delta}(s) = [I + G_\epsilon(s)Q_d(s)]^{-1} \quad (6)$$

Assumption 3 $Q_d(s)$ is rational, i.e., it has no pure time delays in any of its elements.
1. Preliminaries

2. Review of known Results

3. Time Domain Limits in Decentralized Control
   (3a) RGA
   (3b) Time Domain Limits that apply to Decentralized Architectures
   (3c) Time Domain Limits that apply generally

4. Interpretation of Results

5. Bounds on Transient Terms

6. Examples

7. Discussion and Open Problems
Lemma 2 \textit{(SISO cheap control performance limits and a fundamental limitation)}

Consider any open loop stable SISO plant without zeros on the imaginary axis and non zero DC-gain written in the form

\[ G_o(s) = B_p(s)\tilde{G}_o(s)e^{-s\tau} \]  \hspace{1cm} (7)

where \( \tau \) is the pure time delay of the plant, \( \tilde{G}_o(s) \) is a proper, stable and minimum-phase (MP) transfer function, and \( B_p(s) \) is a Blashke product of the form

\[ B_p(s) = \prod_{\ell=1}^{n_z} \frac{-s + z_\ell}{s + z_\ell} \]  \hspace{1cm} (8)

where \( \{z_\ell\}_{\ell=1}^{n_z} \) denotes the set of non-minimum phase (NMP) zeros of \( G_o(s) \).

Assume that this plant is under feedback control achieving zero steady state error for step references. Denote the closed loop error for unit step references by \( e(t) \).
(i) The energy of $e(t)$ is bounded below by

$$\inf \int_0^\infty e^2(t)dt = \tau + 2 \sum_{\ell=1}^{n_z} \frac{1}{Z^\ell}.$$  \hspace{1cm} (9)

(ii) Denote by $\{p_j\}_{j=1}^{n_p}$ and by $\{z_k\}_{k=1}^{n_c}$, $n_c \geq n_z$, the sets of poles and zeros of the complementary sensitivity function $T_o(s)$, respectively. Then

$$\int_0^\infty e(t)dt = \lim_{s \to 0} \left[ \frac{S_o(s)}{s} \right] = \tau - \sum_{j=1}^{n_p} \frac{1}{p_j} + \sum_{k=1}^{n_c} \frac{1}{z_k}.$$ \hspace{1cm} (10)

(iii) The (idealized) controller that achieves the minimum bound (9) is such that

$$\int_0^\infty e(t)dt = \tau + 2 \sum_{\ell=1}^{n_z} \frac{1}{Z^\ell}.$$ \hspace{1cm} (11)
Proof: Outline

(i) Standard

(ii) The closed loop error satisfies

\[ E(s) = \frac{S_o(s)}{s} = \int_0^\infty e(t)e^{-st} dt \tag{12} \]

and since the closed loop has zero steady state error for step references, it follows that

\[ \int_0^\infty e(t)dt = \lim_{s \to 0} \left[ \frac{S_o(s)}{s} \right] \tag{13} \]

Also, since \( G_o(s) \) is stable, all admissible complementary sensitivity functions that achieve zero steady state error for step references can be written in the form (recall assumption 3)

\[ T_o(s) = B_p(s)D(s)e^{-s\tau} \tag{14} \]
where

\[
D(s) = \frac{\prod_{\ell=1}^{n_z} (s + z_\ell) \prod_{k=n_z+1}^{n_c} (s - z_k)}{\prod_{j=1}^{n_p} (s - p_j)} \frac{\prod_{j=1}^{n_p} (-p_j)}{\prod_{\ell=1}^{n_z} z_\ell \prod_{k=n_z+1}^{n_c} (-z_k)}
\]  

(15)

and \(n_p \geq n_c + rd\{G_o(s)\}\) (\(rd\{\}\) denotes relative degree). Therefore,

\[
\lim_{s \to 0} \left[ \frac{S_o(s)}{s} \right] = \lim_{s \to 0} \left[ \frac{1 - B_p(s)D(s)e^{-s\tau}}{s} \right]
\]  

(16)

Using L’Hôpital’s rule and the definitions of \(B_p(s)\) and \(D(s)\), it is straightforward to establish that

\[
\lim_{s \to 0} \left[ \frac{S_o(s)}{s} \right] = \tau - \sum_{j=1}^{n_p} \frac{1}{p_j} + \sum_{k=1}^{n_c} \frac{1}{z_k}
\]  

(17)

which proves the result.
**Theorem** *(Fundamental MIMO limitations for every control architecture)*

Consider a MIMO feedback control loop having stable closed-loop poles located to the left of $-\alpha$ for some $\alpha > 0$. Also assume that zero steady state error occurs for reference step inputs in all channels. Then, for a positive unit reference on the $r^{th}$ channel, the loop (vector) error and (vector) output, denoted respectively by $e_r(t)$ and $y_r(t)$, satisfy the following:

(i) For any plant zero $z_o$ with left directions $h_1^T, \ldots, h_{\mu_z}^T$, satisfying $\Re\{z_o\} > -\alpha$, we have that

$$\int_0^\infty h_i^T e_r(t) e^{-z_o t} \, dt = \frac{h_{ir}}{Z_o}, \quad i = 1, 2, \ldots, \mu_z$$

where $h_{ir}$ is the $r^{th}$ element in $h_i$.

(ii) Also, for $\Re\{z_o\} > 0$,

$$\int_0^\infty h_i^T y_r(t) e^{-z_o t} \, dt = 0, \quad i = 1, 2, \ldots, \mu_z$$
This is a standard result for any stable MIMO design, see for example Control System Design - Goodwin, Graebe and Salgado.

Note, in particular, that if there are no MIMO delays or non-minimum phase zeros, then no significant design limitations apply to centralized designs.
Outline

1. Preliminaries
2. Review of known Results
3. Time Domain Limits in Decentralized Control
   ⇒ (3a) RGA
   (3b) Time Domain Limits that apply to Decentralized Architectures
   (3c) Time Domain Limits that apply generally
4. Interpretation of Results
5. Bounds on Transient Terms
6. Examples
7. Discussion and Open Problems
3. Time Domain Limitations

(3a) RGA

\[ \Lambda = G(s) \otimes [G^{-1}(s)]^T \iff \Lambda_{ij} = G_{ij}(s) \cdot [G^{-1}(s)]_{ji} \quad (20) \]

- \( S = 0 \): gives standard RGA
- \( S = j\omega \): Dynamic RGA

Standard rules state that input-output pairs associated with negative, or large RGA terms, should be avoided and those with RGA terms near 1 should be preferred.

Negative elements in the RGA arise when the d.c. gain of the elements \( G_{ij}(s) \) are significantly larger than \( \text{det}\{G(s)\} \), evaluated at d.c. In turn, negative RGA elements are usually accompanied by positive RGA elements larger than 1. This is due to a property of the RGA matrix, namely that the elements in every row (and in every column) add to 1. We refer to this type of plant as poorly conditioned.
1. Preliminaries
2. Review of known Results
3. Time Domain Limits in Decentralized Control
   (3a) RGA
   ⇒ (3b) Time Domain Limits that apply to Decentralized Architectures
   (3c) Time Domain Limits that apply generally
4. Interpretation of Results
5. Bounds on Transient Terms
6. Examples
7. Discussion and Open Problems
Theorem (Fundamental MIMO limitations in a decentralized architecture) Consider a stable MIMO feedback control loop based on a decentralized architecture and subject to assumptions 1, 2, 3. Assume that zero steady state error occurs for step references in all channels, and that a positive unit step reference is applied on the $r^{th}$ channel. Then, the $j^{th}$ component of the loop error, $e_j^r(t)$, satisfies

$$
\int_0^\infty e_j^r(t) \, dt = \frac{G_{jj}(0)}{G_{rj}(0)} \Lambda_{rj} \left\{ \tau_j - \sum_{k=1}^{n_j} \frac{1}{p_j^k} + \sum_{\ell=1}^{m_j} \frac{1}{z_j^\ell} \right\}
$$

(21)

where $\{p_j^k\}_{k=1}^{n_j}$ and $\{z_j^\ell\}_{\ell=1}^{m_j}$ denotes the sets of poles and zeros, respectively, of the $j^{th}$ diagonal entry of the nominal (diagonal) complementary sensitivity $T_o(s)$. Also, $\Lambda_{rj}$ is the $(r, j)^{th}$ element of the RGA matrix, and $\tau_j$ is the pure delay in $G_{jj}(s)$. 
We note that, to achieve zero steady state error, we require that

\[ Q_d(0) = [G_o(0)]^{-1}. \]  \hfill (22)

This result implies that

\[ S_\Delta(0) = [I + (G(0) - G_o(0)) Q_d(0)]^{-1} = G_o(0)G^{-1}(0) \]  \hfill (23)

For a unit step in the \( r^{th} \) channel, we have that the Laplace transform of the control error in that channel satisfies

\[ E^r(s) = S(s)v_r \frac{1}{s} \]  \hfill (24)

where \( v_r \) is the null column vector save for a unit element in the \( r^{th} \) row.

Since (22) holds, we have that \( E^r(s) \) converges for all \( \Re\{s\} > -\alpha \), and hence

\[ \int_0^\infty e^r_j(t)dt = \lim_{s \to 0} v_j^T E^r(s) = \lim_{s \to 0} v_j^T \frac{S(s)}{s} v_r \]  \hfill (25)

\[ = \lim_{s \to 0} v_j^T \frac{S_o(s)S_\Delta(s)}{s} v_r \]  \hfill (26)
Now we have that

\[ v_j^T \frac{S_0(s)}{s} = \begin{bmatrix} 0 & \ldots & 0 & \frac{S_{0j}(s)}{s} & 0 & \ldots & 0 \end{bmatrix} \]  

(27)

which implies, jointly with (23), that

\[ \int_0^\infty e^{j}_j(t) \, dt = \lim_{s \to 0} \left[ \frac{S_{0j}(s)}{s} \right] G_{jj}(0) [G^{-1}(0)]_{jr} \]  

(28)

The result follows using part \( (ii) \) of lemma 2 and the definition of the RGA.
Remark

If all open loop stable zeros are cancelled in the nominal design (note that this implies that the controller is itself minimum phase), then part (ii) of the previous theorem reduces to

\[
\int_0^\infty e_j^r(t) dt = \frac{G_{jj}(0)}{G_{rj}(0)} \Lambda_{rj} \left\{ \tau_j - \sum_{k=1}^{n'_j} \frac{1}{p^j_k} + \sum_{\ell=1}^{m'_j} \frac{1}{z^j_{\ell}} \right\}
\]  

(29)

where \( \{z^j_{\ell}\}_{\ell=1,\ldots,m'_j} \) is the set of non-minimum phase zeros of \( G_{jj}(s) \) and \( \{p^j_k\}_{k=1,\ldots,n'_j} \), the set of closed loop poles for the \( j^{th} \) nominal loop.
We see that the RGA plays a key role in the above result. However, the theorem gives additional insight since it shows that the RGA is only part of the story in the quantification of the time domain limitations for decentralized control architectures. Other factors which influence the result are the nominal closed loop bandwidths (i.e. the inverse of the poles $p_1^j, p_2^j, \ldots, p_{nj}^j$), the nominal closed loop zeros and the time delays in the diagonal transfer function elements $G_{jj}(s)$. Non-minimum phase zeros in the diagonal elements of the nominal model (obviously not cancelled) play central roles in the third term.
1. Preliminaries
2. Review of known Results
3. Time Domain Limits in Decentralized Control
   (3a) RGA
   (3b) Time Domain Limits that apply to Decentralized Architectures
   ⇒ (3c) Time Domain Limits that apply generally
4. Interpretation of Results
5. Bounds on Transient Terms
6. Examples
7. Discussion and Open Problems
Theorem (Fundamental MIMO limitation in the general MIMO case) Consider a stable full-MIMO control loop, based on a $p \times p$ plant $G(s)$ with non-singular DC gain. The controller is assumed to be parameterized in Youla form, with a rational $p \times p$ full MIMO parameter $Q(s) = [Q_{ij}(s)]$.

(i) Suppose that there is zero steady state error for step references in all channels. Then, for a step reference in the $r^{th}$ channel, the $j^{th}$ component of the control error, denoted by $e_j^r(t)$, satisfies

$$\int_0^\infty e_j^r(t) dt = \sum_{i=1}^p \frac{G_{ji}(0)}{G_{ri}(0)} \Lambda_{ri} \left( \tau_i^r - \sum_{\ell=1}^{n_{p_i}} \frac{1}{p_{i\ell}} + \sum_{k=1}^{n_{z_i}} \frac{1}{z_{ik}} \right)$$

(30)
where $\Lambda_{ri}$ denotes the $(r, i)^{th}$ element of the RGA matrix for $G(s)$, $	au_{ri}^{rj}$ is the time delay in $G_{ji}(s)Q_{ir}(s)$, \{\(z_{ik}^{ri}\)\}_{k=1\ldots,n_{zi}^{ri}} denote the set of zeros of $G_{ji}(s)Q_{ir}(s)$ and \{\(p_{i\ell}^{ri}\)\}_{\ell=1\ldots,n_{pi}^{ri}} the set of poles of $G_{ji}(s)Q_{ir}(s)$.

In particular, if $j = r$, (30) becomes

$$
\int_0^\infty e_j^i(t)\,dt = \sum_{i=1}^{p} \Lambda_{ji} \left( \tau_{ri}^{j} - \sum_{\ell=1}^{n_{pi}^{ji}} \frac{1}{p_{i\ell}^{ji}} + \sum_{k=1}^{n_{zi}^{ji}} \frac{1}{z_{ik}^{ji}} \right)
$$

(31)

(ii) Also, we have

$$
\sum_{i=1}^{p} \frac{G_{ji}(0)}{G_{ri}(0)} \Lambda_{ri} = \begin{cases} 
0 & \text{if } j \neq r \\
1 & \text{if } j = r 
\end{cases}
$$

(32)
Remark

The above result applies to all MIMO designs which stabilize the full MIMO plant $G(s)$. Thus this result also applies to decentralized designs. To use the latter theorem for the evaluation of the accumulated errors in a decentralized design, it is necessary to interpret the Youla parameter $Q(s)$ appropriately. Indeed, let $T_o(s)$ be a nominal (diagonal) complementary sensitivity resulting from a decentralized design. Then the corresponding diagonal Youla parameter is

$$Q_d(s) = G_o^{-1}(s)T_o(s) = G_o^{-1}(s)(I - S_o(s))$$  \hspace{1cm} (33)

We can then evaluate the corresponding Youla parameter for use with $G(s)$ as follows:

$$Q(s) = G^{-1}(s)T(s) = G^{-1}(s)(I - S_\Delta(s) + G_o(s)Q_d(s)S_\Delta(s))$$  \hspace{1cm} (34)

where $T(s)$ is the achieved sensitivity in the real loop. If $Q(s)$ as in equation (34) is utilized in theorem 24 then the result reduces to the expression given in theorem 18 for the decentralized case.
Outline

1. Preliminaries
2. Review of known Results
3. Time Domain Limits in Decentralized Control
   (3a) RGA
   (3b) Time Domain Limits that apply to Decentralized Architectures
   (3c) Time Domain Limits that apply generally

⇒ 4. Interpretation of Results
5. Bounds on Transient Terms
6. Examples
7. Discussion and Open Problems
The main difference between theorem 24 and theorem 18 is that equation (30) implies that, in a not necessarily decentralized architecture, the accumulated error depends on a linear combination of effects; in particular, the coefficients of that linear combination satisfy (32). Hence, a centralized architecture can yield a lower accumulated error by using the MIMO interaction, implicit is this linear combination, in a beneficial fashion in the design. This is not possible in the decentralized case due to the restricted architecture.
Corollary (Comparison between centralized and decentralized designs)

Consider a rational MP plant $G(s)$ (with no delays) having a NMP nominal model $G_o(s)$ satisfying the conditions of theorem 18. Then,

(i) If a centralized architecture is used, then the accumulated errors can be made arbitrarily small.

(ii) If a decentralized architecture is used and $[T_o(s)]_{jj}$ is such that it has as zeros only the NMP zeros of the diagonal elements $G_{jj}$, then the (absolute value of the) accumulated errors are bounded from below by

$$
\left| \int_0^\infty e_j^r(t) \, dt \right| \geq \left| \frac{G_{jj}(0)}{G_{rj}(0)} \Lambda_r \left\{ \tau_j + \tau_{dom} + \sum_{\ell=1}^{m_j'} \frac{1}{z^j_{\ell}} \right\} \right| 
$$

where $\tau_{dom}$ denotes the dominant time constant of the $j^{th}$ loop (a real dominant pole is assumed).
Proof

(i) Since the full MIMO plant $G(s)$ is assumed MP, one can choose the corresponding Youla parameter as

$$Q(s) = G^{-1}(s) \text{diag} \left\{ \frac{1}{(\alpha_i s + 1)^{n_i}} \right\}_{i=1}^{\cdots p}$$  \hspace{1cm} (36)$$

where $\alpha_i > 0$ and $n_i$ are appropriate integers such that $Q(s)$ proper. In this case,

$$T(s) = \text{diag} \left\{ \frac{1}{(\alpha_i s + 1)^{n_i}} \right\}_{i=1}^{\cdots p}$$  \hspace{1cm} (37)$$

which implies that $e_j^f = 0$ for $j \neq r$. For $j = r$ it suffices to use simpler result. Letting $\alpha_i \to 0$ then yields the result.
(ii) Using equation (29) it follows that

\[ \left| \int_0^\infty e_j'(t) \, dt \right| = \left| \frac{G_{jj}(0)}{G_{rj}(0)} \Lambda_{rj} \left\{ \tau_j - \frac{1}{p_j} \sum_{k=1}^{n'} \frac{1}{p_k} + \sum_{\ell=1}^{m'} \frac{1}{z_{j\ell}} \right\} \right|. \]  

(38)

Also, the poles of the nominal \( j^{th} \) loop cannot be made arbitrarily fast, due to the robust stability requirement. It follows that there must be, at least, one relatively slow dominant pole, which we denote \( p_{dom} \). The result then follows if one lets \( \tau_{dom} = -\frac{1}{p_{dom}} \).
The last corollary states that it is always possible, in the centralized control of MP MIMO plants, to achieve lower accumulated errors than when using a decentralized design. This is as expected since $G(s)$ is assumed here to have no MIMO zeros and therefore there are no (important) limitations on the achievable performance for centralized designs. On the other hand, decentralized designs need to deal with several extra issues, namely;

(i) the possibility that the nominal model has NMP zeros, and

(ii) the nominal closed loop poles cannot be made arbitrarily fast, due to robust stability requirements.
Next, consider a more general situation when the full MIMO plant is NMP. Assume first that the RGA has diagonal elements larger than 1 (which implies that negative elements exist in the same row and in the same column). Say that $\Lambda_{jj} > 1$. Assume that a decentralized controller is designed for the given plant. Then, the error in channel $j$, when a unit step reference is applied to that channel, is given by (21) with $j = r$, that is

$$\int_{0}^{\infty} e_j(t) \bigg|_{\text{decent}} \, dt = \Lambda_{jj} \left\{ \tau_j - \sum_{k=1}^{n_j} \frac{1}{p_k^j} + \sum_{\ell=1}^{m_j} \frac{1}{z_{\ell}^j} \right\} > \left\{ \tau_j - \sum_{k=1}^{n_j} \frac{1}{p_k^j} + \sum_{\ell=1}^{m_j} \frac{1}{z_{\ell}^j} \right\}$$

(39)
We next consider the same plant, but under centralized control. Also assume for simplicity that all products \( G_{ji}(s)Q_{ij}(s) \), \( i = 1, 2, \ldots, p \), have the same delays, and the same poles and zeros as \( [T_o(s)]_{jj} \). Then equation (30) together with (32), for \( j = r \), implies that

\[
\int_0^\infty e_j^j(t) \big|_{\text{cent}} \, dt = \left( \tau^{jj} - \sum_{\ell=1}^{n_p^{ij}} \frac{1}{p^{j\ell}} + \sum_{k=1}^{n_z^{jj}} \frac{1}{z^{j\ell}} \right) = \left\{ \tau_j - \sum_{k=1}^{n_j} \frac{1}{p^j_k} + \sum_{\ell=1}^{m_j} \frac{1}{z^j_\ell} \right\}
\]

(40)
This result establishes that centralized control yields a lower accumulated error than the decentralized one. This result relies on the assumption that $\Lambda_{jj} > 1$. If that is not the case, the centralized design can still deliver smaller accumulated errors in the remaining channels. To see this assume that all products $G_{ji}(s)Q_{ij}(s)$, $i = 1, 2, \ldots, p$, have the same delays, and the same poles and zeros. Then using (30) and (32), with $j \neq r$, yields

$$\int_0^{\infty} e_j^r(t) \Big|_{\text{cent}} \, dt = 0.$$

(41)
The comparison is further highlighted when NMP zeros are present. Consider a poorly conditioned $2 \times 2$ plant, with $\Lambda_{11} = \Lambda_{22} > 1$. It will be assumed that this pairing defines the nominal diagonal model $G_o(s)$ used for decentralized design. Suppose that the plant has only one NMP MIMO zero at $s = z$, associated with the canonical left direction $v_j^T$. Note that this implies that $G_{jj}(s)$ has a NMP zero at $s = z$.

Consider a nominal (decentralized) design that cancels all stable zeros of the nominal model and has (very) fast closed loop poles. Under these conditions, theorem 18 implies that:

$$
\int_0^\infty e_j(t)dt = \Lambda_{jj} \frac{1}{z}
$$

(42)
On the other hand, consider a centralized design based on the same criteria as in the decentralized case, i.e.: fast uncancelled closed loop poles and no uncancelled stable zeros. Also, assume that $Q_{ij}(s)$ cancels all of the stable zeros of $G_{ji}(s)$. Under these conditions, according to theorem 24, we have that

$$\int_0^\infty e_j^i(t)dt = \sum_{i=1}^{p} \Lambda_{ji} \frac{1}{Z} = \frac{1}{Z}$$

(43)

Note that in the above we have used the fact that each row sum of the RGA equals 1.
Since the plant is assumed to be poorly conditioned, $\Lambda_{jj} > 1$. Then, (42) implies that the long term average of $e_j^j(t)$, in the decentralized case, will be large relative to the centralized case. Note that this average will be even greater if $z$ is a slow NMP zero. On the other hand, (43) shows that even for poorly conditioned plants, it is possible to achieve good performance with a centralized design, and that this is easier if $z$ is a fast NMP zero. In conclusion, having $\Lambda_{jj} > 1$ may seriously deteriorate the achievable performance in the decentralized case, as compared to the centralized case.

If $0 < \Lambda_{jj} < 1$, then the accumulated error in the $j^{th}$ channel, when using a decentralized architecture, may be smaller than for the case of . However, the effect in the rest of the channels can be always made smaller in the centralized architecture due to the structure of (30) and the constraint (32) for $j \neq r$. 
1. Preliminaries

2. Review of known Results

3. Time Domain Limits in Decentralized Control
   (3a) RGA
   (3b) Time Domain Limits that apply to Decentralized Architectures
   (3c) Time Domain Limits that apply generally

4. Interpretation of Results

⇒ 5. Bounds on Transient Terms

6. Examples

7. Discussion and Open Problems
Accumulated errors, as discussed so far, generate insight into control loop performance. However, accumulated errors are not a norm, and a small value does not necessarily mean that the error itself is small. In this section, lower bounds for the settling time in decentralized control loops are derived.
All results take the form:

\[
\int_0^\infty e(t) \, dt = \Omega
\] (44)

**Case 1: \( \Omega < 0 \)**

In this case, the error must be predominantly negative, that is, the negative accumulation must be larger than the positive accumulation. This implies that \( y(t) \) overshoots the reference \( r(t) \).
Figure: Example of admissible error function (solid-thin) and lower bound (solid-thick) for case 1.
\[ t_2 - \tau_{\text{min}} \geq \frac{|-\Omega + \tau_{\text{min}}|}{\varepsilon(1 + \beta)} = B_1 \]
Case 2: $\Omega > 0$

We next consider the case when the accumulated error is positive due to a NMP zero in $F(s)$, located at $s = c$, $c > 0$. Then,

$$\int_0^\infty e(t)e^{-ct} \, dt = \frac{1}{c} > 0 \quad (46)$$

$$\int_0^\infty y(t)e^{-ct} \, dt = 0 \quad (47)$$

Equation (47) implies that $y(t)$ must be negative in a nonzero time interval, that is, there exists undershoot. In other words, the error $e(t)$ must be larger than 1 in some nonzero time interval.
Figure: Example of admissible error function (solid-thin) and upper bound (solid-thick) for case 2.
\[ t_2 - \tau_{\text{min}} \geq \frac{\Omega - \tau_{\text{min}}}{(1 + \delta)(1 + \gamma)} = B_4 \] (48)
Outline

1. Preliminaries
2. Review of known Results
3. Time Domain Limits in Decentralized Control
   (3a) RGA
   (3b) Time Domain Limits that apply to Decentralized Architectures
   (3c) Time Domain Limits that apply generally
4. Interpretation of Results
5. Bounds on Transient Terms
6. Examples
7. Discussion and Open Problems
Example 1
Consider the following MP plant $G(s)$ with RGA given by $\Lambda$, where

$$G(s) = \begin{bmatrix}
\frac{4}{(s + 1)(s + 4)} & \frac{3}{2(s + 1)(s + 3)} \\
\frac{2}{(s + 1)(s + 4)} & \frac{3}{(s + 1)(s + 3)}
\end{bmatrix} ; \quad \Lambda = \frac{1}{3} \begin{bmatrix}
4 & -1 \\
-1 & 4
\end{bmatrix} \quad (49)$$
(i) Decentralized Control

With the nominal model $G_o(s)$ given by (1), a decentralized controller $C_d(s)$ is designed to achieve a nominal complementary sensitivity given by

$$T_o(s) = \begin{bmatrix} \frac{16}{(s + 4)^2} & 0 \\ 0 & \frac{16}{(s + 4)^2} \end{bmatrix}. \quad (50)$$

If we now compute the accumulated errors for a unit step reference at channel 1, we have based on theorem 18

$$\int_0^\infty e_1^1(t) \big|_{decent} \ dt = \frac{2}{3} \quad (51)$$

$$\int_0^\infty e_2^1(t) \big|_{decent} \ dt = -\frac{1}{3} \quad (52)$$
We next consider a centralized design, based on \( G(s) \), to achieve a true complementary sensitivity equal to the nominal one above, i.e., \( T(s) = T_0(s) \). This latter design uses the Youla-parameter:

\[
Q(s) = \begin{bmatrix}
\frac{48(s + 1)}{9(s + 4)} & -\frac{24(s + 1)}{9(s + 4)} \\
-\frac{32(s + 1)(s + 3)}{9(s + 4)^2} & \frac{64(s + 1)(s + 3)}{9(s + 4)^2}
\end{bmatrix}
\]  

(53)
If we now compute the accumulated errors for unit step reference at channel 1, we have based on theorem 24 that

\[ \int_{0}^{\infty} e_1^1(t)|_{\text{cent}} \, dt = \frac{1}{2} \quad (54) \]

\[ \int_{0}^{\infty} e_2^1(t)|_{\text{cent}} \, dt = 0 \quad (55) \]

Comparing (51), (52) with (54), (55) we see that, the centralized design has the potential for better performance, since the accumulated errors are smaller than are achievable in the decentralized case.
Example 2

Consider the NMP plant

\[
\begin{bmatrix}
\frac{12s-1}{(12s+1)(4500s+1)} & \frac{12s-1}{(12s+1)(4500s+1)} \\
-\frac{2}{3(4500s+1)} & \frac{12s-1}{(12s+1)(4500s+1)}
\end{bmatrix}
\]  \hspace{1cm} (56)

whose RGA is given by

\[
\Lambda = \begin{bmatrix}
3 & -2 \\
-2 & 3
\end{bmatrix}.
\]  \hspace{1cm} (57)

Note that \(G(s)\) has two NMP zeros: one at \(s = 1/60\) with left direction \(h_1 = 2^{-1/2} \begin{bmatrix} -1 & 1 \end{bmatrix}^T\) and the other, at \(s = 1/12\) with left direction \(\tilde{h}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T\).
Figure: Loop errors for $r(t) = [1 \ 0] \mu(t)$ (decentralized case) - Example 2.
Theorem Predicts

\[ J_1(\infty) = \int_0^\infty e_1^1(t) \, dt \]
\[ = \Lambda_{11} \left\{ 0 + \frac{1}{0.08333} + \frac{1}{0.15} + \frac{1}{0.08333} \right\} \]
\[ = 92 \]

\[ J_2(\infty) = \int_0^\infty e_2^1(t) \, dt \]
\[ = \frac{G_{22}(0)}{G_{12}(0)} \Lambda_{12} \left\{ 0 + \frac{1}{0.08333} + \frac{1}{0.15} + \frac{1}{0.08333} \right\} \]
\[ = -61.33 \]
Figure: Loop error for $r(t) = [1 0] \mu(t)$ (centralized case) - Example 2.
Theorem Predicts

\[ J_1(\infty) = \Lambda_{11} \left( 0 + \frac{1}{0.1} + \frac{1}{0.08333} + \frac{1}{0.01667} - \frac{1}{0.06373} + \frac{1}{0.08333} \right) + \]
\[ + \Lambda_{12} \left( 0 + \frac{1}{0.1} + \frac{1}{0.01667} + \frac{1}{0.08333} \right) \approx 70.9 \]

\[ J_2(\infty) = \frac{G_{21}(0)}{G_{11}(0)} \Lambda_{11} \left( 0 + \frac{1}{0.1} + \frac{1}{0.01667} - \frac{1}{0.06373} \right) + \]
\[ + \frac{G_{22}(0)}{G_{12}(0)} \Lambda_{12} \left( 0 + \frac{1}{0.01667} + \frac{1}{0.1} + \frac{1}{0.08333} \right) \approx -55.38 \]
Example 3

The next example illustrates a case where the RGA (when considered in isolation) does not give the full picture of the achievable decentralized performance. (Of course this is known in terms of other considerations but is further exemplified by the accumulated error results.)

\[
G(s) = \begin{bmatrix}
\frac{-10s + 1}{(5s + 1)(6s + 1)} & \frac{30s + 1/2}{(5s + 1)(6s + 1)} \\
-1/2 & \frac{1}{(5s + 1)(6s + 1)} & \frac{-2s + 1}{(5s + 1)(6s + 1)} \\
\end{bmatrix}
\]  

(58)

whose MIMO zeros are located at \( s = -0.0750 \pm 0.2385j \) (i.e. \( G(s) \) is a minimum phase plant). The RGA for this plant is given by

\[
\Lambda = \begin{bmatrix}
0.8 & 0.2 \\
0.2 & 0.8 \\
\end{bmatrix}.
\]  

(59)
Theorem and Robust considerations give

\[
\int_{0}^{\infty} e_1^1(t) \geq 10 \cdot \Lambda_{11} = 8
\]

\[
\int_{0}^{\infty} e_1^2(t) \leq 10 \cdot \frac{G_{11}(0)}{G_{21}(0)} \Lambda_{21} = -4
\]

\[
\int_{0}^{\infty} e_2^1(t) \geq (2 + 33) \cdot \frac{G_{22}(0)}{G_{12}(0)} \Lambda_{12} = 14
\]

\[
\int_{0}^{\infty} e_2^2(t) \geq (2 + 33) \cdot \Lambda_{22} = 28
\]
Note that, since $G(s)$ is minimum phase, there are no significant performance limitations based on the use of a centralized MIMO design. This can be achieved, for example, by choosing

$$Q(s) = G^{-1}(s) \begin{bmatrix} \frac{1}{\alpha_1 s + 1} & 0 \\ 0 & \frac{1}{\alpha_2 s + 1} \end{bmatrix}$$

(60)

where $\alpha_1, \alpha_2 > 0$. This choice for $Q(s)$ implies that

$$\int_0^\infty e_1^1(t) \, dt = \alpha_1, \quad \int_0^\infty e_1^2(t) \, dt = 0$$

(61)

$$\int_0^\infty e_2^1(t) \, dt = 0, \quad \int_0^\infty e_2^2(t) \, dt = \alpha_2$$

(62)
Example 4

The aim of this example is to show how to use the accumulated errors, as an index that may help assessing the performance of decentralized control loops.

\[
G(s) = \begin{bmatrix}
\frac{-10(s+0.4)}{(s+4)(s+1)} & \frac{0.5}{(s+1)} & \frac{-1}{(s+1)} \\
\frac{2}{(s+2)} & \frac{20(s-0.4)}{(s+4)(s+2)} & \frac{1}{(s+2)} \\
\frac{-2.1}{(s+3)} & \frac{3}{(s+3)} & \frac{30(s+0.4)}{(s+4)(s+3)}
\end{bmatrix}
\]  \hspace{1cm} (63)

This model has a non-canonical NMP zero at \( s = 0.2295 \) and a RGA given by

\[
\Lambda = \begin{bmatrix}
2.8571 & -1.2857 & -0.57143 \\
-2.8571 & 3.2381 & 0.61905 \\
1 & -0.95238 & 0.95238
\end{bmatrix}
\]  \hspace{1cm} (64)
The above array suggests the pairing of the \(i^{th}\) input to the corresponding \(i^{th}\) output, if a decentralized control structure is to be considered. Therefore, a suitable nominal model is given by the NMP transfer function

\[
G_o(s) = \text{diag} \left\{ \frac{-10(s + 0.4)}{(s + 4)(s + 1)}, \frac{20(s - 0.4)}{(s + 4)(s + 2)}, \frac{30(s + 0.4)}{(s + 4)(s + 3)} \right\}
\]
The theorem gives the following lower bounds for the accumulated error, due to a unit step change in the first channel:

\[
\int_0^\infty e_1^1(t) \, dt \geq \Lambda_{11} \{0 - 0 + 0\} = 0
\]

\[
\int_0^\infty e_2^1(t) \, dt \geq \frac{G_{22}(0)}{G_{12}(0)} \Lambda_{12} \left\{0 - 0 + \frac{1}{0.4}\right\} = 6.43
\]

\[
\int_0^\infty e_3^1(t) \, dt \geq \frac{G_{33}(0)}{G_{13}(0)} \Lambda_{13} \{0 - 0 + 0\} = 0
\]
Figure: Loop errors of the proposed decentralized design for $r(t) = [1 0 0] \mu(t - 1)$ - Example 4.
<table>
<thead>
<tr>
<th>Integral Errors</th>
<th>Achieved Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Channel 1</td>
<td>0.024</td>
</tr>
<tr>
<td>Channel 2</td>
<td>6.51</td>
</tr>
<tr>
<td>Channel 3</td>
<td>0.47</td>
</tr>
</tbody>
</table>
We have presented time domain constraints on the integral of error in a MIMO system subject to a decentralized architecture.

- Emphasis on additional cost due to decentralized constraint
- Results relate to well known RGA
- However, additional factors also appear
- Also obtained bounds on settling time and undershoot
- Centralized case can utilize interactions to advantage
Outline

1. Preliminaries
2. Review of known Results
3. Time Domain Limits in Decentralized Control
   (3a) RGA
   (3b) Time Domain Limits that apply to Decentralized Architectures
   (3c) Time Domain Limits that apply generally
4. Interpretation of Results
5. Bounds on Transient Terms
6. Examples

⇒ 7. Discussion and Open Problems
7. Discussion and Open Problems

- Open loop unstable case
- Integral of error is not a norm - what about 2 Norm or $H_\infty$ bounds
- Other architectures - e.g. block diagonal, triangular