Convergence Properties of a Continuous-Time Multiple-Model Adaptive Estimator

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***Abstract*— We present and study a Continuous-Time Multiple-Model Adaptive Estimator (CT-MMAE) for state-affine multiple-input-multiple output (MIMO) systems with parametric uncertainty. The CT-MMAE is composed by a bank of local observers (typically Kalman filters) where each observer uses one element of a finite discrete parameter set in its implementation. The state estimate is given by a weighted sum of the estimates produced by the bank of observers. We show, for the case where the unknown noise and disturbance are $\mathcal{L}_2$ signals, and under appropriate observability assumptions, that if the actual plant parameter is identical to one of its discrete values, the state estimate converges globally asymptotically to the true value and the plant model is correctly identified. If the actual plant parameter vector does not belong to the finite discrete parameter set, we provide upper bounds to state and parameter estimation errors. Some deterministic and stochastic simulation results are presented and discussed.**

I. Introduction

During the past decade a number of papers have described the use of multiple model architectures for adaptive control, e.g. [1]–[6] (see also [7] and the references therein). Some have utilized a discrete-time probabilistic approach, [1], [2] while others [3]–[6] employed deterministic continuous-time methodologies. All of these papers utilize a multiple-model architecture for the identification system. The stochastic discrete-time multiple-model estimation architecture (MMAE) was first proposed by Magill [8] and numerous results on its properties have been obtained over the years; key properties can be found e.g. in [9], [10]. The stochastic continuous-time MMAE (CT-MMAE) was introduced in [11], [12], but no further research has been carried out on this subject, to the best of our knowledge. All stochastic MMAE architectures utilize a bank of Kalman filters and (under the usual linear-gaussian assumptions) generate the true conditional mean of the state as well as its conditional covariance, [9]–[12]. Following the work of Krener [13] one can treat the continuous-time Kalman filters as deterministic observers and view, and analyze, the CT-MMAE architecture in a deterministic setting.

Our motivation for this paper is to examine some of the CT-MMAE properties from both a deterministic and stochastic perspective. Such insights may be useful in placing the diverse proposed approaches in adaptive control [1]–[6] under a common theoretical umbrella.

This work was supported by project GREX / CEC-IST (Contract No. 035223), the FREESUBNET RTN of the CEC, and the FCT-ISR/IST plurianual funding program (through the POS-C Program initiative in cooperation with FEDER).

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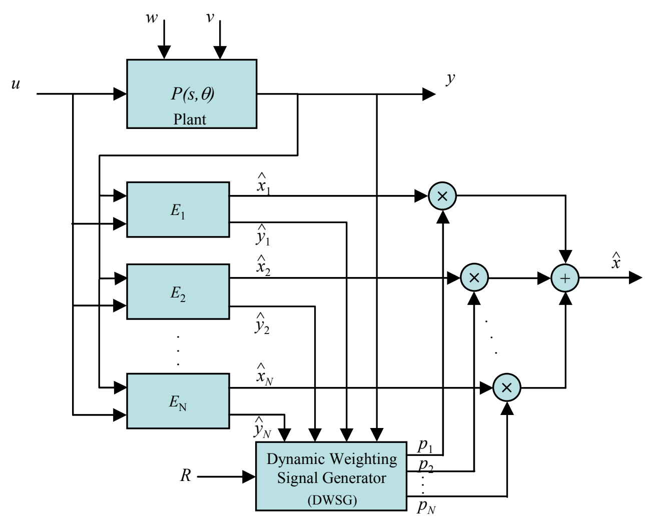


Fig. 1. The CT-MMAE architecture.

Figure 1 shows the architecture of the CT-MMAE that we study and was motivated by the results by Dunn [11], [12]. The estimators E_i can be viewed either as stochastic Kalman filters or deterministic Krener observers [13]. In a stochastic setting, the signals p_i(t) correspond to true posterior probabilities [11]. However, the dynamics of the weight signals $p_i(t)$ can be also analyzed in a deterministic setting.

The contribution of this paper is twofold. First, we show for the case where the unknown noise and disturbance signals acting on the plant are in $\mathcal{L}_2$, that under appropriate observability assumptions, if the actual plant parameter is identical to one of its discrete values, the state estimate converges globally asymptotically to the true value and the plant model is correctly identified. If the actual plant parameter vector does not belong to the finite discrete parameter set, we provide upper bounds to state and parameter estimation errors. Second, we show from simulation results (deterministic and stochastic), for a specific linear system, that when the unknown parameter is not a member of the selected finite discrete parameter set used to design the Kalman filters there is a intriguing difference behavior between the continuous MMAE and the discrete MMAE.

The structure of the paper is as follows. In Section II we pose the deterministic version of our problem. In Section III we define the details of the CT-MMAE of Fig. 1. Section IV summarizes our main results. In Section V we present some deterministic and stochastic simulations which demonstrate some intriguing patterns. Conclusions and suggestions for future research are summarized in Section VI.

Notation: Let $x \in \mathbb{R}^n$ and P be a symmetric, positive definite $n \times n$ matrix. $\|x\|$ denotes the standard Euclidean norm and $\|x\|_P := \left(x'Px\right)^{1/2}$. A piecewise continuous function $\phi : [0,T) \to \mathbb{R}^n$, $T \in (0,\infty]$ is in $\mathcal{L}_2$ if $\int_0^T \|\phi(\tau)\|^2 \, d\tau < c$ for some constant c.

II. Problem Statement

In this section we model the CT-MMAE in a purely deterministic setting. We consider state-affine multiple-input-multiple-output (MIMO) systems of the form

$$\dot{x} = A(t,\theta)x + B(t,\theta)u + G(t)w, \tag{1a}$$

$$y = C(t,\theta)x + v, \tag{1b}$$

where $x \in \mathbb{R}^n$ denotes the state of the system, $u \in \mathbb{R}^m$ its control input, $y \in \mathbb{R}^q$ its measured noisy output, $w \in \mathbb{R}^r$ an input plant disturbance that cannot be measured, and $v \in \mathbb{R}^p$ measurement noise. The matrices $A(t,\theta)$, $B(t,\theta)$, and $C(t,\theta)$ are assumed piecewise continuous, uniformly bounded in time, and contain *unknown constant parameters* denoted by the vector $\theta \in \mathbb{R}^l$. The initial condition $x(0)$ of (1a) and the signals w and v are assumed deterministic but unknown.

The problem under consideration is *to design and analyze a continuous multiple-model adaptive observer which estimates the continuous-time state vector $x(t)$ and identifies the unknown parameter θ from measured $u(t)$ and $y(t)$.*

III. The Continuous-Time Multiple-Model Adaptive Observer

This section proposes a CT-MMAE, whose architecture is shown in Fig. 1. Consider a finite set of candidate parameter values $\Theta := \{\theta_1, \theta_2, \ldots, \theta_N\}$. We propose the following CT-MMAE:

$$\hat{x}(t) := \sum_{i=1}^{N} p_i(t)\hat{x}_i(t), \tag{2}$$

$$\hat{\theta}(t) := \sum_{i=1}^{N} p_i(t)\theta_i(t), \tag{3}$$

where $\hat{x}(t)$ and $\hat{\theta}(t)$ are the estimate of the state x and parameter vector θ at time t, respectively. In (2), each $\hat{x}_i$, $i = 1, \ldots, N$ correspond to a "local" state estimate generated by a (deterministic) Kalman-Bucy filter or minmax filter [13] of the type[1]

$$\dot{P}_i = A_iP_i + P_iA_i' + G_iQG_i' - P_iC_i'R^{-1}C_iP_i, \qquad P_i(0) = P_{0i} \tag{4a}$$

$$\dot{\hat{x}}_i = A_i\hat{x}_i + B_iu + P_iC_i'R^{-1}\big(y - C_i\hat{x}_i\big), \qquad \hat{x}_i(0) = \hat{x}_{0i} \tag{4b}$$

[1]For simplicity of notation, we will drop the arguments of the matrices.

where $A_i(t) := A(t,\theta_i)$ and the same notation applies for B_i and C_i. The dynamic weights $p_i \in \mathbb{R}$, $i = 1, \ldots, N$ in (2)–(3) satisfy

$$\dot{p}_i = p_i\Big(C_i\hat{x}_i - \sum_{j=1}^{N} p_jC_j\hat{x}_j\Big)'R^{-1}\Big(y - \sum_{j=1}^{N} p_jC_j\hat{x}_j\Big), \quad p_i(0) = p_{0i}. \tag{5}$$

The structure of the key equation (5), which generates the time-evolution of the weights p_i, is identical to that proposed by Dunn [11], [12] for the stochastic version of the problem. In (4), the matrices P_{0i}, $Q(t)$ and $R(t)$ play the same role as the covariances and white-noise intensities of the corresponding Kalman-Bucy filter models. In (5), the initial conditions p_{0i} must be chosen such that $p_{0i} \in (0,1)$ and $\sum_{i=1}^{N} p_{0i} = 1$. In the stochastic case they can be interpreted [11], [12] as the *a priori* probabilities of θ_i being the actual parameter vector θ at time $t = 0$.

IV. Main Results

In this section we summarize our main results regarding the CT-MMAE using a deterministic framework. We first show that the overall sum of the dynamic weights p_i is always unity for all $t \geq 0$.

Proposition 1: Suppose that $p_{0i} \in (0,1)$ and $\sum_{i=1}^{N} p_{0i} = 1$. Then, each $p_i(t)$, $i = 1, \ldots N$ is nonnegative, uniformly bounded and contained in the interval $[0,1]$ for every $t \geq 0$. Furthermore,

$$\sum_{i=1}^{N} p_i(t) = 1, \quad \forall t \geq 0$$

Proof: See the Appendix. ■

The following result establishes the convergence of each local state estimation error $\tilde{x}_i(t) := \hat{x}_i(t) - x(t)$, $i = 1, \ldots, N$.

Lemma 1: Suppose that there exist positive constants $\delta_1, \delta_2 \in (0,\infty)$ such that

$$\delta_1 I \leq G(t)Q(t)G(t)' \leq \delta_2 I. \tag{6}$$

Then, if P remains uniformly bounded, there exist positive constants c_i, λ_i, γ_i^w, γ_i^v, γ_i^ϕ such that

$$\|\tilde{x}_i(t)\| \leq c_ie^{-\lambda_it}\|\tilde{x}_i(0)\| + \gamma_i^w \sup_{\tau\in[0,t)} \|w(\tau)\| + \gamma_i^v \sup_{\tau\in[0,t)} \|v(\tau)\| + \gamma_i^\phi \sup_{\tau\in[0,t)} \|\phi_i(t,x(\tau),u(\tau))\|, \tag{7}$$

$\forall t \geq 0$, where

$$\phi_i(t,x,u) := \Delta A_ix + \Delta B_iu - L_i\Delta C_ix,$$

$L_i := P_iC_i'R_i^{-1}$ is the Kalman or observer gain, and $\Delta(\cdot)_i := (\cdot)_i - (\cdot)$ denotes the mismatch between the model used to derive the i^{th} Kalman-filter estimator and the true system (1).

Proof: See the Appendix. ■

Remark 1: If the true parameter vector θ belongs to Θ, i.e., $\theta_{i^\star} = \theta$, for some $i^\star \in \{1, 2, \dots N\}$, and if the assumptions of Lemma 1 hold, the state estimate $\hat{x}_{i^\star}$ converges exponentially fast to the state x in the absence of disturbance input w and measurement noise v because $\phi_{i^\star}$ in (7) is zero. The same can be concluded for w and v nonzero but $\mathcal{L}_2$ signals. In general, when the disturbance, noise, and ϕ_i are bounded signals, $\hat{x}_{i^\star}$ converges to a neighborhood of the true state x. □

The following Lemma provides conditions for the convergence of the dynamic weights $p_i(t)$ and, consequently, for the asymptotic identification of the correct model.

Lemma 2: Let $i^\star \in \{1, 2, \dots, N\}$ be an index of a parameter vector in Θ, $\mathcal{I} := \{1, 2, \dots, N\} \backslash i^\star$ an index set, $\Upsilon^n := \{v \in \mathbb{R}^n : v_i \geq 0, i = 1, \dots, n\}$ a nonnegative vector space, and $M(t)$ a $n \times (N-1)$ matrix whose columns are the $N-1$ "local" output estimation errors $\tilde{y}_i := \hat{y}_i - y$, $\forall i \in \mathcal{I}$. Suppose that there exist positive constants ϵ, μ and T such that for all $t \geq 0$ the following persistent excitation (PE)-like conditions hold

$$\frac{1}{T}\int_t^{t+T} \|\tilde{y}_{i^\star}(\tau)\|^2_{R^{-1}} d\tau \leq \epsilon, \tag{8a}$$

$$\frac{1}{T}\int_t^{t+T} \left\|M(\tau)v\right\|^2_{R^{-1}} d\tau \geq \mu, \quad \forall v \in \Upsilon^n, \|v\| = 1. \tag{8b}$$

Then, $p_{i^\star}(t)$ converges to the residual set $(1 - \sqrt{\frac{2\epsilon}{\mu}}, 1]$. Furthermore, if in the limit $T \to \infty$, ϵ is zero, then $p_{i^\star}(t) \to 1$ as $t \to \infty$.

Proof: See the Appendix. ■

Remark 2: The significance of Lemma 2 is that for the case of $\theta \in \Theta$, the correct deterministic "model" is identified. We remark that in the stochastic version of the CT-MMAE problem [11], [12], no probabilistic convergence results of the same type were derived. However, our simulations show that this convergence property is true for both the deterministic and the stochastic case.

Remark 3: The local output estimation error $\tilde{y}_i$ is the output of the following dynamic system

$$\begin{aligned}\begin{bmatrix} \dot{\tilde{x}}_i \\ \dot{x} \end{bmatrix} &= \begin{bmatrix} A_i - L_i C_i & \Delta A_i - L_i \Delta C_i \\ 0 & A \end{bmatrix}\begin{bmatrix} \tilde{x}_i \\ x \end{bmatrix} + \begin{bmatrix} \Delta B_i \\ B \end{bmatrix} u \\ &\quad + \begin{bmatrix} -G \\ G \end{bmatrix} w + \begin{bmatrix} L_i \\ 0 \end{bmatrix} v \\ \tilde{y}_i &= \begin{bmatrix} C_i & \Delta C_i \end{bmatrix}\begin{bmatrix} \tilde{x}_i \\ x \end{bmatrix}\end{aligned}$$

It is clear that for $i = i^\star$, if $\Delta(\cdot) = 0$ and w and v are vanishing signals then

$$\lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} \|\tilde{y}_{i^\star}(\tau)\|^2_{R^{-1}} d\tau = 0,$$

which implies from Lemma 2 that $p_{i^\star} \to 1$. □

We now state the main result of the paper.

Theorem 1: Suppose that (1) is asymptotically stable, (6) holds, P remains uniformly bounded, and the PE-like assumptions (8) hold for some $i = i^\star \in \{1, \dots, N\}$. Then the CT-MMAE enjoys the following properties:

1) For the particular case that the true parameter θ belongs to the discrete set Θ and w and v are $\mathcal{L}_2$ signals, then the state estimation error $\tilde{x}(t) := \hat{x}(t) - x(t)$ and the parameter estimation error $\tilde{\theta}(t) := \hat{\theta}(t) - \theta$ satisfy

$$\lim_{t\to\infty} \tilde{x}(t) = 0, \quad \lim_{t\to\infty} \tilde{\theta}(t) = 0.$$

2) For the general case, $\tilde{x}(t)$ enjoys the following input-to-state (ISS) like condition

$$\begin{aligned}\|\tilde{x}(t)\| &\leq c e^{-\lambda t}\|\tilde{x}(0)\| + \gamma_w \sup_{\tau\in[0,t)} \|w(\tau)\| \\ &\quad + \gamma_v \sup_{\tau\in[0,t)} \|v(\tau)\| + \gamma_u \sup_{\tau\in[0,t)} \|u(\tau)\|, \end{aligned} \tag{9}$$

$\forall t \geq 0$, and $\tilde{\theta}(t)$ converges to the residual set

$$\Big[0, (1 - \sqrt{\frac{2\epsilon}{\mu}})\|\tilde{\theta}_{i^\star}\| + \sqrt{\frac{2\epsilon}{\mu}} \max_{i\in\mathcal{I}} \|\tilde{\theta}_i\|\Big] \tag{10}$$

where $\tilde{\theta}_i := \theta_i - \theta$, $i = 1, \dots, N$.

Proof: We start by proving the second statement. Inequality (9) follows from

$$\tilde{x} = \sum_{i=1}^N p_i \hat{x}_i - \sum_{i=1}^N p_i x = \sum_{i=1}^N p_i \tilde{x}_i \tag{11}$$

and the fact that each $\tilde{x}_i$ can be viewed as a cascade of two ISS systems: system (14) (see inequality (7)) together with

$$\|\phi_i\| \leq \big(\|\Delta A_i\| + \|L_i \Delta C_i\|\big)\|x\| + \|\Delta B_i\|\|u\|$$

and system (1) which is ISS with respect to the inputs u and w. To conclude (10), we first notice that

$$\tilde{\theta} = \sum_{i=1}^N p_i \theta_i - \sum_{i=1}^N p_i \theta = \sum_{i=1}^N p_i \tilde{\theta}_i \tag{12}$$

where we have used the fact that $\sum_{i=1}^N p_i = 1$. Thus,

$$\|\tilde{\theta}\| \leq \sum_{i=1}^N p_i \|\tilde{\theta}_i\| \leq p_{i^\star}\|\tilde{\theta}_{i^\star}\| + (1 - p_{i^\star}) \max_{i\in\mathcal{I}} \|\tilde{\theta}_i\|$$

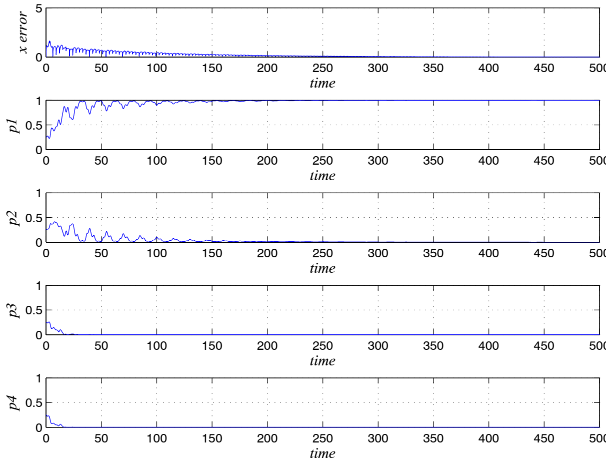
and therefore, using Lemma 2 it follows that $\tilde{\theta}$ will converge to the residual set (10). The first Statement of Theorem 1 can be concluded from (11) and (12) together with the results $p_{i^\star} \to 1$ and $\tilde{x}_{i^\star} \to 0$ as $t \to \infty$ of Lemma 2. ■

V. Illustrative example

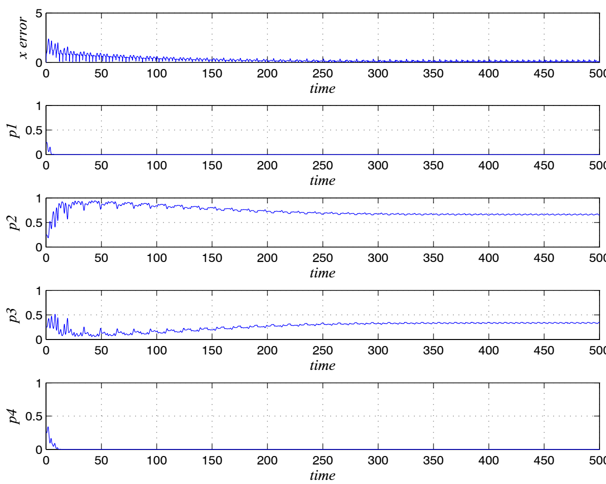
To illustrate the results of the paper we consider a third-order system composed by a first-order pre-filter $\frac{5}{s+5}$ in cascade with a second-order linear harmonic oscillator $\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$. A state-space representation of the plant, including the disturbance and noise inputs, is given by

$$\begin{aligned}\dot{x} &= \begin{bmatrix} -5 & 0 & 0 \\ 0 & 0 & 1 \\ \omega_n^2 & -\omega_n^2 & -2\xi\omega_n \end{bmatrix} x + \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} w \\ y &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x + v\end{aligned}$$

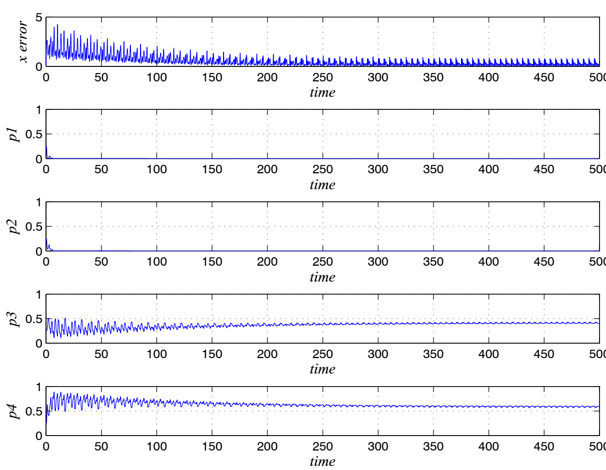
where the damping ratio $\xi = 0.1$ is fixed but the undamped natural frequency $\omega_n \in [0.5, 5]$ is assumed to be an unknown parameter. Figures 2 and 3 show the simulations results



(a) $\omega_n = 0.7\, rad/s$



(b) $\omega_n = 1.2\, rad/s$



(c) $\omega_n = 2.7\, rad/s$

Fig. 2. Deterministic case: Time evolution of the norm of the state estimation error $\|\tilde{x}\|$ and the dynamic weights p_i, $i = 1, 2, 3, 4$, respectively.

obtained with three different values for ω_n using deterministic and stochastic disturbances, respectively. The set of candidate parameter values was $\Theta = \{0.7, 1.0, 2.0, 4.0\}$ defining a set of four "models". For each model a steady-state Kalman filter was designed using the following values for the white disturbance and noise covariances: $E\{w(t)w(\tau)\} = Q\delta(t-\tau)$, $Q = 100$, $E\{v(t)v(\tau)\} = R\delta(t-\tau)$, $R = 0.1$. The dynamic weights p_i were initialized to $p_i(0) = 0.25$, $i = 1, \ldots, 4$. The results shown in Fig. 2 assume that the input control signal and the $\mathcal{L}_2$ disturbance and noise signals are

$$u = \mathrm{sqr}(T_u), \quad w = e^{-\lambda t}\,\mathrm{sqr}(T_w), \quad v = e^{-\lambda t}\sin(\omega t)$$

where $\mathrm{sqr}(T)$ denotes a square wave of amplitude 1.0 and period T, $T_u = 10\, s$, $T_w = 6\, s$, $\lambda = 0.01$, and $\omega = 50\, rad/s$.

As can be seen, for the case of $\omega_n = 0.7$ in Fig. 2(a) (the true plant parameter is identical to one of the elements of Θ) the convergence of the estimation error $\tilde{x}$ to zero is achieved and the dynamic weights p_i converge to the correct model, i.e., $p_1(t) \to 1$. Similar behaviour (not shown) was noted when the true plant parameter was a member of the discrete set Θ. These simulation results agree with Lemma 2.

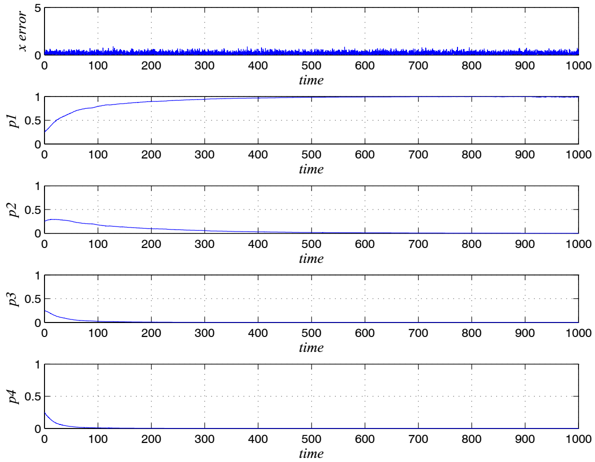
Figure 2(b) shows the behaviour when $\omega_n = 1.2$ while Fig. 2(c) uses the value $\omega_n = 2.7$. In both cases, these truth values are not a member of the selected finite set Θ. The state estimation error converges to near zero, but the weights p_i do not converge to any of the finite models. In the situation of Fig. 2(b) the parameter estimation (3) yields $\hat{\omega}_n \simeq 1.33$ which is close, but not identical to $\omega_n = 1.2$. In Fig 2(c) the parameter estimate yields $\hat{\omega}_n \simeq 3.1$ which is close, but not identical to $\omega_n = 2.7$. These asymptotic errors illustrate the presence of the residual set (10).

Figure 3 presents numerical averages for 5 Monte-Carlo simulations with $u = 0$, and w and v white noise with covariances intensities $Q = 100$ and $R = 0.1$, respectively. We did these simulations to see whether or not there was a significant difference between the deterministic and the stochastic versions of the CT-MMAE. By comparing Figs. 2 and 3 we conclude, that for this example, there were no significant differences.

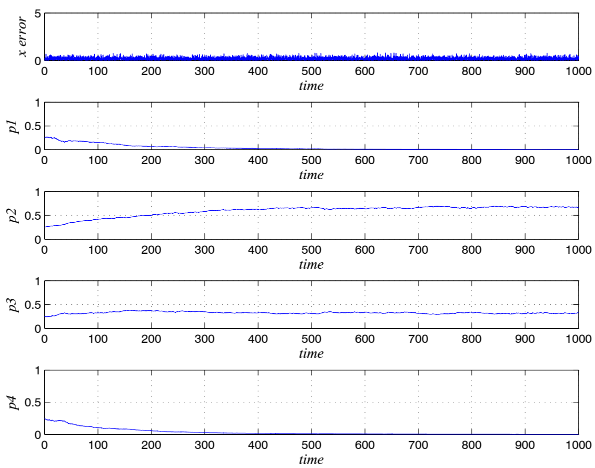
In the simulations of Fig. 3(a) we note that $p_1(t)$ converges to the "true model". In this case, the $p_i(t)$ are true conditional probabilities [11], [12] but their convergence to the true model was not proven in [12] and [11]. The simulations in Figs. 3(b)–3(c) demonstrate (again for this specific system) that the posterior probabilities converge somewhere between the "nearest" finite models. This behaviour *is different* from the stochastic discrete-time MMAE case for which it has been proven that one of the posterior probabilities will converge almost surely to a "nearest" probabilistic neighbor (see [9], [10]) using an information-theoretic distance metric.

VI. Conclusions

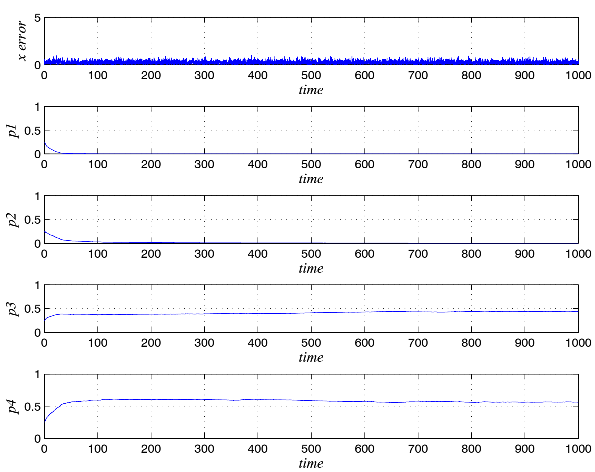
We presented and analyzed a CT-MMAE system for state-affine MIMO systems with parametric uncertainty. We showed that for the case where the unknown noise and disturbance are $\mathcal{L}_2$ signals and under appropriate persistent-excitation and observability assumptions, if the actual plant parameter is identical to one of its discrete values, the state estimate converges globally asymptotically to the true value



(a) $\omega_n = 0.7\, rad/s$



(b) $\omega_n = 1.2\, rad/s$



(c) $\omega_n = 2.7\, rad/s$

Fig. 3. Stochastic case: Time evolution of the norm of the state estimation error $\|\tilde{x}\|$ and the dynamic weights p_i, $i = 1, 2, 3, 4$, respectively.

and the plant model is correctly identified. If the actual plant parameter vector does not belong to the finite discrete parameter set, the estimates (state and parameter vector) converge to a neighborhood of the true values.

The simulation results show some interesting behaviour. In both the deterministic and the stochastic simulations, when the unknown parameter belongs to the finite set used to design the Kalman filters, one of the signals $p_i(t)$ converges to unity thereby identifying the correct model; see Figs. 2(a), 3(a). However, as shown in Figs. 2(b)-2(c) and 3(b)-3(c), when the unknown parameter is between finite values, the signals $p_i(t)$ display an "averaging" behaviour which differs from the available stochastic discrete-time MMAE results, [9], [10], regarding convergence to the "nearest probabilistic neighbor" almost surely. This intriguing difference between the continuous-time and discrete-time MMAE requires further research.

Another point that is a topic of current research is to extend the results of the paper taking into account that the plant may have unstructured unmodeled dynamics.

APPENDIX

Proposition 1

Proof: Defining $p(t) := \sum_{i=1}^{N} p_i(t)$ and computing its time-derivative we obtain

$$\begin{aligned}\dot{p} &= \Big(\sum_{i=1}^{N} p_i \hat{y}_i - \sum_{i=1}^{N} p_i \hat{y}\Big) R^{-1}(y - \hat{y}) \\ &= -(1-p)\hat{y}R^{-1}\tilde{y}\end{aligned}$$

where

$$\hat{y}_i := C_i \hat{x}_i \tag{13a}$$

$$\hat{y} := \sum_{i=1}^{N} p_i C_i \hat{x}_i \tag{13b}$$

$$\tilde{y} := \hat{y} - y \tag{13c}$$

Therefore, if $p(0) = 1$, then $\dot{p}$ and p satisfy $\dot{p}(t) = 0$, $p(t) = 1$, $\forall t \geq 0$. If $p_i(0) > 0$, we can also conclude that $p_i(t) \geq 0$, $\forall t \geq 0$ because by contradiction, if it was not the case, then there would be a finite time T_i such that $p_i(T_i) = 0$. But in that case $\dot{p}_i(T_i) = 0$ and, consequently, p_i would be zero for all $t \geq T_i$. The boundedness $p_i(t) \in [0, 1]$, $\forall t \geq 0$ follows immediately from the fact that $p_i \geq 0$ and $p = 1$ for all $t \geq 0$. ■

Lemma 1

Proof: From (1a) and (4b) we conclude that

$$\dot{\tilde{x}}_i = (A_i - L_i C_i)\tilde{x}_i - Gw + L_i v + \phi_i. \tag{14}$$

Consider now the evolution of P_i^{-1} given by

$$\dot{P}_i^{-1} = -P_i^{-1}A_i - A_i' P_i^{-1} - P_i^{-1} G Q G' P_i^{-1} + C_i' R^{-1} C_i \tag{15}$$

which follows from (4a) and the fact that $\frac{d}{dt}P_i^{-1} = -P_i^{-1}\dot{P}_i P_i^{-1}$. Defining $V_i(\tilde{x}_i) := \tilde{x}_i' P_i^{-1} \tilde{x}_i$, computing its time-derivative, and using (14)–(15), yields

$$\begin{aligned}\dot{V}_i = &-\tilde{x}_i'(P_i^{-1} G Q G' P_i^{-1} + C_i' R^{-1} C_i)\tilde{x}_i - 2\tilde{x}_i' P_i^{-1} G w \\ &+ 2\tilde{x}_i' C_i' R^{-1} v + 2\tilde{x}_i' P_i^{-1} \phi_i\end{aligned}$$

By completing the squares, we further conclude that

$$\begin{aligned}\dot V_i = &-\frac{1}{2}\tilde x_i'(P_i^{-1}GQG'P_i^{-1}+C_i'R^{-1}C_i)\tilde x_i\\&-\big\|\frac{1}{2}Q^{\frac{1}{2}}G'P_i^{-1}\tilde x_i+2Q^{-\frac{1}{2}}w\big\|^2+4\|w\|^2_{Q^{-1}}\\&-\frac{1}{2}\|C_i\tilde x_i+2v\|^2_{R^{-1}}+2\|v\|^2_{R^{-1}}\\&-\big\|\frac{1}{2}G'P_i^{-1}\tilde x_i+2G'(GQG')^{-1}\phi_i\big\|^2_Q\\&+4\|G'(GQG')^{-1}\phi_i\|^2_Q\\\le&-\frac{1}{2}\tilde x_i'(P_i^{-1}GQG'P_i^{-1}+C_i'R^{-1}C_i)\tilde x_i+4\|w\|^2_{Q^{-1}}\\&+2\|v\|^2_{R^{-1}}+4\|G'(GQG')^{-1}\phi_i\|^2_Q\end{aligned}$$

Here, given a positive semi-definite matrix Q we denote by $Q^{\frac{1}{2}}$ any matrix such that $(Q^{\frac{1}{2}})'Q^{\frac{1}{2}}=Q$. In case P is uniformly bounded, P^{-1} is uniformly positive definite and so is $P^{-1}GQG'P^{-1}$. In this case,

$$\begin{aligned}\dot V_i\le&-\frac{1}{2}\delta\mu V_i+4\|w\|^2_{Q^{-1}}+2\|v\|^2_{R^{-1}}\\&+4\|G'(GQG')^{-1}\phi_i\|^2_Q\end{aligned}$$

where $\mu>0$ is some constant that satisfies $P^{-1}(t)\ge\mu I,\ \forall t\ge 0$. It is now straightforward to conclude that the ISS-like bound (7) holds. ■

Lemma 2

Proof: From (5) and using the definitions (13a)–(13c) we obtain for $i=i^\star$

$$\dot p_{i^\star}=\psi(t)p_{i^\star} \tag{16}$$

where

$$\psi(t):=\frac{1}{2}\big(\|\tilde y-\tilde y_{i^\star}\|^2_{R^{-1}}+\|\tilde y\|^2_{R^{-1}}-\|\tilde y_{i^\star}\|^2_{R^{-1}}\big).$$

Notice that $\tilde y:=\hat y-y$ satisfies

$$\tilde y=\sum_{i=1}^N p_iC_i\hat x_i-\sum_{i=1}^N p_iy=\sum_{i=1}^N p_i\tilde y_i.$$

Thus, using the inequality[2] $\|a-b\|^2_P\ge(1-\kappa)\|a\|^2_P-(\frac{1}{\kappa}-1)\|b\|^2_P$, $\kappa\in(0,1)$, we conclude that

$$\begin{aligned}\psi(t)=&\frac{1}{2}\Big(\big\|(p_{i^\star}-1)\tilde y_{i^\star}+\sum_{i\in\mathcal I}p_i\tilde y_i\big\|^2_{R^{-1}}\\&+\big\|p_{i^\star}\tilde y_{i^\star}+\sum_{i\in\mathcal I}p_i\tilde y_i\big\|^2_{R^{-1}}-\|\tilde y_{i^\star}\|^2_{R^{-1}}\Big)\\\ge&\frac{1}{2}(1-\kappa)\big\|\sum_{i\in\mathcal I}p_i\tilde y_i\big\|^2_{R^{-1}}\\&-\frac{1}{2}\big(\frac{1}{\kappa}-1\big)(p_{i^\star}-1)^2\|\tilde y_{i^\star}\|^2_{R^{-1}}\\&+\frac{1}{2}(1-\kappa)\big\|\sum_{i\in\mathcal I}p_i\tilde y_i\big\|^2_{R^{-1}}\\&-\frac{1}{2}\Big(p_{i^\star}^2\big(\frac{1}{\kappa}-1\big)+1\Big)\|\tilde y_{i^\star}\|^2_{R^{-1}}\\=&(1-\kappa)\big\|\sum_{i\in\mathcal I}p_i\tilde y_i\big\|^2_{R^{-1}}\\&-\frac{1}{2}\Big(\big(\frac{1}{\kappa}-1\big)(2p_{i^\star}^2-2p_{i^\star}+1)+1\Big)\|\tilde y_{i^\star}\|^2_{R^{-1}}\\\ge&(1-\kappa)\big\|\sum_{i\in\mathcal I}p_i\tilde y_i\big\|^2_{R^{-1}}-\frac{1}{2\kappa}\|\tilde y_{i^\star}\|^2_{R^{-1}}\end{aligned}$$

From this inequality and the PE assumptions (8) we further conclude that

$$\begin{aligned}\frac{1}{T}\int_t^{t+T}\psi(\tau)\,d\tau\ge&\frac{1-\kappa}{T}\int_t^{t+T}(1-p_{i^\star})^2\|M(\tau)v\|^2_{R^{-1}}\,d\tau\\&-\frac{1}{2\kappa T}\int_t^{t+T}\|\tilde y_{i^\star}\|^2_{R^{-1}}\,d\tau,\ \forall v\in\Upsilon^2,\ \|v\|=1\\\ge&(1-\kappa)\big(1-\max_{t\ge0}p_{i^\star}(t)\big)^2\mu-\frac{1}{2\kappa}\epsilon\\\ge&\frac{1}{2}\big(1-\max_{t\ge0}p_{i^\star}(t)\big)^2\mu-\epsilon.\end{aligned}$$

Returning to (16). Let $\bar t\ge T$ be an arbitrarily finite time and $\bar n$ the largest integer that satisfies $\bar n\le\frac{\bar t}{T}$. Then, we obtain

$$\begin{aligned}p_{i^\star}(t)=&p_{i^\star}(0)\exp\Big(\frac{1}{2}\int_0^{t-\bar t}\psi(\tau)\,d\tau+\int_{t-\bar t}^t\psi(\tau)\,d\tau\Big)\\\ge&p_{i^\star}(0)\exp\Big(\int_0^{t-\bar t}\psi(\tau)\,d\tau\\&+\sum_{j=1}^{\bar n}\int_{t-\bar t+(j-1)T}^{t-\bar t+jT}\psi(\tau)\,d\tau\Big)\\\ge&p_{i^\star}(0)\exp\Big(\int_0^{t-\bar t}\psi(\tau)\,d\tau\\&+\bar n\big[\frac{1}{2}\big(1-\max_{t\ge0}p_{i^\star}(t)\big)^2\mu-\epsilon\big]T\Big),\quad\forall t\ge\bar t\end{aligned}$$

From the definition of $\bar n$ we conclude that $\bar n\ge\frac{\bar t}{T}-1$ and consequently

$$\begin{aligned}&\bar n\big[\frac{1}{2}\big(1-\max_{t\ge0}p_{i^\star}(t)\big)^2\mu-\epsilon\big]T\ge\\&\qquad\big(\frac{\bar t}{T}-1\big)\big[\frac{1}{2}\big(1-\max_{t\ge0}p_{i^\star}(t)\big)^2\mu-\epsilon\big]T\\&\qquad=\big[\frac{1}{2}\big(1-\max_{t\ge0}p_{i^\star}(t)\big)^2\mu-\epsilon\big](\bar t-T).\end{aligned}$$

Thus,

$$p_{i^\star}(t)=p_{i^\star}(0)\exp\Big(\frac{1}{2}\int_0^{t-\bar t}\psi(\tau)\,d\tau\Big)e^{\left[\frac{1}{2}\left(1-\max_{t\ge0}p_{i^\star}(t)\right)^2\mu-\epsilon\right](\bar t-T)},$$

and, in particular, for $t=\bar t$ we have

$$p_{i^\star}(\bar t)\ge p_{i^\star}(0)e^{\left[\frac{1}{2}\left(1-\max_{t\ge0}p_{i^\star}(t)\right)^2\mu-\epsilon\right](\bar t-T)}.$$

It is now straightforward to conclude that $p_{i^\star}$ will increase its value to at least $1-\sqrt{\frac{2\epsilon}{\mu}}$. Also, if in the limit as $T\to\infty$, ϵ can be taken zero, then $p_{i^\star}\to1$ as $t\to\infty$. ■

[2] $\|a-b\|^2_P=(1-\kappa)\|a\|^2_P+(1-\frac{1}{\kappa})\|b\|^2_P+\kappa\|a\|^2_P+\frac{1}{\kappa}\|b\|^2_P-2a'Pb=(1-\kappa)\|a\|^2_P+(1-\frac{1}{\kappa})\|b\|^2_P+\|\sqrt{\kappa}a-\frac{1}{\sqrt{\kappa}}b\|^2_P\ge(1-\kappa)\|a\|^2_P-(\frac{1}{\kappa}-1)\|b\|^2_P$, $\kappa>0$

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