Pre-IFAC 2008 workshop: “Cooperative Control of Multiple Autonomous Vehicles”

Design of stable collective motions on manifolds

R. Sepulchre -- University of Liege, Belgium

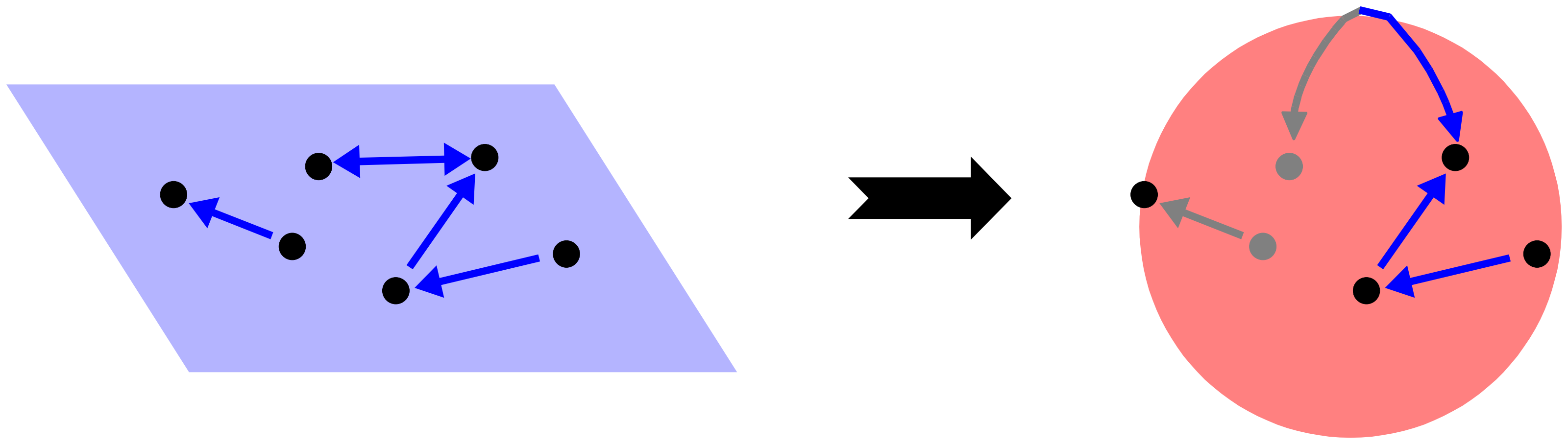
collaborators: N. Leonard & D. Paley -- Princeton University
L. Scardovi -- University of Liege / Princeton
S. Bonnabel -- University of Liege

speaker: A. Sarlette -- University of Liege

The world is not flat.

People typically disagree.

The world is not flat.



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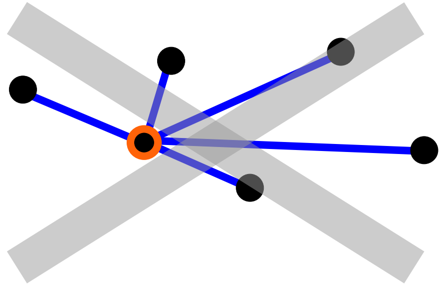

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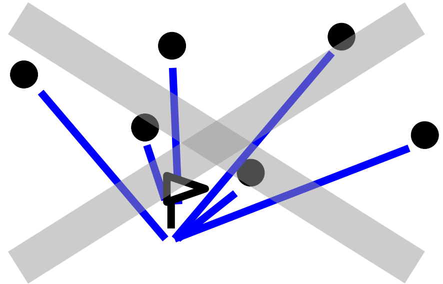

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Design of stable collective motions on manifolds

Outline. Motivating examples → problem setting

Reaching consensus on manifolds

A general control design method for collective motion on Lie groups

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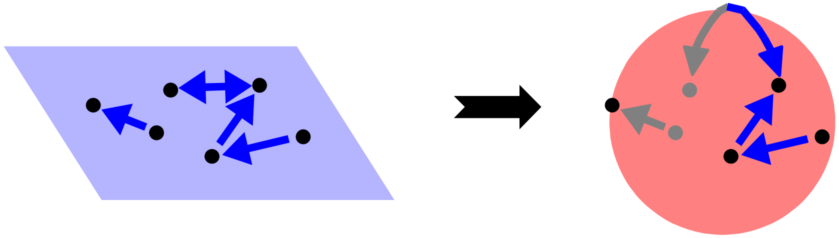
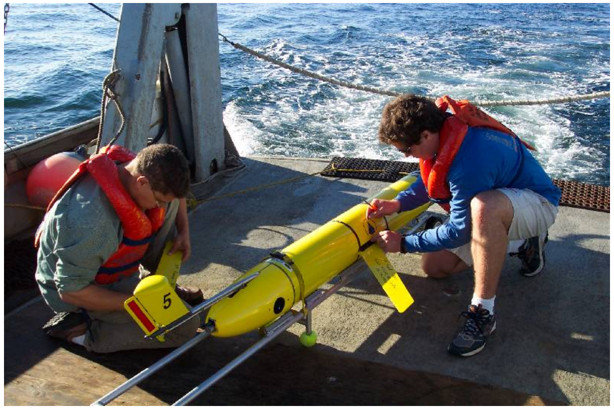
Coordination problems often involve nonlinear manifolds



I. Distributed autonomous sensor networks can be used e.g. to collect ocean data

Autonomous Ocean Sampling Network (Naomi Leonard et al.)



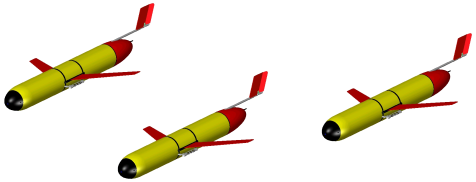
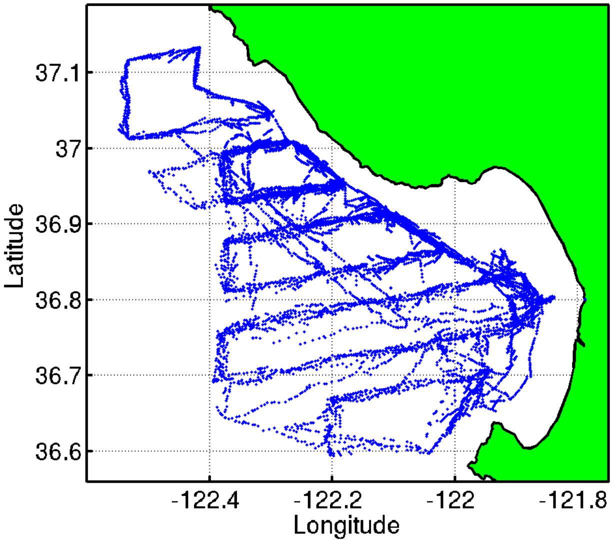


Photo by Norbert Wu



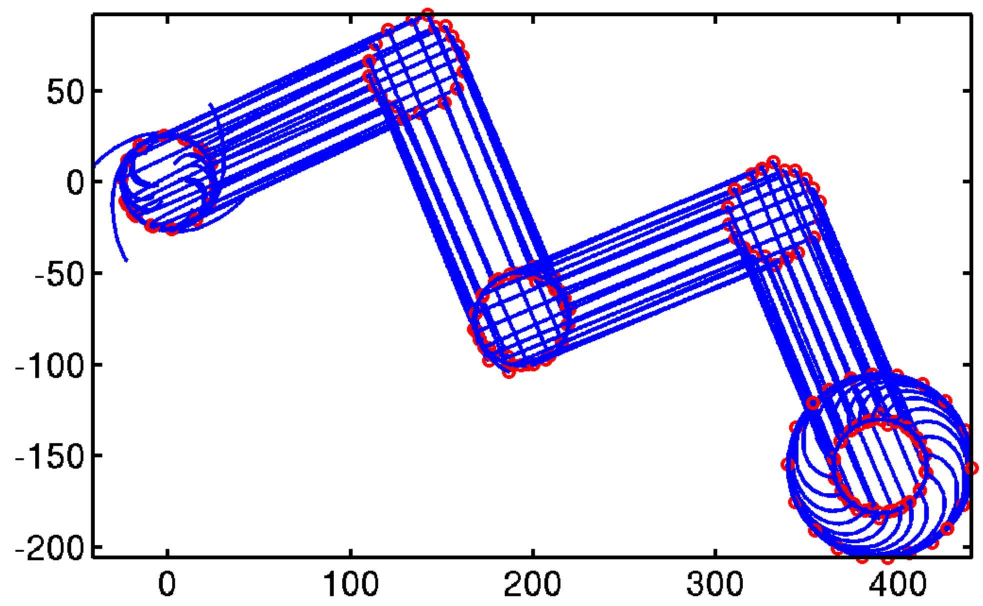
Control of the swarm is based on templates of distributed stable collective motion

Collective motion, sensor networks and ocean sampling,
N.Leonard, D.Paley, F.Lekien et al., IEEE Proceedings, 2006

Autonomous gliders,
sparse communication

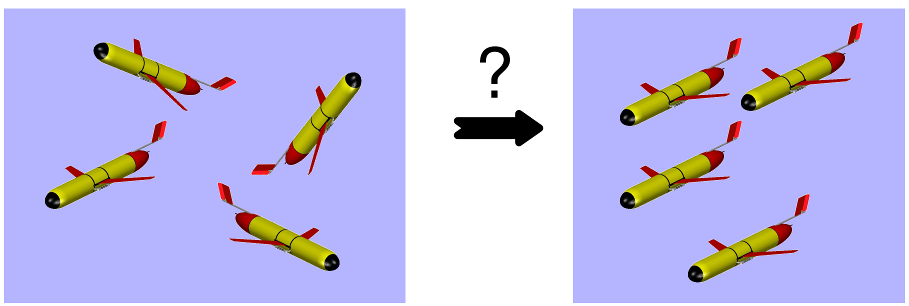
Buoyancy driven,
constant speed $\approx$40cm/s

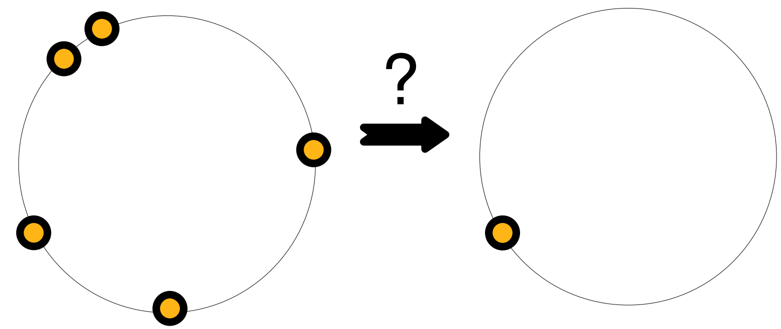
Collective path planning
with simplified model



Collective motion in the plane involves nonlinear manifolds

Common direction for straight motion → agreement on circle



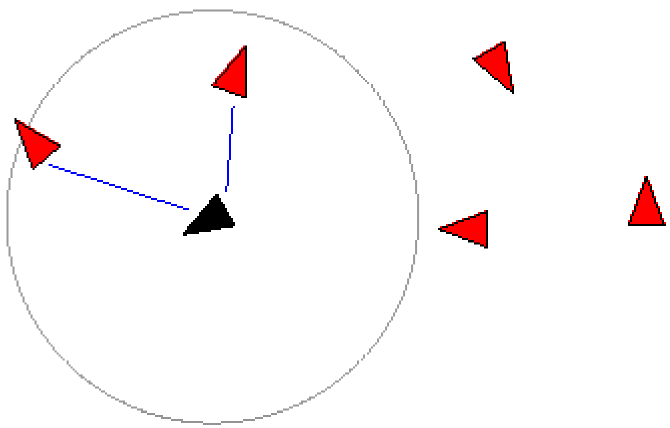


General motion “in formation” → Lie group $SE(2)$

translations $\mathbb{R}^2$
rotations S^1
} non-trivial coupling

Vicsek et al. proposed a similar model for heading synchronization

Novel type of phase transition in a system of self-driven particles, T.Vicsek, A Czirok et al., Physical Review Letters, 1995



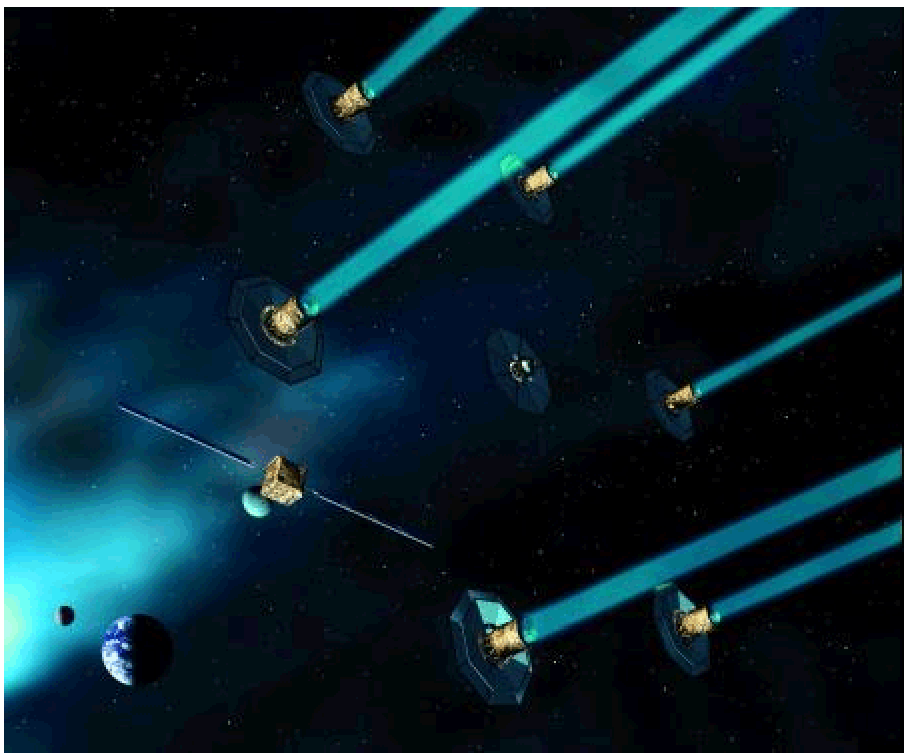
unit velocity : $(x_k)_+ = x_k + e^{i\theta_k}$

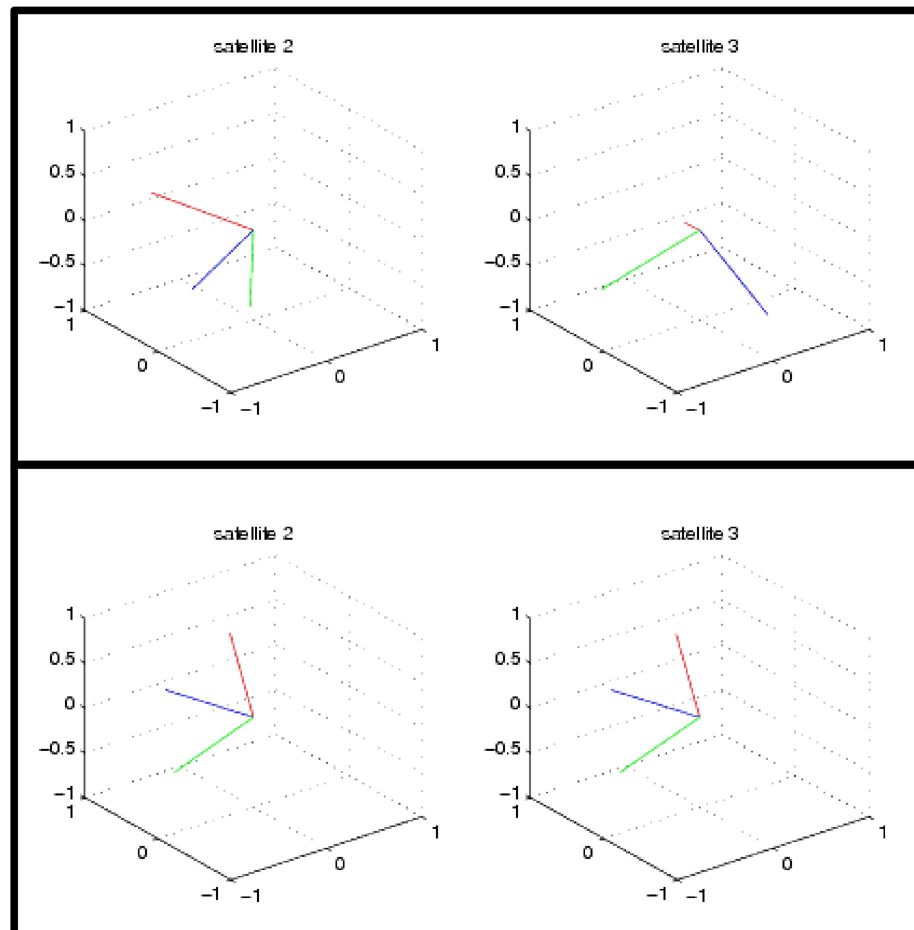
"average" direction : $(\theta_k)_+ = \arg\measuredangle \left(e^{i\theta_k} + \sum_{j \rightsquigarrow k} e^{i\theta_j} \right)$

proximity graph (open question): communicate if closer than *R*

II. Satellite formations e.g. for interferometry require attitude synchronization

Darwin space interferometer (ESA / NASA, concept under revision)





Collective motion of satellites involves nonlinear manifolds

Kinematic model :

$$\dot{Q}_k = Q_k [\omega_k]^{\wedge}$$

→ orientation matrices Q_k evolve on the Lie group $SO(3)$

Dynamic model :

$$J \dot{\omega}_k = (J \omega_k) \times \omega_k + \tau_k$$

→ simplest dynamics involve nonlinear link between torques τ_k and velocities ω_k

III. Agreement on the circle also appears for phase synchronization of oscillator networks

Flashing fireflies

Laser tuning

Huygens' clocks

Cell / neuron action

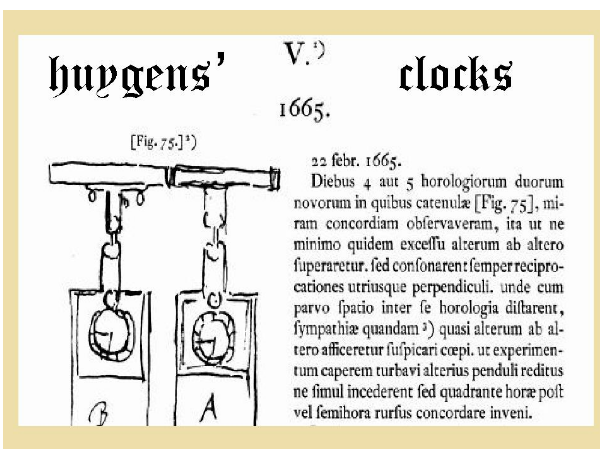
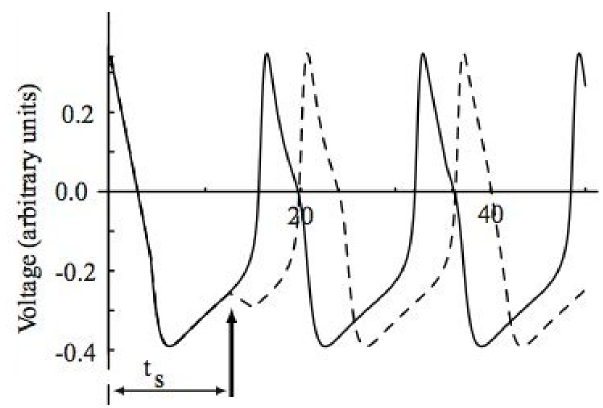


Photo by Michael Schatz



Two types of synchronization on the circle: phase synch. and frequency synch.

Phase variables $\theta_k \in$ circle, $k = 1,2,\ldots N$

Phase synchronization : $\theta_k = \theta_j \quad \forall \; k, j$

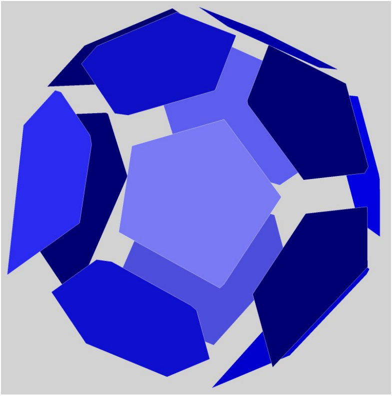
Frequency synchronization : $\dot{\theta}_k = \dot{\theta}_j \quad \forall \; k, j$

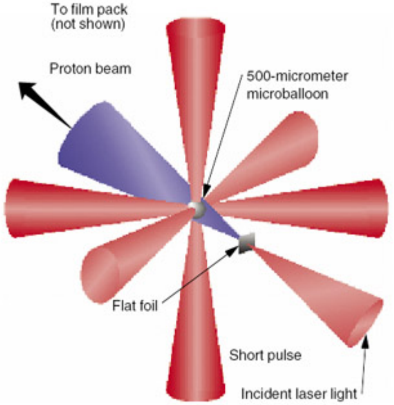
Kuramoto model
$$\dot{\theta}_k = \omega_k + \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_k)$$

Self-entrainment of population of coupled nonlinear oscillators, Y.Kuramoto, Lecture notes in Physics, vol. 39, Springer 1975

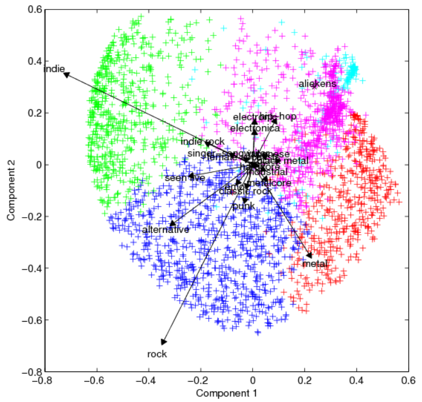
IV. Coordination on manifolds relates to many other engineering problems

Packing





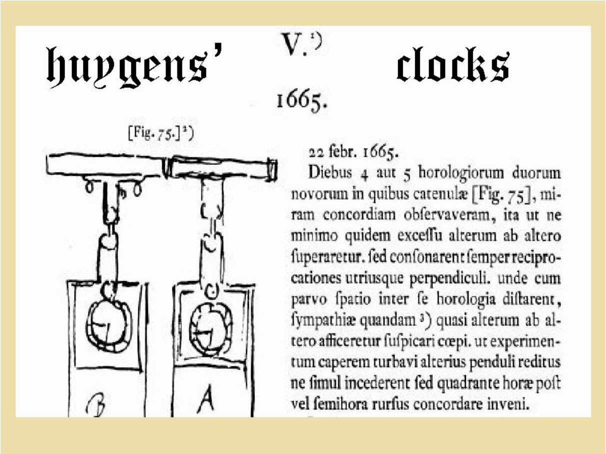
Clustering

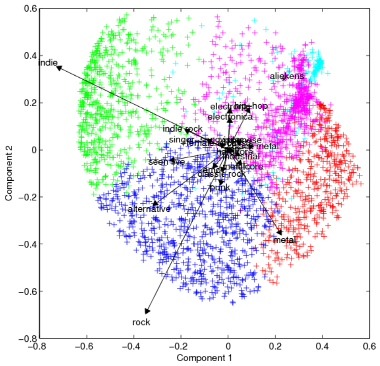


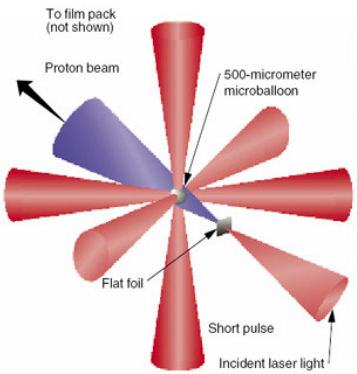
- points on spheres

- lines, subspaces of $\mathbb{R}^n$ (Grassmann manifold)

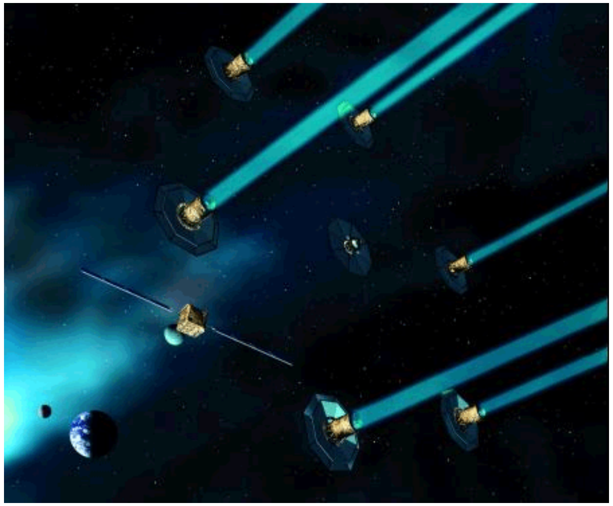
Applications: optimal coding, beam / sensor placement, numerical integration, data mining, ...

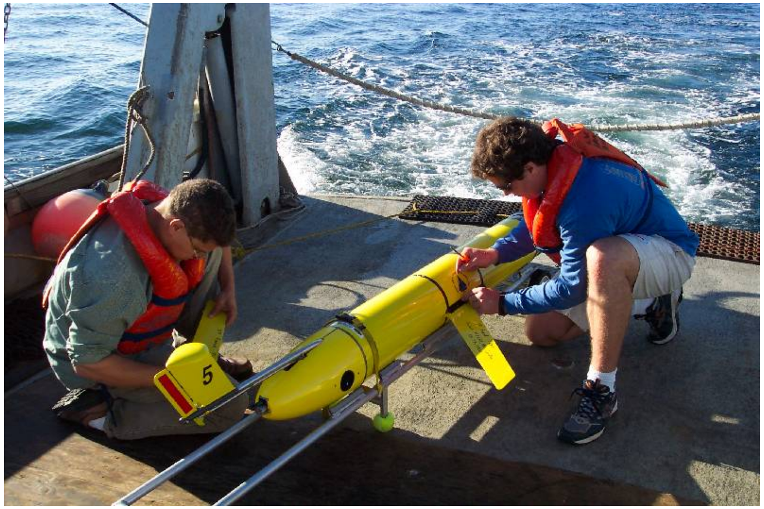






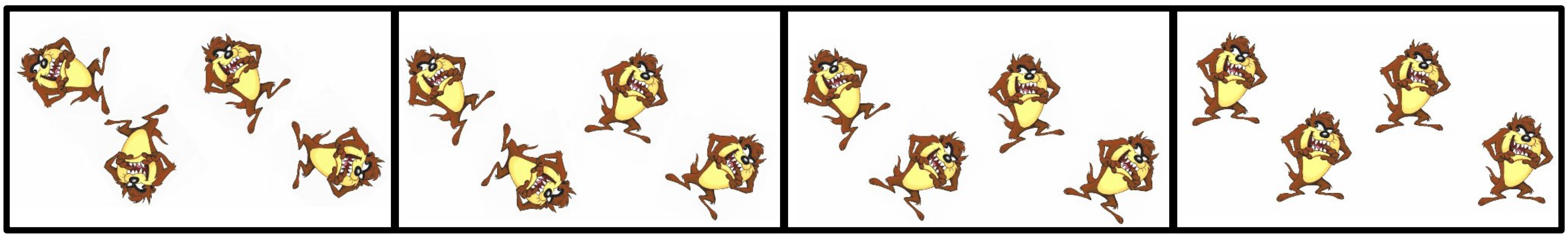
Coordination problems often involve nonlinear manifolds





Coordination on manifolds consists of two different tasks

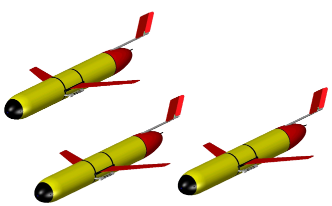
Synchronization: reach the same point on a manifold

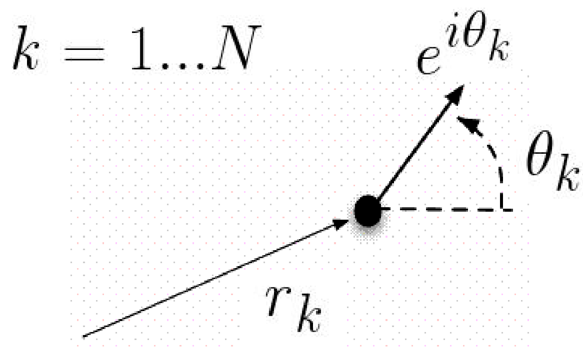


Coordinated motion: move “in formation” on a manifold

Collective motions on *SE*(2) as a representative example

N autonomous rigid bodies moving in the plane at unit speed





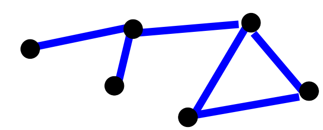
$$\dot{r}_k = e^{i\theta_k}$$

$$\dot{\theta}_k = u_k$$

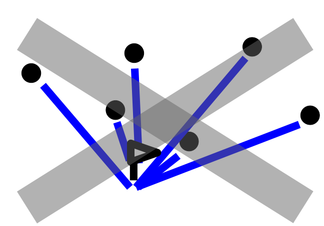
(curvature control)

Goal : design feedback control to stabilize collective motions

Restrictions : limited communication



no reference, no leader



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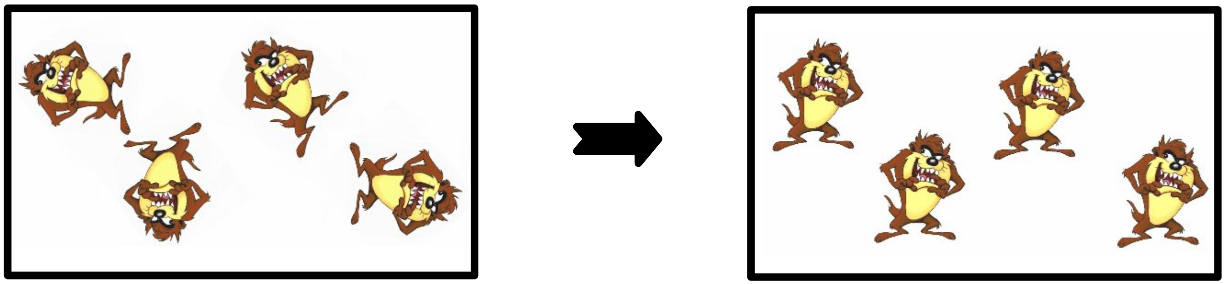
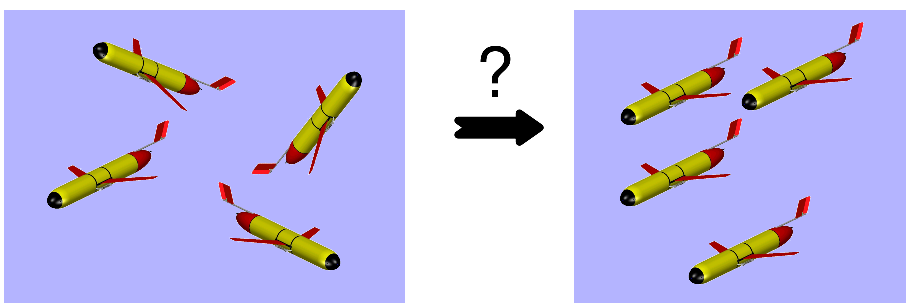
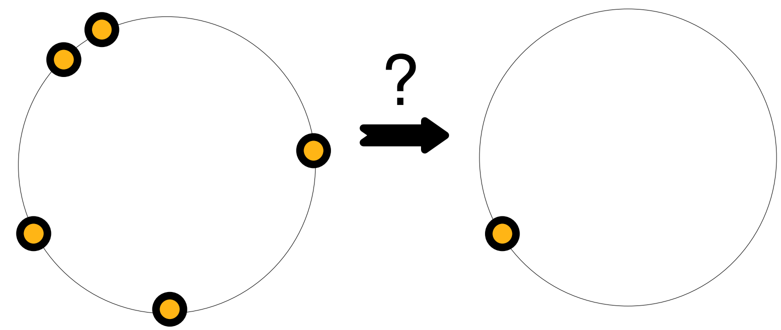


Motivation : parallel motion in the plane requires agreement on heading direction





Goals : Global phase synchronization on the circle S^1

Extension to higher dimension: sphere S^2, $SO(3)$

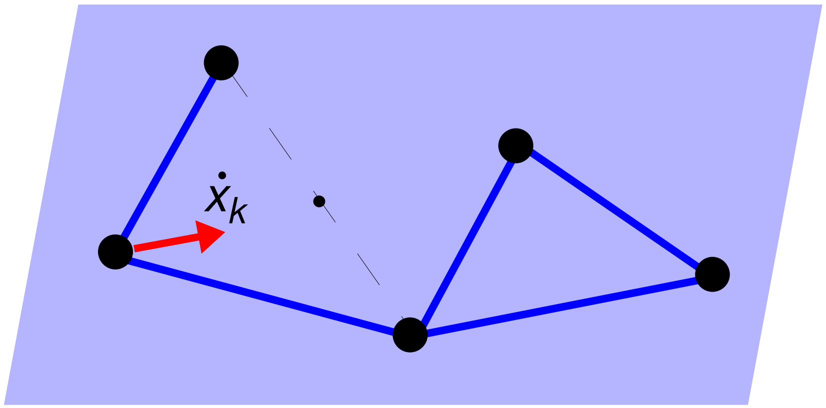
Related things: mean; balanced configurations

Agreement algorithms on vector spaces are ~easy

Goal : agree on $x_k \in \mathbb{R}^n$

Distributed algorithm

$$\dot{x}_k = \sum_{j \rightsquigarrow k} (x_j - x_k)$$



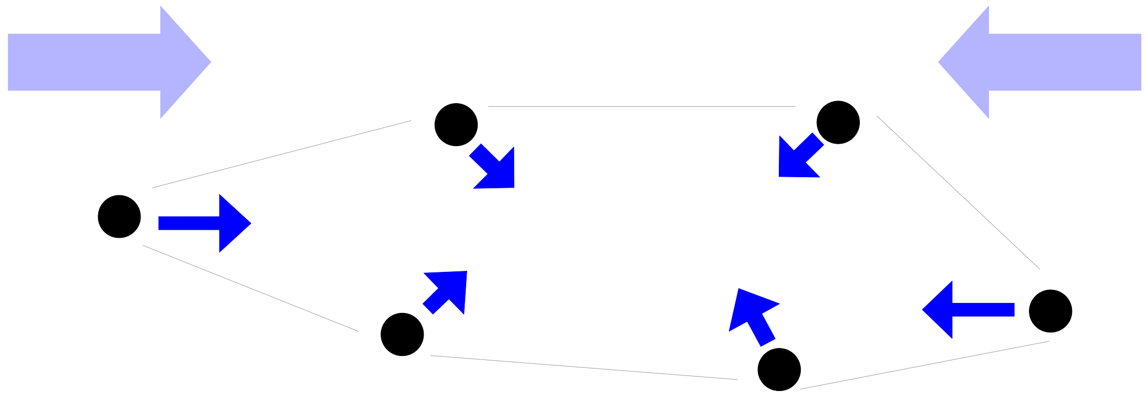
For fixed undirected interconnections

gradient of cost function $\frac{1}{2} \sum_k \sum_{j \rightsquigarrow k} \|x_j - x_k\|^2$

converges to average $\bar{x} = \frac{1}{N} \sum_k x_k(0)$

Convexity of vector spaces ensures exponential synchronization for varying & directed graphs

Stability of multi-agent systems with time-dependent communication links,
L.Moreau, IEEE Trans. Automatic Control vol. 50(2), 2005

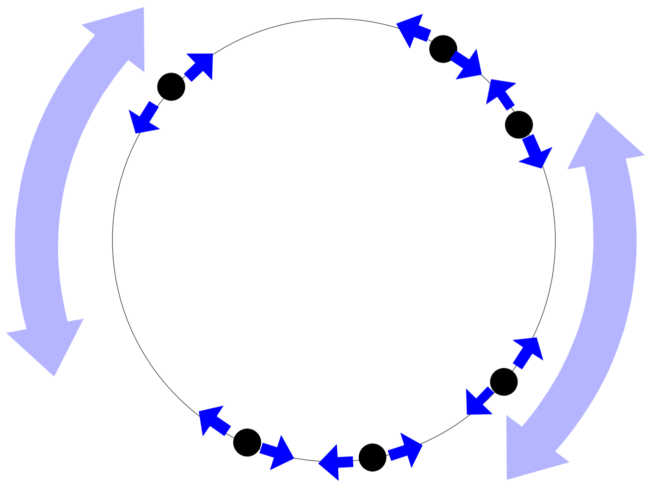


Uniformly connected interconnection : ∃ T and *k* such that

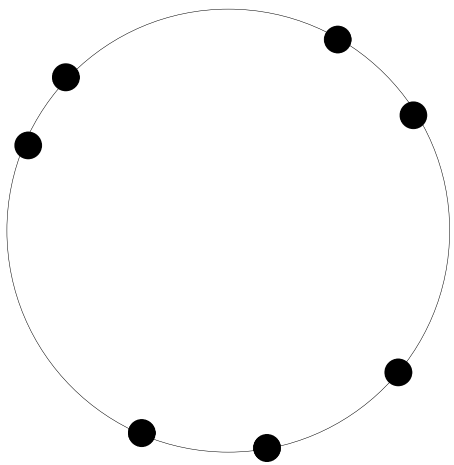
the union of links during [t, t+T] is root-connected to *k*

Synchronization on the circle is not so obvious

Goal : agree on $\theta_k \in S^1$



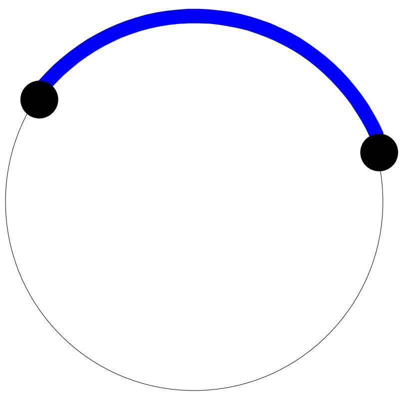
How will agents move towards neigbors ?



Where is the average position ?

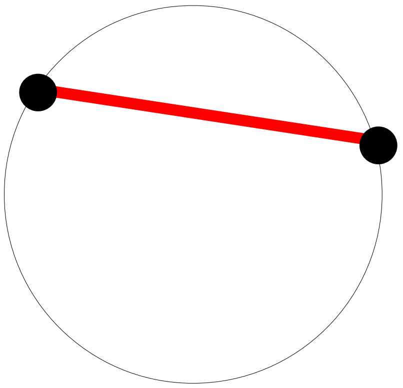
An alternative distance measure yields convenient properties

Geodesic distance



$$d_g(\theta_k, \theta_j) = |\theta_k - \theta_j|$$

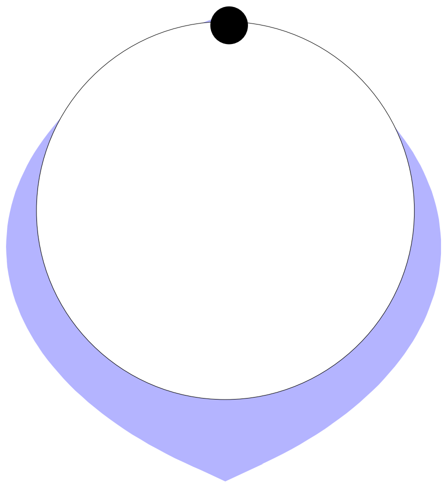
Chordal distance



$$d_c(\theta_k, \theta_j) = 2\sin\left|\frac{\theta_k - \theta_j}{2}\right|$$

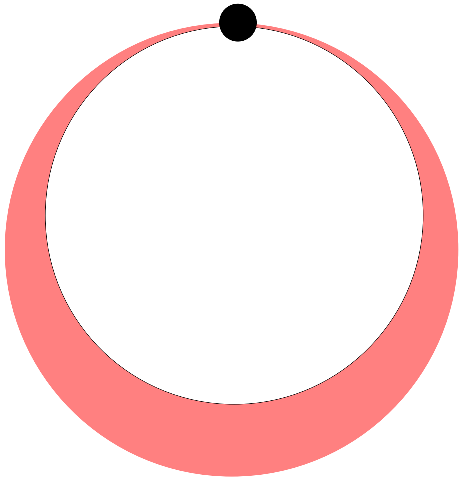
An alternative distance measure yields convenient properties

Geodesic distance



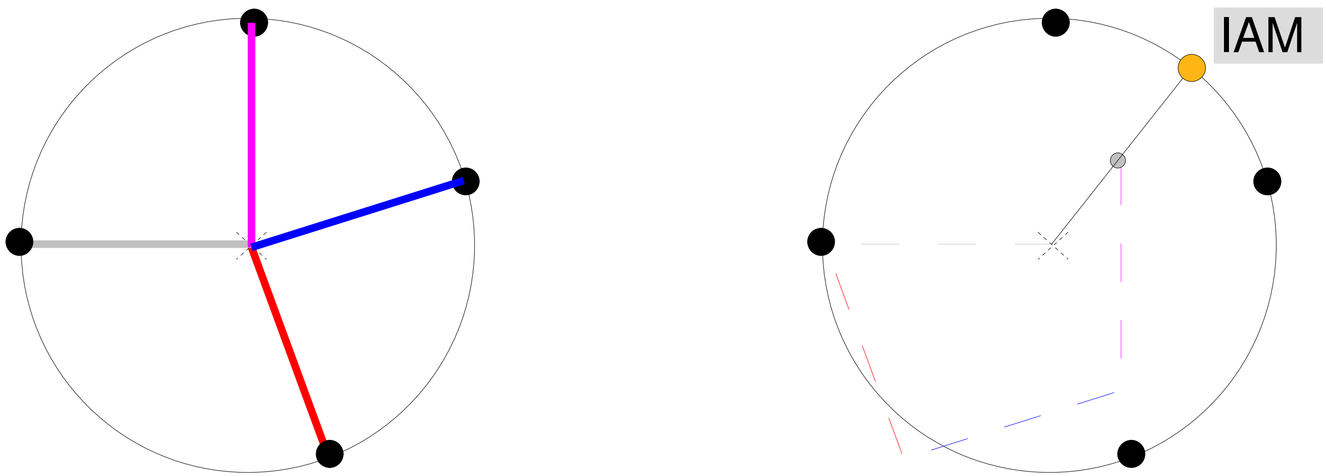
not continuously differentiable

Chordal distance



smooth

The mean associated to the chordal distance is easily computable in closed form



Induced arithmetic mean $:= \quad \arg\measuredangle\left(\sum_k e^{i\theta_k} \right)$

similar to Vicsek $\quad (\theta_k)_+ = \arg\measuredangle\left(e^{i\theta_k} + \sum_{j \rightsquigarrow k} e^{i\theta_j} \right)$

A gradient algorithm can be derived for fixed undirected interconnections

Cost function with chordal distance $\frac{1}{2}\sum_{k}\sum_{j\rightsquigarrow k}\left(d_c(\theta_k,\theta_j)\right)^2$

Gradient algorithm

$$\dot{\theta}_k = \sum_{j\rightsquigarrow k} \sin(\theta_j - \theta_k)$$

similar to Kuramoto

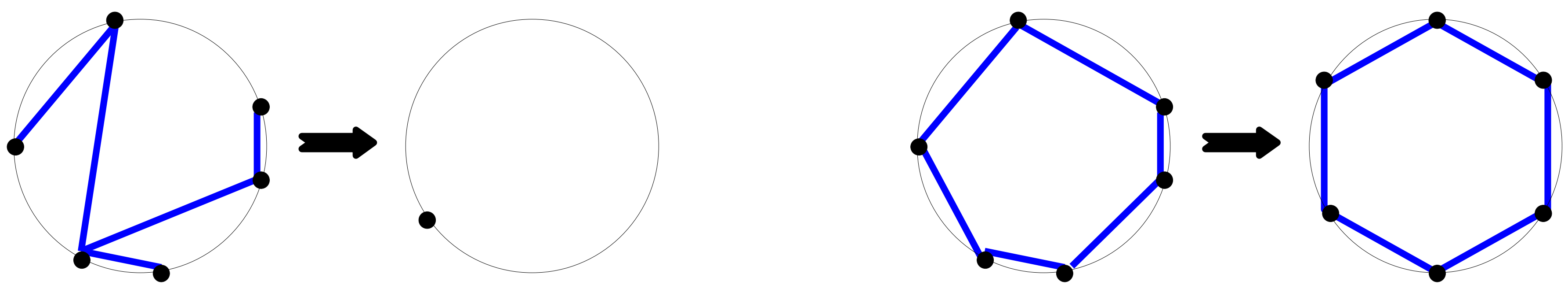
$$\dot{\theta}_k = \omega_k + \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_k)$$

Convergence properties are weaker on the circle than for vector spaces

For fixed undirected graph

All solutions converge to an equilibrium

Local minima different from synchronization exist depending on interconnections



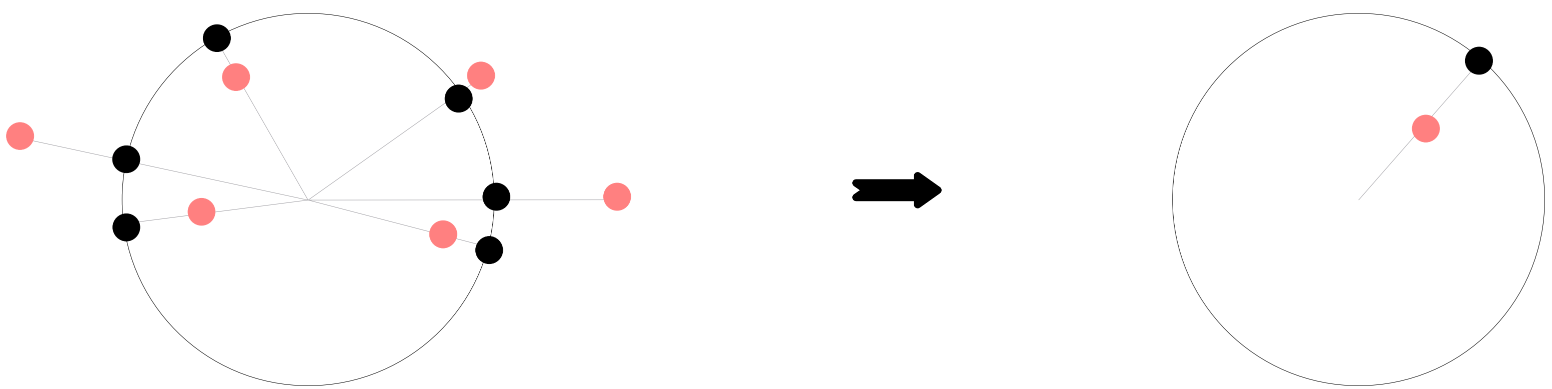
For directed / varying graph, convergence is only local

An alternative algorithm with auxiliary variables recovers vector space properties

Idea : associate to each agent an auxiliary variable $x_k \in \mathbb{R}^2$

1. synchronize the x_k (vector space consensus)

2. each θ_k tracks the projection of x_k on S^1



The alternative algorithm achieves global convergence for directed & varying graphs

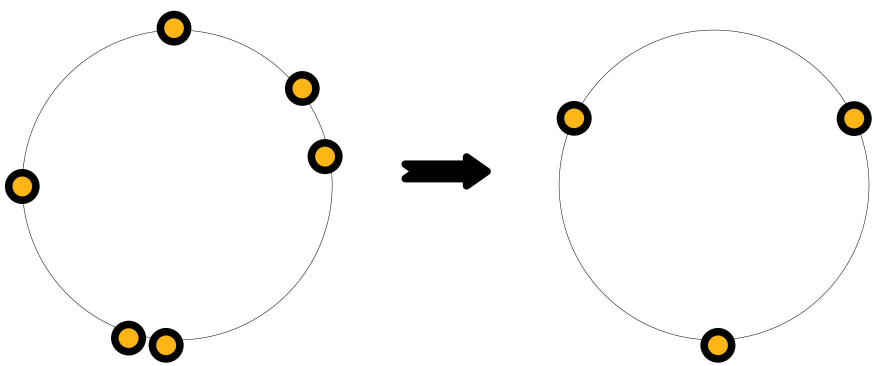
No reference frame
→ variables linked to the agents $\quad y_k := x_k \, e^{-i\theta_k}$

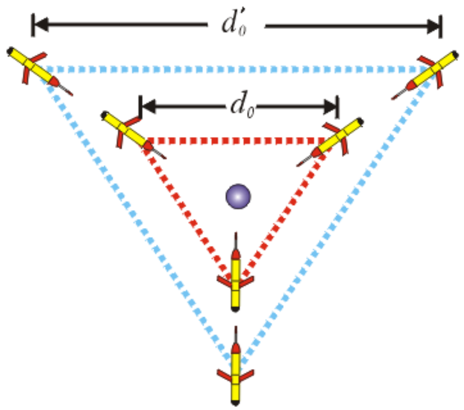
Algorithm
$$\dot{y}_k = \sum_{j \rightsquigarrow k} \left(y_j \, e^{i(\theta_j - \theta_k)} - y_k \right) - i\dot{\theta}_k \, y_k$$
$$\dot{\theta}_k = y_k - \mathbf{e}_1 \mathbf{e}_1^T y_k$$

Convergence : for uniformly connected interconnections, this algorithm ensures (almost) global synchronization.

These results have been extended in various ways

Beyond synchronization: stabilize balancing, splay states





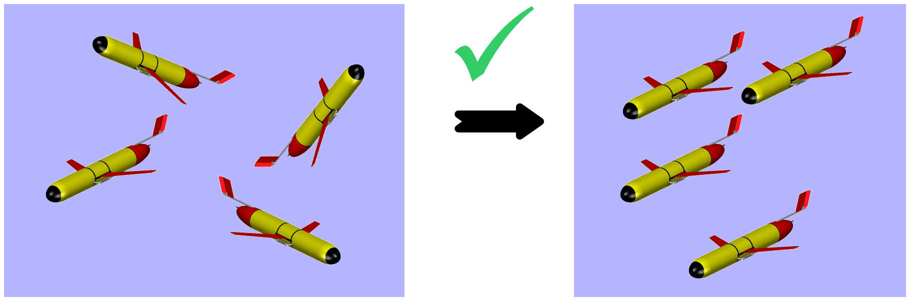
Beyond the circle : compact homogeneous manifolds

S^2 : sphere (heading in 3D)

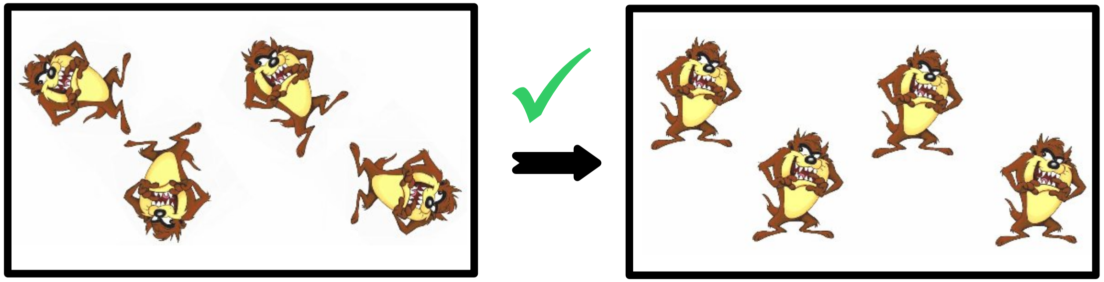
$SO(3)$: rotation matrices (satellite attitudes)

The developed geometric methods solve the problem of reaching the same point on manifolds

Heading for parallel motion



Synchronization of body orientations



What about more complex motions in formation ?

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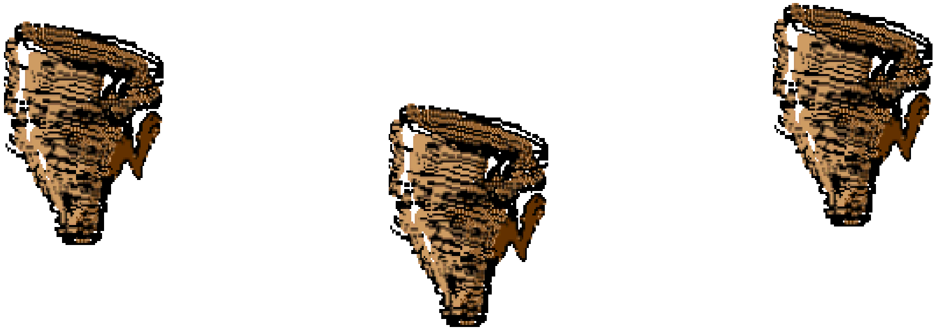
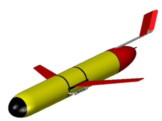
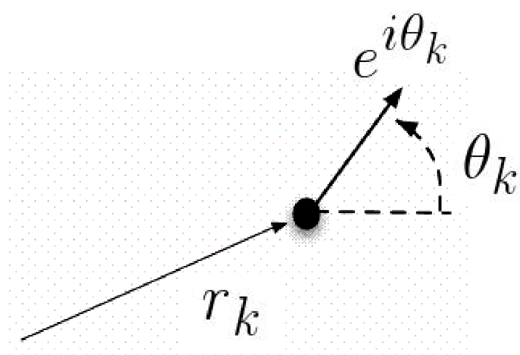


SE(2) is a manifold with translation operation = a Lie group

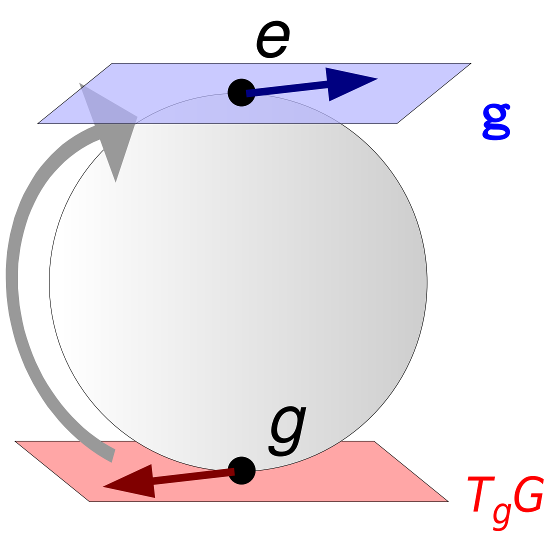


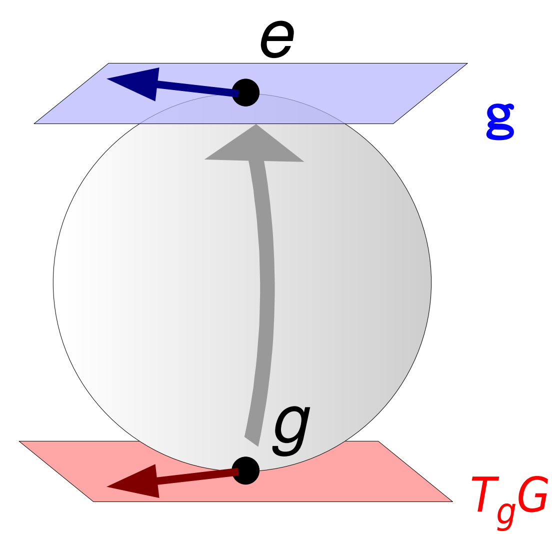


$$g_k = \begin{pmatrix} \mathrm{R}_{\theta_k} & r_k \\ 0 \quad 0 & 1 \end{pmatrix} \in SE(2)$$

$$g_1\, g_2 = \begin{pmatrix} R_{\theta_1} & r_1 \\ 0 \quad 0 & 1 \end{pmatrix} \begin{pmatrix} R_{\theta_2} & r_2 \\ 0 \quad 0 & 1 \end{pmatrix} = \begin{pmatrix} R_{\theta_1+\theta_2} & r_1 + R_{\theta_1} r_2 \\ 0 \quad 0 & 1 \end{pmatrix}$$

Velocities on Lie groups are compared thanks to the translation operation





$$\dot{g} = g\,\xi^l \qquad\qquad \dot{g} = \xi^r g$$

$$\Rightarrow \xi^r = g\,\xi^l\,g^{-1} =: \mathrm{Ad}(g)\,\xi^l \quad \text{with } \xi^l \text{ and } \xi^r \in \mathrm{T_eG} =: \mathfrak{g}$$

Lie group velocities have a physical meaning on *SE*(2)

$$\frac{d}{dt} g_k = \begin{pmatrix} \mathrm{R}_{\theta_k} & r_k \\ 0 \quad 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\omega_k & v_k^l \\ \omega_k & 0 & \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_k & v_k^r \\ \omega_k & 0 & \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{R}_{\theta_k} & r_k \\ 0 \quad 0 & 1 \end{pmatrix}$$

ω_k rotation rate in the plane

$v^l{}_k$ linear velocity in body frame (steering control : $v^l{}_k = \mathbf{e}_1$)

$v^r{}_k$ if $\omega_k = 0$, velocity in inertial frame
if $\omega_k \neq 0$, characterizes position of the center of curvature

Two types of relative positions on Lie groups yield to definitions of “collective motion”

$g_k^{-1}\, g_j \;=\;$ on $SE(2)$ $\begin{pmatrix} R_{\theta_j-\theta_k} & R_{-\theta_k}(r_j - r_k) \\ 0 \quad 0 & 1 \end{pmatrix}$

Def : Left-invariant coordination : constant $g_k^{-1}\, g_j$

$g_j\, g_k^{-1} \;=\;$ on $SE(2)$ $\begin{pmatrix} R_{\theta_j-\theta_k} & r_j - R_{\theta_j-\theta_k} r_k \\ 0 \quad 0 & 1 \end{pmatrix}$

Def : Right-invariant coordination : constant $g_j\, g_k^{-1}$

Collective motion corresponds to equal Lie group velocities

Thm : Left-invariant coordination $\Leftrightarrow$ $\xi_k{}^r = \xi_j{}^r$

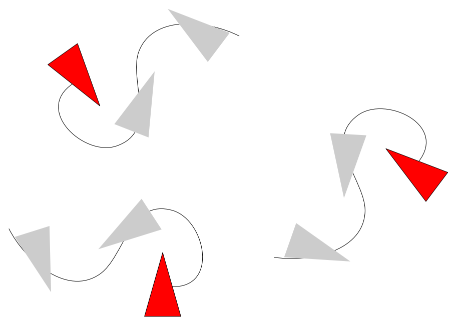
$$
\begin{aligned}
\frac{d}{dt}(g_k^{-1} g_j) &= g_k^{-1}\; g_j\, \xi_j^l \;-\; \xi_k^l\, g_k^{-1}\; g_j \\
&= g_k^{-1}\; (g_j\, \xi_j^l\, g_j^{-1} - g_k \xi_k^l g_k^{-1})\; g_j \\
&= g_k^{-1}\; (\xi_j^r - \xi_k^r)\; g_j
\end{aligned}
$$

Thm : Right-invariant coordination $\Leftrightarrow$ $\xi_k{}^l = \xi_j{}^l$

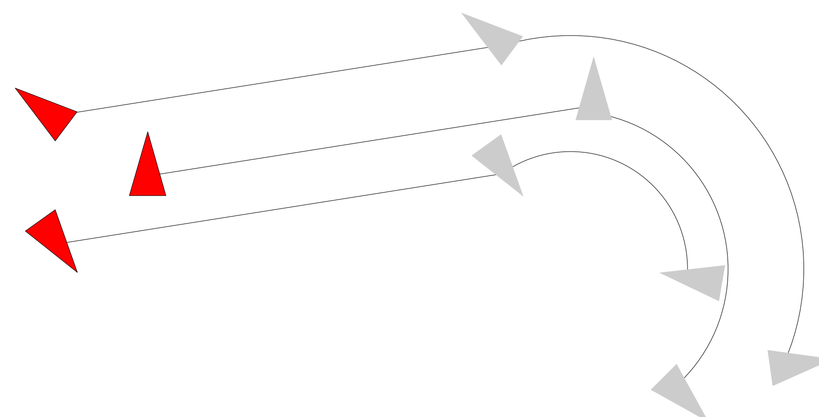
Advantage : Lie group velocities are in $\mathbf{g}$ $\equiv$ vector space

Both types of collective motion have a physical meaning on $SE(2)$

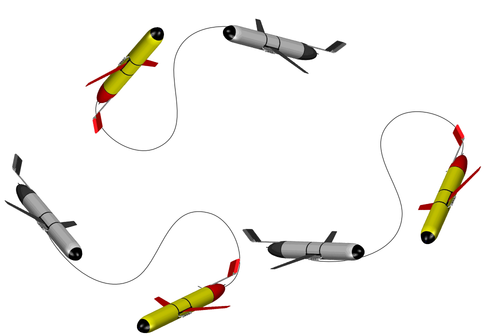
right coordination:
same velocity in body frame

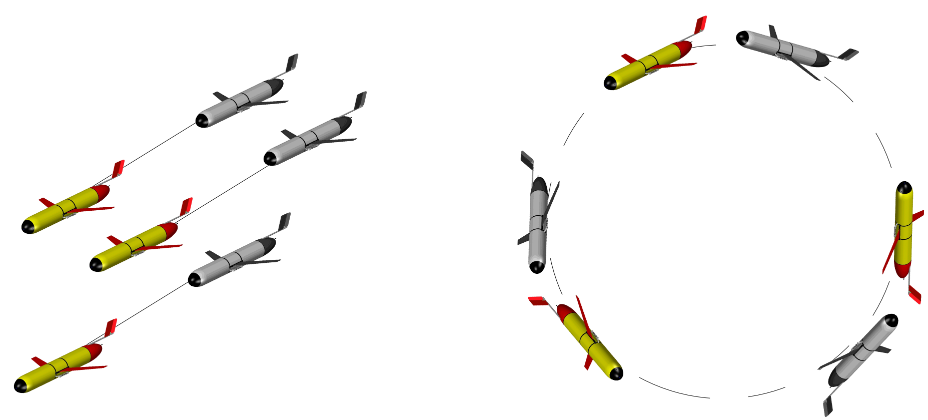


left coordination:
constant relative position & heading



Steering control implies additional constraints





Agents can use only
left-invariant variables for control

no reference tracking

⇒ **relative** positions and headings in the plane for $SE(2)$

= left relative Lie group positions $g_k^{-1} g_j$

⇒ velocities in body frame for $SE(2)$

= left Lie group velocities ξ_k^l

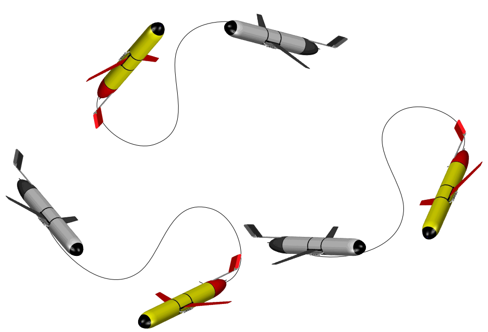
Right-invariant coordination is straightforward with vector space consensus

Right-invariant coordination $\Leftrightarrow$ equal $\xi_k^{\,l}$

Vector space consensus algorithm

$$\dot{\xi}_k^{\,l} = \sum_{j \rightsquigarrow k} (\xi_j^{\,l} - \xi_k^{\,l})$$

For steering control on *SE*(2) :



$v_k^{\,l} = v_j^{\,l} = \mathbf{e}_1$ already

agree on rotation rate

$$\dot{\omega}_k^{\,l} = \sum_{j \rightsquigarrow k} (\omega_j^{\,l} - \omega_k^{\,l})$$

Left-invariant coordination is not so obvious and involves control of positions

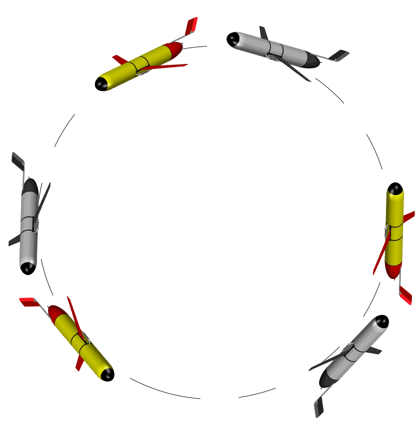
Analogy with right-invariant coordination

$$\frac{d}{dt}(Ad_{g_k}\,\xi_k^l) \;=\; \sum_{j \rightsquigarrow k} \left(Ad_{g_j}\,\xi_j^l - Ad_{g_k}\,\xi_k^l \right)$$

$$\Leftrightarrow \quad \frac{d}{dt}\,\xi_k^l \;=\; \sum_{j \rightsquigarrow k} \left(Ad_{g_k^{-1}\,g_j}\,\xi_j^l \;-\; \xi_k^l \right)$$

However, this does not satisfy steering control constraints.

Not just velocities, but positions must be controlled to agree on circle centers



The solution involves a combination of consensus and Lyapunov-derived control

Consensus algorithm to agree on "desired" rotation rate

$$\dot{w}_k = \sum_{j \rightsquigarrow k} (w_j - w_k)$$

Cost function for positions :

circle center differences *assuming desired rotation rate*

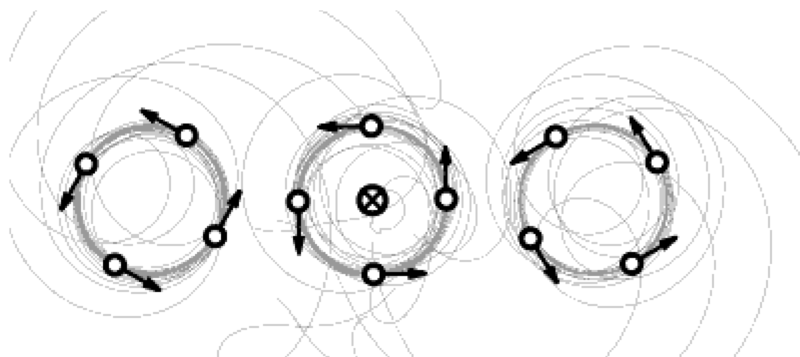
$$V = \frac{1}{2} \sum_k \sum_{j \rightsquigarrow k} \left\| Ad_{g_k} \begin{pmatrix} v_k^l \\ w_k \end{pmatrix} - Ad_{g_j} \begin{pmatrix} v_j^l \\ w_j \end{pmatrix} \right\|^2$$

derived algorithm for left-invariant coordination

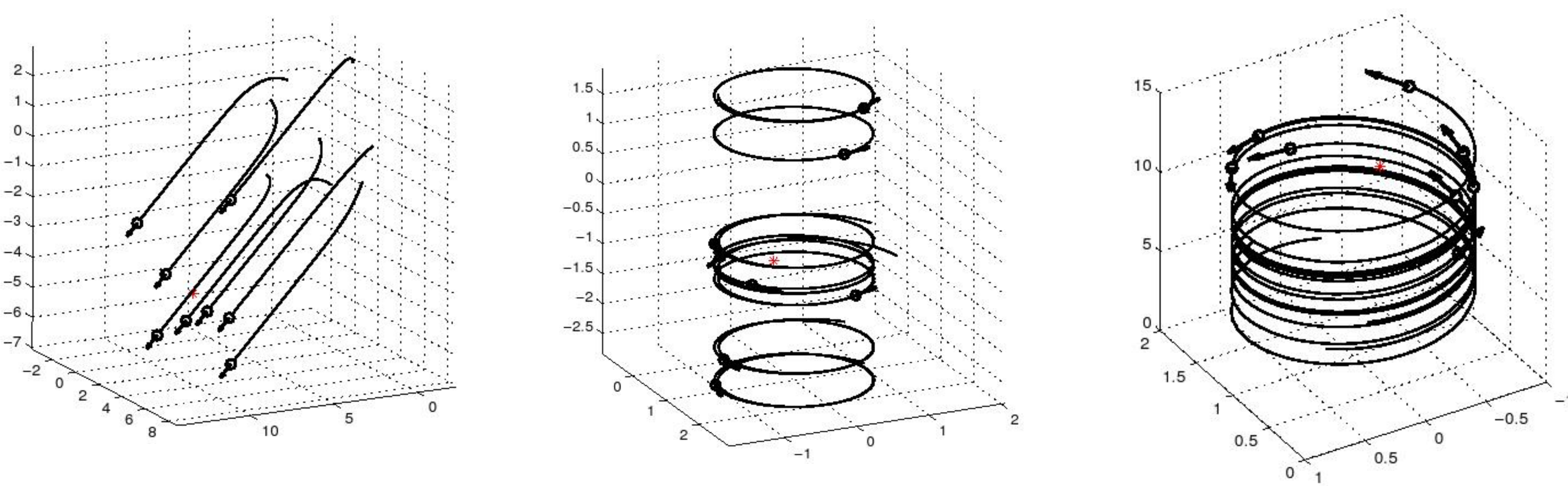
$$\omega_k = w_k \left(1 + \left(R_{-\theta_k} \textstyle\sum_{j \rightsquigarrow k} (r_k - r_j)\right) \cdot \mathbf{e}_1 \right)$$

The geometric setting facilitates extensions

Coupling collective motion with particular configurations



Other Lie groups, e.g. $SE(3)$: rigid bodies in 3 dimensions



The Lie group framework allows to characterize and design control for coordinated motion

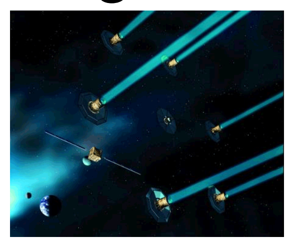
“Relative positions”, “coordination”, “movement in formation” are defined by Lie group properties

Collective motion $\Leftrightarrow$ synchronization of Lie group velocities
(consensus in vector space)

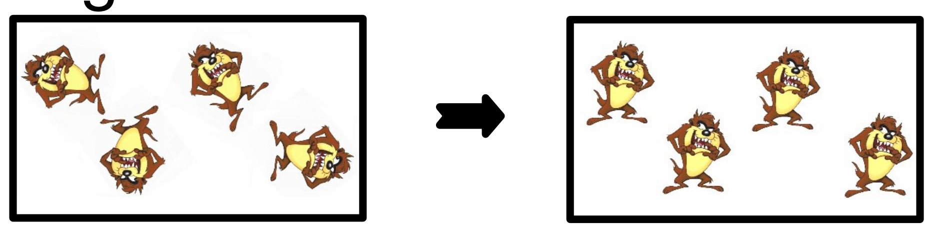
Motion “in formation” is not trivial for underactuated agents, but combining consensus and geometric Lyapunov functions yields appropriate controls.

Design of stable collective motions on manifolds

Outline. Motivating examples → problem setting



Reaching consensus on manifolds



A general control design method for collective motion on Lie groups

Conclusion:

Appropriate geometric tools allow to solve the ubiquitous problem of control design to stabilize collective motions on manifolds.

Remaining issues : Convergence analysis of other models

- Behavior of simple algorithms (Kuramoto, Vicsek,..)
- State-dependent communication graphs
- Coupling all these planning algorithms together / with other task / with complex dynamics

Acknowledgments

Control of coordinated motion

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Consensus on the circle

Luca Scardovi (Liege / Princeton),
Vincent Blondel (UC Louvain), Emre Tuna

Optimization algorithms on manifolds

Pierre-Antoine Absil (UC Louvain), Robert Mahony (ANU)

More on the subject...

Agreement / consensus on manifolds

**Consensus optimization on manifolds*, A.Sarlette & R.Sepulchre, to be publ. SIAM/SICON

Collective motion in 2D and 3D

Stabilization of planar collective motion with all-to-all communication,
R.Sepulchre, D.Paley & N.Leonard, IEEE Trans. Automatic Control vol. 52(5), 2007

Stabilization of planar collective motion with limited communication,
R.Sepulchre, D.Paley & N.Leonard, IEEE Trans. Automatic Control vol. 53(3), 2008

**Stabilization of three-dimensional collective motion,*
L.Scardovi, N.Leonard & R.Sepulchre, submitted to Comm.Inf.Syst., 2008

Collective motion on Lie groups (general theory)

Coordinated motion on Lie groups,
A.Sarlette, S.Bonnabel & R.Sepulchre, to be submitted to IEEE Trans. Automatic Control