Trajectory tracking for nonholonomic and underactuated vehicles by the transverse function approach

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Outline

- Recalls on nonholonomic and underactuated vehicles
- Control issues in the trajectory tracking problem
- The transverse function approach
- Simulation and experimental results
- Conclusion
Nonholonomic and underactuated vehicles

Unicycle and Slider (or Hovercraft):

\[
\begin{align*}
\dot{p} &= R(\theta) \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \\
\dot{\theta} &= \omega \\
\dot{v}_1 &= u_1 \\
\dot{\omega} &= u_2
\end{align*}
\]

\[
\begin{align*}
\dot{p} &= R(\theta) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
\dot{\theta} &= \omega \\
\dot{v}_1 &= v_2 \omega + u_1 \\
\dot{v}_2 &= -v_1 \omega \\
\dot{\omega} &= u_2
\end{align*}
\]

with

\[
p = \begin{pmatrix} x \\ y \end{pmatrix}, \quad R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]
Nonholonomic and underactuated vehicles

Nonholonomic vehicles:

- Unicycle
- Car
- More generally

\[ \dot{q} = \sum_{i=1}^{m} v_i X_i(q) \] \[ \text{with } m < n = \dim(q) \]

Characteristic properties:

- Controllable systems (LARC),
- Linearized system at any equilibrium (i.e. any configuration) is not controllable,
- Equilibrium points are not As. Stabilizable by \( C^0 \) feedback \( v(q) \) (Brockett's Theorem)
- Symmetry group: \( q \in M = G \times S \) with \( G = SE(2) \) (a Lie group) and the equations are left-invariant w.r.t. the group operation on \( G \).
Nonholonomic and underactuated vehicles

Underactuated vehicles:

- General form of the dynamic equations:

\[
\begin{aligned}
\dot{q} &= \sum_{i=1}^{n} v_i X_i(q) \\
\dot{v} &= Q(q, v) + \sum_{i=1}^{m} u_i b_i(q, v, t) \quad \text{with } m < n = \dim(q)
\end{aligned}
\]

- Large variety of systems and properties (see e.g. Bullo & Lewis, 2005), depending on the type of actuation, and on the external forces.

- In this talk, to “simplify”, Euler-Poincaré equations:

\[
\begin{aligned}
\dot{g} &= \sum_{i=1}^{n} v_i X_i(g) \\
\dot{v} &= Q(v) + \sum_{i=1}^{m} u_i b_i 
\end{aligned}
\]

with \( g \in G \), a Lie group, \( X_1, \ldots, X_m \) left-invariant v.f. on \( G \), \( Q \) quadratic.
Nonholonomic and underactuated vehicles

\[
\begin{align*}
\dot{g} &= \sum_{i=1}^{n} v_i X_i(g) \\
\dot{v} &= Q(v) + \sum_{i=1}^{m} u_i b_i
\end{align*}
\]

This class of systems has many similarities with nonholonomic systems:

- Assuming controllability, these are critical systems, i.e. the linearized system at any equilibrium is not controllable (and not asymptotically stabilizable either),
- Equilibrium points are not As. Stabilizable by $C^0$ feedback $u(g, v)$,
- Symmetry group: $g \in G$ with $G = SE(2), SE(3), SO(3)$.

Examples:

- Slider, $G = SE(2)$
- Spacecraft with two torque control inputs (like gas jets), $G = SO(3)$
- Rigid body in space (without gravity) with $m < 6$ control inputs, $G = SE(3)$
Control issues in the trajectory tracking problem

- Trajectory tracking problem: given a “smooth” reference trajectory $q_r(t)$, find a feedback law $v(...)$ or $u(...)$ which makes $q_r(t)$ “stable”
- We won’t talk about output feedback control (used e.g. in platooning)

Which kind of stability?

Asymptotic Stability (in the sense of Lyapunov) raises several issues...
Control issues in the trajectory tracking problem

For fixed points, i.e. \( q_r(t) = \text{cste} \):

- Many approaches for asymptotic stabilization have been developed:
  - Time-varying periodic feedback \( v(q, t) \),
  - Discontinuous feedback \( v(q) \),
  - Hybrid-feedback (continuous/discrete) \( v(q(kT), t) \),

- But the main issue, ROBUSTNESS, has not really been addressed...

- Non-linear version of the Stability/Precision dilemma:
  - High gains are necessary for precision (or fast convergence),
  - But they have a negative influence on robustness,
  - This problem is highly amplified by the uncontrollability of the system’s linearization,
  - \( \Rightarrow \) in practice a very precise positionning is not possible:
Control issues in the trajectory tracking problem

For non-stationnary trajectories:

- Several results for non-holonomic systems under specific assumptions on \( q_r(.) \) (ex. persistent excitation),
- Relying on the controllability properties along non-stationnary trajectories (Sontag, 1992)
- Very few results for underactuated systems,
- For underactuated systems, difficulty to specify admissible trajectories
- Moreover, there is no universal controller (Lizárraga, 2004):
  \[
  \dot{x} = Ax + Bu, \quad \dot{x}_r = Ax_r + Bu_r \\
  u = u_r + K(x - x_r) \text{ is a universal controller, i.e.}
  \]

\[
\frac{d}{dt}(x - x_r) = (A + BK)(x - x_r)
\]

\( \mapsto \) asymptotic stability for any \( x_r \) if \( A + BK \) is Hurwitz-stable

- No such feedback exists for our systems!!!
- When the reference motion is not known in advance, the control strategy is unclear!
The transverse function approach

- **Principle:** relax the stabilization objective, from asymptotic to practical stability

- **What for?**
  - Design universal controllers,
  - Improve the robustness properties, e.g. for the stabilization of fixed points supress or limit oscillations resulting from modeling errors.

- **Additional benefits of the approach:**
  - We can use it to stabilize non-admissible trajectories,
  - this can be used as a path-planner,
  - useful for some tracking applications,
  - Tools for robustness analyses.

- **How to do that?**
  - Introduce \( n - m \) additional virtual inputs,
  - This can be viewed as a generalization of time-varying periodic feedbacks (Samson, 1990),
  - This can be viewed as a generalization of the tunable frequency oscillator (Dixon & all, 2000).
The transverse function approach

Nonholonomic systems: (Morin & Samson, 2000, 2003, etc)

\[ \dot{g} = \sum_{i=1}^{m} v_i X_i(g) \]

- Let \( f: \alpha \mapsto f(\alpha) \) with \( \alpha = (\alpha_1, \ldots, \alpha_p) \in S^1 \times \cdots \times S^1 \).

\[ \frac{d}{dt} (g - f(\alpha)) = \sum_{i} X_i(g) v_i - \sum_{j} \frac{\partial f}{\partial \alpha_j} \dot{\alpha}_j \]

- When \( g \) is close to \( f(\alpha) \),

\[ \frac{d}{dt} (g - f(\alpha)) \approx \sum_{i} X_i(f(\alpha)) v_i - \sum_{j} \frac{\partial f}{\partial \alpha_j} \dot{\alpha}_j \]

\[ \approx H(\alpha) \bar{v} \]

\[ H(\alpha) = \left( X_1(f(\alpha)), \ldots, X_m(f(\alpha)), -\frac{\partial f}{\partial \alpha_1}(\alpha), \ldots, -\frac{\partial f}{\partial \alpha_p}(\alpha) \right), \quad \bar{v} = \begin{pmatrix} v \\ \dot{\alpha} \end{pmatrix} \]

- If \( \text{rank}(H) = n \), \( \bar{v} = -k H(\alpha)^\dagger (g - f(\alpha)) \) yields

\[ \frac{d}{dt} (g - f(\alpha)) \approx -k (g - f(\alpha)) \]
The transverse function approach

- Existence of t.f.: Yes, if the system is controllable
- Calculation of t.f.: Yes, formulas derived
- Extension to trajectory tracking: Direct for systems on Lie groups:
  - Tracking error: $\tilde{g}(t) = g_r(t)^{-1} g(t)$
  - Error model:
    $$\dot{\tilde{g}} = \sum_{i=1}^{m} X_i(\tilde{g})v_i + P(\tilde{g}, t)$$
    $$z = \tilde{g}f(\alpha)^{-1} \approx \tilde{g} - f(\alpha)$$
    $$\dot{z} = A(z, \alpha)H(\alpha)\dot{v} + \bar{P}(z, \alpha, t)$$

- Extension to underactuated systems:
  - Add additional virtual inputs at the dynamic level,
  - Ongoing research, solutions for many systems: Slider, Underactuated spacecraft, Rigid body in space with 3 forces, etc
  - Different ways to proceed (Morin & Samson, 2005; Lizárraga & Sosa, 2005)...
  - See our Poster Presentation on Friday for details!
Simulation and experimental results

Nonholonomic systems

- Unicycle (See Artus, Morin & Samson, 2003, 2004)

- Car
Simulation and experimental results (cont’)

Underactuated systems

- Spacecraft

- Slider
Conclusion

- Hard problems for critical systems:
  - Robustness to modeling errors
  - The Grail:
    - A universal controller,
    - which does not yield “unnecessary” manoeuvres,
    - and conserves this property under small modeling errors.

- A few other works & perspectives:
  - Tuning of the transient behavior,
  - Sensor-based control (robustness to measurement errors),
  - Continue to develop the approach for “Euler-Poincaré” equations,
  - Adaptation and evaluation for non-critical underactuated systems (ex: the vertical Hovercraft, lots of VTOLs, etc),
  - and lots more...