Nonlinear Control Systems

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7. Feedback Linearization

IST-DEEC PhD Course

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Feedback Linearization

Given a nonlinear system of the form

$$\dot{x} = f(x) + G(x)u$$
$$y = h(x)$$

Does exist a state feedback control law

$$u = \alpha(x) + \beta(x)v$$

and a change of variables

$$z = T(x)$$

that transforms the nonlinear system into a an equivalent linear system ($\dot{z} = Az + Bv$) ?

Feedback Linearization

Example: Consider the following system

$$\dot{x} = Ax + B\gamma(x)\big(u - \alpha(x)\big)$$

where $\gamma(x)$ is nonsingular for all x in some domain D.

Then,

$$u = \alpha(x) + \beta(x)v, \quad \text{with } \beta(x) = \gamma^{-1}(x)$$

yields

$$\dot{x} = Ax + Bv$$

If we would like to stabilize the system, we design

$$v = -Kx \quad \text{such that } A - BK \text{ is Hurwitz}$$

Therefore

$$u = \alpha(x) - \beta(x)Kx$$

Feedback Linearization

Example: Consider now this example:

$$\dot{x}_1 = a \sin x_2$$
$$\dot{x}_2 = -x_1^2 + u$$

How can we do this? We cannot simply choose u to cancel the nonlinear term $a \sin x_2$!

However, if we first change the variables

$$z_1 = x_1$$
$$z_2 = a \sin x_2 = \dot{x}_1$$

then

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = a \cos x_2 \dot{x}_2 = a \cos x_2 (-x_1^2 + u)$$

Therefore with

$$u = x_1^2 + \frac{1}{a \cos x_2} v, \quad -\pi/2 < x_2 < \pi/2$$

we obtain the linear system

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = v$$

Feedback Linearization

- A continuously differentiable map $T(x)$ is a diffeormorphism if $T^{-1}(x)$ is continuously differentiable. This is true if the Jacobian matrix $\frac{\partial T}{\partial x}$ is nonsingular $\forall x \in D$.

- $T(x)$ is a global diffeormorphism if and only if $\frac{\partial T}{\partial x}$ is nonsingular $\forall x \in \mathbb{R}^n$ and $T(x)$ is proper, that is, $\lim_{\|x\| \to \infty} \|T(x)\| = \infty$.

Definition

A nonlinear system

$$\dot{x} = f(x) + G(x)u \tag{1}$$

where $f : D \to \mathbb{R}^n$ and $G : D \to \mathbb{R}^{n \times p}$ are sufficiently smooth on a domain $D \subset \mathbb{R}^n$ is said to be feedback linearizable (or input-state linearizable) if there exists a diffeomorphism $T : D \to \mathbb{R}^n$ such tat $D_z = T(D)$ contains the origin and the change of variables $z = T(x)$ transforms (1) into the form

$$\dot{z} = Az + B\gamma(x)\big(u - \alpha(x)\big)$$

Feedback Linearization

Suppose that we would like to solve the tracking problem for the system

$$\dot{x}_1 = a \sin x_2$$
$$\dot{x}_2 = -x_1^2 + u$$
$$y = x_2$$

If we use state feedback linearization we obtain

$$\begin{array}{rcl} z_1 & = & x_1 \\ z_2 & = & a \sin x_2 = \dot{x}_1 \\ u & = & x_1^2 + \frac{1}{a \cos x_2} v \end{array} \quad \longrightarrow \quad \begin{array}{rcl} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & v \\ y & = & \sin^{-1}(z_2/a) \end{array}$$

which is not good!
Linearizing the state equation does not necessarily linearize the output equation.

Notice however if we set $u = x_1^2 + v$ we obtain

$$\dot{x}_2 = v$$
$$y = x_2$$

There is one catch: The linearizing feedback control law made x_1 unobservable from y. We have to make sure that x_1 whose dynamics are given by $\dot{x}_1 = a \sin x_2$ is well behaved. For example, if $y = y_d = cte \ \longrightarrow \ x_1(t) = x_1(0) + ta \sin y_d$. It is unbounded!

Input-Output Linearization

SISO system

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

where f, g, h are sufficiently smooth in a domain $D \subset \mathbb{R}^n$. The mappings $f : D \to \mathbb{R}^n$ and $g : D \to \mathbb{R}^n$ are called vector fields on D.

Computing the first output derivative...

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}\left[f(x) + g(x)u\right] =: L_f h(x) + L_g h(x)u$$

In the sequel we will use the following notation:

$$L_f h(x) = \frac{\partial h}{\partial x} f(x) \longrightarrow \text{Lie Derivative of } h \text{ with respect to } f$$

$$L_g L_f h(x) = \frac{\partial (L_f h)}{\partial x} g(x)$$

$$L_f^0 h(x) = h(x)$$

$$L_f^2 h(x) = L_f L_f h(x) = \frac{\partial (L_f h)}{\partial x} f(x)$$

$$L_f^k h(x) = L_f L_f^{k-1} h(x) = \frac{\partial (L_f^{k-1} h)}{\partial x} f(x)$$

Input-Output Linearization

$$\dot{y} = L_f h(x) + L_g h(x) u$$

If $L_g h(x) u = 0$ then $\dot{y} = L_f h(x)$ (independent of u).

Computing the second derivative...

$$y^{(2)} = \frac{\partial (L_f h)}{\partial x} \left[f(x) + g(x) u \right] = L_f^2 h(x) + L_g L_f h(x) u$$

If $L_g L_f h(x) u = 0$ then $\dot{y}^{(2)} = L_f^2 h(x)$ (independent of u).

Repeating this process, it follows that if

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, \rho - 1$$
$$L_g L_f^{\rho-1} h(x) \neq 0$$

then u does not appear in $y, \dot{y}, \ldots, y^{(\rho-1)}$ and

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{(\rho-1)} h(x) u$$

Input-Output Linearization

$$y^{(\rho)} = L_f^\rho h(x) + L_g L_f^{\rho-1} h(x) u$$

Therefore, by setting

$$u = \frac{1}{L_g L_f^{\rho-1} h(x)} [-L_f^\rho h(x) + v]$$

the system is input-output linearizable and reduces to

$$y^{(\rho)} = v \;\longrightarrow\; \text{chain of } \rho \text{ integrators}$$

Definition

The nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

is said to have relative degree ρ, $1 \le \rho \le n$, in the region $D_0 \subset D$ if for all $x \in D_0$

$$\begin{aligned} L_g L_f^{i-1} h(x) &= 0, \quad i = 1, 2, \ldots, \rho - 1 \\ L_g L_f^{\rho-1} h(x) &\neq 0 \end{aligned}$$

Examples

Example 1: Van der Pol system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u, \quad \epsilon > 0$$

1. $y = x_1$

   Calculating the derivatives...

   $$\dot{y} = \dot{x}_1 = x_2$$
   $$\ddot{y} = \dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u$$

   Thus the system has relative degree $\rho = 2$ in $\mathbb{R}^2$.

2. $y = x_2$

   Then

   $$\dot{y} = \dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u$$

   In this case the system has relative degree $\rho = 1$ in $\mathbb{R}^2$.

3. $y = x_1 + x_2^2$

   Then

   $$\dot{y} = x_2 + 2x_2(-x_1 + \epsilon(1 - x_1^2)x_2 + u)$$

   In this case the system has relative degree $\rho = 1$ in $D_0 = \{x \in \mathbb{R}^2 : x_2 \neq 0\}$.

Examples

Example 2:

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 + u \\ y &= x_1 \end{aligned}$$

Calculating the derivatives...

$$\dot{y} = \dot{x}_1 = x_1 = y \ \longrightarrow \ y^{(n)} = y = x_1, \ \forall n \geq 1$$

The system does not have a well defined relative degree!

Why? Because the output $y(t) = x_1(t) = e^t x_1(0)$ is independent of the input u.

Examples

<u>Example 3:</u>

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

where $m < n$ and $b_m \neq 0$.

A state model of the system is the following

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & \dots & 0 \\ 0 & 0 & 1 & 0 & \cdots & \vdots \\ \vdots & & \ddots & & \ddots & 0 \\ 0 & \cdots & & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & & \cdots & -a_{n-1} \end{bmatrix}_{n\times n} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1} \quad C = \begin{bmatrix} b_0 & b_1 & \cdots & b_m & 0 & \cdots 0 \end{bmatrix}_{1\times n}$$

What is the relative degree ρ ?

Examples

$$\dot{y} = CAx + CBu$$

If $m = n - 1 \longrightarrow CB = b_m \neq 0 \longrightarrow \rho = 1$

Otherwise, $CB = 0$

$$y^{(2)} = CA^2x + CABu$$

Note that CA is obtained by shifting the elements of C one position to the right and CA^i by shifting i positions.

Therefore,

$$CA^{i-1}B = 0, \quad \text{for } i = 1, 2, \ldots n - m - 1$$

$$CA^{n-m-1}B = b_m \neq 0$$

$$y^{(n-m)} = CA^{n-m}x + CA^{n-m-1}Bu \longrightarrow \rho = n - m$$

In this case the relative degree of the system is the difference between the degrees of the denominator and numerator polynomials of $H(s)$.

Consider again the linear system given by the transfer function

$$H(s) = \frac{N(s)}{D(s)} \quad \text{with} \quad \begin{cases} \deg D &= n \\ \deg N &= m < n \\ \rho &= n - m \end{cases}$$

$D(s)$ can be written as

$$D(s) = Q(s)N(s) + R(s)$$

where the degree of the quotient $\deg Q = n - m = \rho$ and the degree of the reminder $\deg R < m$

Thus

$$H(s) = \frac{N(s)}{Q(s)N(s) + R(s)} = \frac{\frac{1}{Q(s)}}{1 + \frac{1}{Q(s)}\frac{R(s)}{N(s)}}$$

and therefore we can conclude that $H(s)$ can be represented as a negative feedback connection with $1/Q(s)$ in the forward path and $R(s)/N(s)$ in the feedback path.

Note that the ρ-order transfer function $1/Q(s)$ has no zeros and can be realized by

$$\begin{aligned} \dot{\xi} &= (A_c + B_c\lambda^T)\xi + B_c b_m e \\ y &= C_c\xi \end{aligned}$$

where

$$\xi = \begin{bmatrix} y & \dot{y} & \dots & y^{(\rho-1)} \end{bmatrix}^T \in \mathbb{R}^\rho$$

and (A_c, B_c, C_c) is a canonical form representation of a chain of ρ integrators:

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}_{\rho\times\rho} \qquad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\rho\times 1} \qquad C_c = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}_{1\times\rho}$$

$$B_c\lambda^T = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ & \lambda^T & \end{bmatrix}, \quad \lambda \in \mathbb{R}^\rho$$

$$\frac{R(s)}{N(s)} \quad \longrightarrow \quad \begin{array}{rcl} \dot{\eta} & = & A_0\eta + B_0 y \\ w & = & C_0\eta \end{array}$$

The eigenvalues of A_0 are the zeros of the polynomial $N(s)$, which are the zeros of the transfer function $H(s)$.

Thus, the system $H(s)$ can be realized by the state model

$$\begin{aligned} \dot{\eta} &= A_0\eta + B_0 C_c \xi \\ \dot{\xi} &= A_c\xi + B_c(\lambda^T\xi - b_m C_0\eta + b_m u) \\ y &= C_c\xi \end{aligned}$$

Note that $y = C_c\xi$ and

$$\dot{\xi} = A_c\xi + B_c(\lambda^T\xi - b_m C_0\eta + b_m u) \quad \longleftrightarrow \quad \begin{array}{rcl} \dot{\xi}_1 & = & \xi_2 \\ \dot{\xi}_2 & = & \xi_3 \\ \vdots & & \\ \dot{\xi}_\rho & = & \lambda^T\xi - b_m C_0\eta + b_m u \end{array}$$

and therefore $y^{(\rho)} = \lambda^T\xi - b_m C_0\eta + b_m u$

$$y^{(\rho)} = \lambda^T \xi - b_m C_0 \eta + b_m u$$

Thus, setting

$$u = \frac{1}{b_m}[-\lambda^T \xi + b_m C_0 \eta + v]$$

results in

$$\begin{aligned}
\dot{\eta} &= A_0 \eta + B_0 C_c \xi \;\longrightarrow\; \text{Internal dynamics: It is unobservable from the output } y \\
\dot{\xi} &= A_c \xi + B_c v \;\longrightarrow\; \text{chain of integrators} \\
y &= C_c \xi
\end{aligned}$$

Suppose we would like to stabilize the output y at a constant reference r, that is, $\xi \to \xi^\star = (r, 0, \ldots, 0)^T$.
Defining $\zeta = \xi - \xi^\star$ we obtain

$$\dot{\zeta} = A_c \zeta + B_c v$$

Therefore, setting

$$v = -K\zeta = -K(\xi - \xi^\star)$$

with $(A_c - B_c K)$ Hurwitz we obtain the closed-loop system

$$\begin{aligned}
\dot{\eta} &= A_0 \eta + B_0 C_c (\xi^\star + \zeta) \\
\dot{\zeta} &= (A_c - B_c K)\zeta \\
y &= C_c \xi
\end{aligned}$$

where the eigenvalues of A_0 are the zeros of $H(s)$. If $H(s)$ is minmum phase (zeros in the open left-half complex plan) then A_0 is Hurwitz.

Feedback Linearization

Can we extend this result

$$\begin{aligned} \dot{\eta} &= A_0\eta + B_0 C_c \eta \\ \dot{\xi} &= A_c\xi + B_c\left(\lambda^T\xi - b_m C_0\eta + b_m u\right) \\ y &= C_c x \end{aligned}$$

for the nonlinear system (SISO)

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

that is find a $z = T(x)$, where

$$z = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix}$$

such that $T(x)$ is a diffeomorphism on $D_0 \subset D$ and $\frac{\partial \phi_i}{\partial x} g(x) = 0$, for $1 \leq i \leq n-\rho$, $\forall x \in D$. Note that

$$\dot{\eta} = \frac{\partial \phi_i}{\partial x}\dot{x} = \frac{\partial \phi_i}{\partial x} f(x) + \frac{\partial \phi_i}{\partial x} g(x) u$$

Does exist such $T(x)$?

Normal form

Theorem (13.1)

Consider the SISO system

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}$$

and suppose that it has relative degree $\rho \leq n$ in D. Then, for every $x_0 \in D$, there exists a such diffeomorphism $T(x)$ on a neighborhood of x_0.

Using this transformation we obtain the system re-written in normal form:

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A_c\xi + B_c\gamma(x)[u - \alpha(x)] \\ y &= C_c\xi\end{aligned}$$

where $\xi \in \mathbb{R}^\rho$, $\eta \in \mathbb{R}^{n-\rho}$ and (A_c, B_c, C_c) is the canonical form representation of a chain of integrators, and

$$f_0(\eta, \xi) := \frac{\partial \phi}{\partial x} f(x)\bigg|_{x=T^{-1}(z)} \qquad \gamma(x) = L_g L_f^{\rho-1} h(x) \qquad \alpha(x) = -\frac{L_f^\rho h(x)}{L_g L_f^{\rho-1} h(x)}$$

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A_c\xi + B_c\gamma(x)[u - \alpha(x)] \\ y &= C_c\xi\end{aligned}$$

The external part can be linearized by

$$u = \alpha(x) + \beta(x)v$$

with $\beta(x) = \gamma^{-1}(x)$. The internal part is described by

$$\dot{\eta} = f_0(\eta, \xi)$$

Setting $\xi = 0$ result

$$\dot{\eta} = f_0(\eta, 0) \quad \longrightarrow \text{This is called the zero-dynamics}$$

Note that for the linear case we have $\dot{\eta} = A_0\eta$, where the eigenvalues of A_0 are the zeros of $H(s)$.

Definition

The system is said to be minimum phase if $\dot{\eta} = f_0(\eta, 0)$ has an asymptotically stable equilibrium point in the domain of interest.

The zero dynamics can be characterized in the original coordinates by notting that

$$y(t) = 0, \forall t \geq 0 \;\Rightarrow\; \xi(t) = 0 \;\Rightarrow\; u(t) = \alpha(x(t))$$

where the first implication is due to the fact that $\xi = [y,\ \dot{y},\ ...]^T$ and the second due to $\dot{\xi} = A_0\xi + B_0\gamma(x)[u - \alpha(x)]$.

Thus, when $y(t) = 0$, the solution of the state equation is confined to the set

$$Z^* = \left\{ x \in D_0 : h(x) = L_f h(x) = ... = L_f^{\rho-1} h(x) = 0 \right\}$$

and the input

$$u = u^*(x) := \left. \alpha(x) \right|_{x \in Z^*}$$

that is

$$\dot{x} = f^*(x) := [f(x) + g(x)\alpha(x)]_{x \in Z^*}$$

In the special case that $\rho = n \Rightarrow \eta$ does not exist. In that case the system has no zero dynamics and by default is said to be minimum phase.

Example

Example 1

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 + u$$
$$y = x_2$$

It is in the normal form ($\xi = y,\ \eta = x_1$)

Zero-dynamics?
$\dot{x}_1 = 0$, which does not have an asymptotic stable equilibrium point. Hence, the system is not minimum phase.

Example 2

$$\dot{x}_1 = -x_1 + \frac{2 + x_3}{1 + x_3^2}u$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = x_2 x_3 + u$$
$$y = x_2$$

What is the relative degree and the zero dynamics?

$$\dot{x}_1 = -x_1 + \frac{2 + x_3}{1 + x_3^2}u$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = x_2 x_3 + u$$
$$y = x_2$$

Computing the time-derivative...

$$\dot{y} = \dot{x}_2 = x_3$$
$$\ddot{y} = \dot{x}_3 = x_1 x_3 + u$$

Thus, the relative degree is $\rho = 2$. Analyzing the zero-dynamics we have

$$y = 0$$
$$\dot{y} = 0$$
$$\ddot{y} = 0$$

we have $x_2 = x_3 = 0$ and from the last we have $u = -x_1 x_3 = 0$. Therefore, $\dot{x}_1 = -x_1$ and the system is minimum phase.

Full-State Linearization

The single-input system

$$\dot{x} = f(x) + g(x)u$$

with f, g sufficiently smooth in a domain $D \subset \mathbb{R}^n$ is feedback linearizable if there exists a sufficiently smooth $h : D \to \mathbb{R}$ such that the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

has relative degree n in a region $D_0 \subset D$.

This implies that the normal form reduces to

$$\begin{aligned} \dot{z} &= A_c z + B_c \gamma(x)[u - \alpha(x)] \\ y &= C_c z \end{aligned}$$

Note that

$$z = T(x)$$

Thus

$$\dot{z} = \frac{\partial T}{\partial x}\dot{x}$$

which is equivalent to

$$A_c z + B_c \gamma(x)[u - \alpha(x)] = \frac{\partial T}{\partial x}[f(x) + g(x)u]$$

Splinting in two we obtain

$$\frac{\partial T}{\partial x}f(x) = A_c T(x) - B_c \gamma(x)\alpha(x) \tag{2}$$

$$\frac{\partial T}{\partial x}g(x) = B_c \gamma(x) \tag{3}$$

Equation (2) is equivalent to

$$\begin{aligned}
\frac{\partial T_1}{\partial x}f(x) &= T_2(x)\\
\frac{\partial T_2}{\partial x}f(x) &= T_3(x)\\
&\vdots\\
\frac{\partial T_{n-1}}{\partial x}f(x) &= T_n(x)\\
\frac{\partial T_n}{\partial x}f(x) &= -\alpha(x)\gamma(x)
\end{aligned}$$

and (3) is equivalent to

$$\frac{\partial T_1}{\partial x} g(x) = 0$$
$$\frac{\partial T_2}{\partial x} g(x) = 0$$
$$\vdots$$
$$\frac{\partial T_{n-1}}{\partial x} g(x) = 0$$
$$\frac{\partial T_n}{\partial x} g(x) = \gamma(x) \neq 0$$

Setting $h(x) = T_1$, we see that

$$T_{i+1}(x) = L_f T_i(x) = L_f^i h(x), \; i = 1, 2, ..., n-1$$

and

$$\begin{aligned} & L_g L_f^{i-1} h(x) = 0, \; i = 1, 2, ..., n-1 \\ & L_g L_f^{n-1} \neq 0 \end{aligned} \quad (4)$$

Therefore we can conclude that if $h(.)$ satisfies (4) the system is feedback linearizable.

The existence of $h(.)$ can be characterized by necessary and sufficient conditions on the vector fields f and g. First we need some terminology.

Definition

Given two vector fields f and g on $D \subset \mathbb{R}^n$, the Lie Bracket $[f, g]$ is the vector field

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

Note that

$$\begin{aligned} & [f, g] = -[g, f] \\ & f = g = cte \Rightarrow [f, g] = 0 \end{aligned}$$

Adjoint representation

$$\begin{aligned} ad_f^0 g(x) &= g(x) \\ ad_f^1 g(x) &= [f, g](x) \\ ad_f^k g(x) &= [f, ad_f^{k-1} g](x), \; k \geq 1 \end{aligned}$$

Example 1

$$f(x) = \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix}, \qquad g(x) = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$$

Then,

$$\begin{aligned}
[f,g](x) &= \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix} f(x) - \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} g(x) \\
&= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ x_1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
ad_f^2 g = [f, ad_f g] &= \frac{\partial ad_f g}{\partial x} f(x) - \frac{\partial f}{\partial x} ad_f g(x) \\
&= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix} \\
&= \begin{bmatrix} -x_1 - 2x_2 \\ x_1 + x_2 - \sin x_1 - x_1 \cos x_1 \end{bmatrix}
\end{aligned}$$

Example 2: $f(x) = Ax$ and $g(x) = g$ is a constant vector field.

Then,

$$ad_f g(x) = [f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) = -Ag$$

$$ad_f^2 g(x) = [f, ad_f g](x) = \frac{\partial ad_f g}{\partial x} f - \frac{\partial f}{\partial x} ad_f g = -A(-Ag) = A^2 g$$

$$ad_f^k g = (-1)^k A^k g$$

Definition

For vector fields $f_1, f_2, ..., f_k$ on $D \subset \mathbb{R}^n$, a distribution Δ is a collection of all vector spaces $\Delta(x) = span\{f_1(x), ..., f_k(x)\}$, where for each fixed $x \in D$, $\Delta(x)$ is the subspace of $\mathbb{R}^n$ spanned by the vectors $f_1(x), ..., f_k(x)$.

The dimension of $\Delta(x)$ is defined by

$$dim(\Delta(x)) = rank[f_1(x), f_2(x), ..., f_k(x)]$$

which may depend on x.

If $f_1, f_2, ..., f_k$ are linearly independent, then $dim(\Delta(x)) = k, \ \forall x \in D$. In this case, we say that Δ is a nonsingular distribution on D. A distribution Δ is involutive if

$$g_1 \in \Delta, g_2 \in \Delta \Rightarrow [g_1, g_2] \in \Delta.$$

If Δ is a nonsingular distribution on D, generated by $f_1, ..., f_k$ then it is involutive if and only if $[f_i, f_j] \in \Delta, \ \forall 1 \le i, j \le k$

Example 3

Let $D = \mathbb{R}^3$, $\Delta = span\{f_1, f_2\}$ and

$$f_1 = \begin{bmatrix} 2x_2 \\ 1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}$$

$dim(\Delta(x)) = rank[f_1, f_2] = 2, \ \forall x \in D$

is Δ involutive?

$$[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2x_2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Checking that $[f_1, f_2] \in \Delta$ is the same to see if $[f_1, f_2]$ can be generated by f_1, f_2, that is if $rank[f_1(x), f_2(x), [f_1, f_2](x)] = 2, \forall x \in D$. But

$$rank \begin{bmatrix} 2x_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & x_2 & 1 \end{bmatrix} = 3, \ \forall x \in D$$

Hence, Δ is not involutive.

Theorem 13.2

Theorem

The system $\dot{x} = f(x) + g(x)u$, with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ is feedback linearizable if and only if there is a domain $D_0 \subset D$ such that

1. *The matrix $G(x) = [g(x), ad_f g(x), ..., ad_f^{n-1} g]$ has rank n $\forall x \in D_0$.*
2. *The distribution $D = span\{g, ad_f g(x), ..., ad_f^{n-2} g\}$ is involutive in D_0.*

Example

$$\dot{x} = f(x) + gu, \quad f(x) = \begin{bmatrix} a \sin x_2 \\ -x_1^2 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we have seen that

$$ad_f g = [f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = -\begin{bmatrix} 0 & a \cos x_2 \\ -2x_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -a \cos x_2 \\ 0 \end{bmatrix}$$

The matrix $G = [g, ad_f g] = \begin{bmatrix} 0 & -a \cos x_2 \\ 1 & 0 \end{bmatrix}$ has rank $G = 2, \ \forall \cos x_2 \neq 0$. The distribution $D = span\,\{g\}$ is involutive. Thus, we can conclude that there exists a $T(x)$ in $D_0 = \{x \in \mathbb{R}^2 : \cos x_2 \neq 0\}$ that allow us to do feedback linearization.

Now we have to find $h(x)$ that satisfies

$$\frac{\partial h}{\partial x} g = 0; \quad \frac{\partial (L_f h)}{\partial x} g \neq 0; \quad h(0) = 0$$

$$\begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\partial h}{\partial x_2} = 0$$

Thus, $h(.)$ must be independent of x_2.

$$L_f h(x) = \frac{\partial h}{\partial x} f(x) = \begin{bmatrix} \frac{\partial h}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} a \sin x_2 \\ -x_1^2 \end{bmatrix} = \frac{\partial h}{\partial x_1} a \sin x_2$$

$$\frac{\partial L_f h}{\partial x} g = \begin{bmatrix} \frac{\partial}{\partial x_1} \left(\frac{\partial h}{\partial x_1} a \sin x_2 \right) & \frac{\partial}{\partial x_2} \left(\frac{\partial h}{\partial x_1} a \sin x_2 \right) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\partial h}{\partial x_1} a \cos x_2 \neq 0$$

In conclusion, $\frac{\partial h}{\partial x_1} \neq 0$ and $\frac{\partial h}{\partial x_2} = 0$.
Examples of such $h(x)$ include $h(x) = x_1$ or $h(x) = x_1 + x_1^3$. Given $h(x)$ we can now perform input-output linearization.

Example 2

A single link manipulator with flexible points

$$\dot{x} = f(x) + gu$$

$$f(x) = \begin{bmatrix} x_2 \\ -a\sin x_1 - b(x_1 - x_3) \\ x_4 \\ c(x_1 - x_3) \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix} \qquad a, b, c, d > 0$$

$$G = [g, ad_f g, ad_f^2 g, ad_f^3 g]$$

$$ad_f g = [f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = 0 - \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a\cos x_1 - b & 0 & b & 0 \\ 0 & 0 & 0 & 1 \\ c & 0 & -c & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -d \\ 0 \end{bmatrix}$$

$$ad_f^2 g = [f, ad_f g] = \frac{\partial (ad_f g)}{\partial x} f - \frac{\partial f}{\partial x} ad_f g$$

$$= 0 - \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a\cos x_1 - b & 0 & b & 0 \\ 0 & 0 & 0 & 1 \\ c & 0 & -c & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -d \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ bd \\ 0 \\ -cd \end{bmatrix}$$

$$ad_f^3 g = [f, ad_f^2 g] = \frac{\partial ad_f^2 g}{\partial x} f - \frac{\partial f}{\partial x} ad_f^2 g$$

$$= 0 - \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a\cos x_1 - b & 0 & b & 0 \\ 0 & 0 & 0 & 1 \\ c & 0 & -c & 0 \end{bmatrix} \begin{bmatrix} 0 \\ bd \\ 0 \\ -cd \end{bmatrix} = \begin{bmatrix} -bd \\ 0 \\ cd \\ 0 \end{bmatrix}$$

Thus,

$$G = \begin{bmatrix} 0 & 0 & 0 & -bd \\ 0 & 0 & bd & 0 \\ 0 & -d & 0 & cd \\ d & 0 & -cd & 0 \end{bmatrix} \rightarrow \text{rank } G = 4, \ \forall x \in \mathbb{R}^4$$

The distribution $\Delta = span\left\{g, ad_f g, ad_f^2\right\}$ is involutive since $g, ad_f g, ad_f^2$ are constant vector fields ($[g, ad_f g] = 0$).
Therefore, we can conclude that there exists an $h(x) : \mathbb{R}^4 \rightarrow \mathbb{R}$ and a $T(x)$ that make the system full state feedback linearizable. In particular, $h(x)$ must satisfy

$$\frac{\partial L_f^{i-1} h}{\partial x} g = 0, \ i = 1, 2, 3.$$

$$\frac{\partial L_f^3 h}{\partial x} g \neq 0, \quad h(0) = 0$$

For $i = 1$, we have $\frac{\partial h}{\partial x} g = 0 \Rightarrow \frac{\partial h}{\partial x_4} = 0$ and so $h(x)$ must be independent of x_4.

$L_f h(x) = \frac{\partial h}{\partial x} f(x) = \frac{\partial h}{\partial x_1} x_2 + \frac{\partial h}{\partial x_2}[-a \sin x_1 - b(x_1 - x_3)] + \frac{\partial h}{\partial x_3} x_4$.
From

$$\frac{\partial L_f h}{\partial x} g = 0 \Rightarrow \frac{\partial L_f h}{\partial x_4} = 0 \Rightarrow \frac{\partial h}{\partial x_3} = 0$$

Thus, $h(x)$ is independent of x_3.
Thus,

$$L_f h(x) = \frac{\partial h}{\partial x_1} x_2 + \frac{\partial h}{\partial x_2}[-a \sin x_1 - b(x_1 - x_3)]$$

.

$$\begin{aligned} L_f^2 h(x) = L_f L_f h(x) = \frac{\partial (L_f h)}{\partial x} f(x) = \frac{\partial (L_f h)}{\partial x_1} x_2 + \frac{\partial (L_f h)}{\partial x_2} [-a \sin x_1 - b(x_1 - x_3)] \\ + \frac{\partial (L_f h)}{\partial x_3} x_4 + \frac{\partial (L_f h)}{\partial x_4} c(x_1 - x_3) \end{aligned}$$

For $i = 2$,

$$\frac{\partial \left(L_f^2 h\right)}{\partial x} g = 0 \Rightarrow \frac{\partial \left(L_f h\right)}{\partial x_3} = 0 \Rightarrow \frac{\partial h}{\partial x_2} = 0$$

Thus h is independent of x_2.

Hence,

$$L_f^3 h(x) = L_f L_f^2 h = \frac{\partial\left(L_f^2 h\right)}{\partial x} f(x) = \frac{\partial\left(L_f^2 h\right)}{\partial x_1} x_2 + \frac{\partial\left(L_f^2 h\right)}{\partial x_1}\left[-a \sin x_1 - b(x_1 - x_3)\right] + \frac{\partial\left(L_f^2 h\right)}{\partial x_3} x_4 + \frac{\partial\left(L_f^2 h\right)}{\partial x_4} c(x_1 - x_3)$$

Also, from condition

$$\frac{\partial L_f^3 h(x)}{\partial x} g \neq 0 \Rightarrow \frac{\partial L_f^2 h}{\partial x_3} \neq 0 \Rightarrow \frac{\partial L_f h}{\partial x_2} \neq 0 \Rightarrow \frac{\partial h}{\partial x_1} \neq 0$$

Therefore, let $h(x) = x_1$. Then, the change of variables

$$\begin{aligned}
z_1 &= h(x) = x_1 \\
z_2 &= L_f h(x) = x_2 \\
z_3 &= L_f^2 h(x) = -a \sin x_1 - b(x_1 - x_3) \\
z_4 &= L_f^3 h(x) = -a \cos x_1\, \dot{x}_1 - b\dot{x}_1 + b\dot{x}_3 = -a x_2 \cos x_1 - b x_2 + b x_4
\end{aligned}$$

transforms the state equation into

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= -(a \cos z_1 + b + c) z_3 + a(z_2^2 - c) \sin z_1 + bd\, u
\end{aligned}$$

Example - field controlled DC motor

Consider the system

$$\dot{x} = f(x) + gu$$

with

$$f(x) = \begin{bmatrix} -ax_1 \\ -bx_2 + k - cx_1x_3 \\ \theta x_1 x_2 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Computing the

$$ad_f g = [f, g] = \begin{bmatrix} a \\ cx_3 \\ -\theta x_2 \end{bmatrix}, \quad ad_f^2 g = [f, ad_f g] = \begin{bmatrix} a^2 \\ (a+b)\, cx_3 \\ (b-a)\, \theta x_2 - \theta k \end{bmatrix}$$

$$G = [g, ad_f g, ad_f^2 g] = \begin{bmatrix} 1 & a & a^2 \\ 0 & cx_3 & (a+b)\, cx_3 \\ 0 & -\theta x_2 & (b-a)\, \theta x_2 - \theta k \end{bmatrix}$$

Rank of G?

$\det G = c\theta(-k + 2bx_2)x_3.$

Hence G has rank 3 for $x_2 \neq \frac{k}{2b}$ and $x_3 \neq 0$. Let's check the distribution $D = span\{g, ad_f g\}$

$$[g, ad_f g] = \frac{\partial(ad_f g)}{\partial x} g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -\theta & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence D is involutive because $[g, ad_f g] \in D$.

Therefore, the conditions of Theorem 13.2 are satisfied in particular for the domain

$$D_0 = \left\{ x \in \mathbb{R}^3 : x_2 > \frac{k}{2b}, x_3 > 0 \right\}$$

$h(.) = ?$

$$\dot{x} = f(x) + gu$$

with

$$f(x) = \begin{bmatrix} -ax_1 \\ -bx_2 + k - cx_1x_3 \\ \theta x_1 x_2 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Equilibrium points: $x_1 = 0$, $x_2 = \frac{k}{b}$, $x_3 = cte$.

Suppose we are interested in the desired operating point $x^* = [0 \ \frac{k}{b} \ w_0]^T$.

Then, $h(x)$ must satisfy $(n = 3)$

$$\frac{\partial h}{\partial x} g = 0; \quad \frac{\partial (L_f h)}{\partial x} g = 0; \quad \frac{\partial (L_f^2 h)}{\partial x} g \neq 0$$

and $h(x^*) = 0$.

$$\frac{\partial h}{\partial x} g = 0 \Rightarrow \frac{\partial h}{\partial x_1} = 0$$

that is, h must be independent of x_1.

$L_f h(x) = ...$

State feedback control

Consider the system

$$\dot{x} = f(x) + G(x)u$$

and let $z = T(x) = [T_1(x),\ T_2(x)]^T$ such that

$$\dot{\eta} = f_0(\eta, \xi)$$
$$\dot{\xi} = A\xi + B\gamma(x)[u - \alpha(x)]$$

Suppose that (A, B) is controllable, $\gamma(x)$ is nonsingular $\forall x \in D$, $f_0(0, 0) = 0$ and $f_0(\eta, \xi), \alpha(x), \gamma(x) \in C^1$.

Goal: Design a state feedback control law to stabilize the origin $z = 0$.

Setting $u = \alpha(x) + \beta(x)v$, $\beta(x) = \gamma^{-1}(x)$, we obtain the triangular system

$$\dot{\eta} = f_0(\eta, \xi)$$
$$\dot{\xi} = A\xi + Bv$$

Let $v = -K\xi$ with $(A - BK)$ Hurwitz then we can conclude the following:

State feedback control

Lemma (13.1)

The origin $z = 0$ is asymptotically stable (AS) if the origin of $\dot{\eta} = f_0(\eta, 0)$ is AS (that is, if the system is minimum phase).

Proof.

By the converse theorem $\exists V_1(\eta) : \frac{\partial V_1}{\partial \eta} f_0(\eta, 0) \leq -\alpha_3(\|x\|)$ in some neighborhood of $\eta = 0$, where $\alpha_3 \in \mathcal{K}$. Let $P = P^T > 0$ be the solution of the Lyapunov equation $P(A - BK) + (A - BK)^T P = -I$.

$$V = V_1 + k\sqrt{\xi^T P \xi}, \quad k > 0$$

$$\begin{aligned}
\dot{V} &\leq \frac{\partial V_1}{\partial \eta} f_0(\eta, \xi) + \frac{k}{2\sqrt{\xi^T P \xi}} \xi^t [P(A - BK) + (A - BK)^T P]\xi \\
&\leq \frac{\partial V_1}{\partial \eta} f_0(\eta, 0) + \frac{\partial V_1}{\partial \eta}[f_0(\eta, \xi) - f_0(\eta, 0)] - \frac{k\xi^T \xi}{2\sqrt{\xi^T P \xi}} \\
&\leq -\alpha_3(\|\eta\|) + k_1\|\xi\| - kk_2\|\xi\|, \quad k_1, k_2 > 0
\end{aligned}$$

where the last inequality follows from restricting the state to be in any bounded neighborhood of the origin and using the continuous differentiability property of V_1 and f_0. Thus, choosing $k > k_1/k_2$ ensures that $\dot{V} < 0$, which follows that the origin is AS. □

State feedback control

Lemma (13.2)

The origin $z = 0$ is GAS if $\dot{\eta} = f_0(\eta, \xi)$ is ISS.

Note that if $\dot{\eta} = f_0(\eta, 0)$ is GAS or GES does NOT imply ISS. But, if it is GES + globally Lipschitz then it is ISS.
Otherwise, we have to prove ISS by further analysis.

Example:

$$\begin{aligned} \dot{\eta} &= -\eta + \eta^2 \xi \\ \dot{\xi} &= v \end{aligned}$$

Zero dynamics: $\dot{\eta} = -\eta \longrightarrow \eta = 0$ is GES
but $\dot{\eta} = -\eta + \eta^2 \xi$ is not ISS, e.g., if $\xi(t) = 1$ and $\eta(0) \geq 2$ then $\dot{\eta}(t) \geq 2, \forall t \geq 0$, which implies that η grows unbounded.

However with $v = -K\xi$, $K > 0$ we achieve AS.

To view this, let $\nu = \eta\xi$, then

$$\begin{aligned}\dot{\nu} &= \eta\dot{\xi} + \dot{\eta}\xi \\ &= \eta\upsilon - \eta\xi + \eta^2\xi^2 \\ &= -K\eta\xi - \eta\xi + \eta^2\xi^2 \\ &= -(1+K)\nu + \upsilon^2 = -[(1+K) - \nu]\nu^2\end{aligned}$$

Thus, with $\nu(0) < 1 + K \Rightarrow \upsilon \to 0$ and therefore we can also conclude that $\exists T \geq t_0 : \upsilon(t) \leq \frac{1}{2},\ \forall t \geq T$.

Consider now $V = 1/2\eta^2$. Then

$$\begin{aligned}\dot{V} &= \eta\dot{\eta} \\ &= -\eta^2 + \eta^3\xi \\ &= -\eta^2(1 - \eta\xi) = -\eta^2(1-\nu) < 0,\ \forall t \geq T\end{aligned}$$

Thus, $\eta \to 0$ and note also that $\dot{\xi} = -K\xi$. Therefore, the control law $v = -K\xi$ can achieve semiglobal stabilization.

One may think that we can assign the eigenvalues of $(A - BK)$ to the left half-complex plane to make $\dot{\xi} = (A - BK)\xi$ decay to zero arbitrarily fast. BUT this may have consequences: the zero-dynamics may go unstable! This is due to the peaking phenomenon.

Example

$$\begin{aligned} \dot{\eta} &= -\tfrac{1}{2}(1+\xi_2)\eta^3 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v \end{aligned}$$

Setting $v = -K\xi = -k^2\xi_1 - 2k\xi_2 \;\longrightarrow\; A - BK = \begin{bmatrix} 0 & 1 \\ -k^2 & -2k \end{bmatrix}$
and the eigenvalues are $-k, -k$. Note that

$$e^{(A-BK)t} = \begin{bmatrix} (1+kt)e^{-kt} & te^{-kt} \\ -k^2te^{-kt} & (1-kt)e^{-kt} \end{bmatrix}$$

which shows that as $k \to \infty$, $\xi(t)$ will decay to zero arbitrary fast.
However, the element $(2,1)$ reaches a maximum value k/e at $t = \frac{1}{k}$. There is a peak of order k! Furthermore, the interaction of peaking with nonlinear growth could destabilize the system.

E.g., consider the initial conditions

$$\begin{aligned} \eta(0) &= \eta_0 \\ \xi_1(0) &= 1 \\ \xi_2(0) &= 0 \end{aligned}$$

Then,

$$\xi_2(t) = -k^2 t e^{-kt}$$

and

$$\begin{aligned} \dot{\eta} &= -\frac{1}{2}(1+\xi_2)\eta^3 \\ &= -\frac{1}{2}(1-k^2 t e^{-kt})\eta^3 \end{aligned}$$

In this case the solution $\eta(t)$ is given by

$$\eta^2(t) = \frac{\eta_0^2}{1+\eta_0^2[t+(1+kt)e^{-kt}-1]}.$$

which has a finite escape if $\eta_0 > 1$.

Tracking

$$\dot{\eta} = f_0(\eta, \xi)$$
$$\dot{\xi} = A_0\xi + B_0\gamma(x)[u - \alpha(x)]$$
$$y = C_0\xi$$

Goal: Design u such that y asymptotically tracks a reference signal $r(t)$. Assume that $r(t), \dot{r}(t), ..., r^{(\rho)}$ are bounded and available on-line. Note that the reference signal could be the output of a pre-filter.

Example: If $\rho = 2$, the pre-filter could be

$$G(s) = \frac{w_n^2}{s^2 + 2\xi w_n s + w_n}$$

Then

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -w_n^2 y_1 - 2\xi w_n y_2 + w_n^2 y_d$$
$$r = y_1$$

Note that in this case $\dot{r} = \dot{y}_1 = y_2$ and $\ddot{r} = \dot{y}_2$.

Consider a system with relative degree ρ. Let

$$\mathbf{r} = \begin{bmatrix} r \\ \vdots \\ r^{(\rho-1)} \end{bmatrix}, \text{ and } e = \begin{bmatrix} \xi_1 - r \\ \vdots \\ \xi_\rho - r^{(\rho-1)} \end{bmatrix} = \xi - \mathbf{r}$$

Then, the error system is given by

$$\begin{aligned}\dot{\eta} &= f_0(\eta, e + \mathbf{r}) \\ \dot{e} &= A_c e + B_c(\gamma(x)[u - \alpha(x)] - r^{(\rho)})\end{aligned}$$

Setting $u = \alpha(x) + \beta(x)[v - r^{(\rho)}]$ with $\beta = \frac{1}{\gamma(x)}$ it follows that

$$\begin{aligned}\dot{\eta} &= f_0(\eta, e + \mathbf{r}) \\ \dot{e} &= A_c e + B_c v\end{aligned}$$

Thus, selecting $v = -Ke$ with $(A_c - B_c K)$ Hurwitz we can conclude that the states of the closed-loop system

$$\begin{aligned}\dot{\eta} &= f_0(\eta, e + \mathbf{r}) \\ \dot{e} &= (A_0 - B_0 K)e\end{aligned}$$

are bounded if $\dot{\eta} = f_0(\eta, e + R)$ is ISS and that $e \to 0$ as $t \to \infty$.