Nonlinear Control Systems

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3. Fundamental properties

IST-DEEC PhD Course

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Example

Consider the system

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0$$

- Does it have a solution over an interval $[t_0, t_1]$?

  That is, does exist a continuous function $x : [t_0, t_1] \to \mathbb{R}^m$ such that $\dot{x}(t)$ is defined and satisfies $\dot{x}(t) = f(t, x(t)),\ \forall t \in [t_0, t_1]$?

- Is it unique? or is possible to have more than one solution?

- ... and if we restrict $f(t, x)$ to be continuous in x and piecewise continuous in t. Is this sufficient to guarantee existence and uniqueness?

  No!, e.g.

  $$\dot{x} = x^{1/3}, \qquad x(0) = 0$$

  It has solution $x(t) = \left(\frac{2t}{3}\right)^{3/2}$.
  But is not unique, since $x(t) = 0$ is another solution!

Lipschitz condition

A function $f(x)$ is said to be *locally Lipschitz* on a domain (open and connected set) $D \subset \mathbb{R}^n$ if each point of D has a neighborhood D_0 such that $f(.)$ satisfies

$$\|f(x) - f(y)\| \leq L \|x - y\|, \quad \forall x, y \in D_0$$

(with the same Lipschitz constant L). The same terminology is extended to a function $f(t, x)$, provided that the Lipschitz constant holds uniformly in t for all t in a given interval.

Remark: For $f : \mathbb{R} \longrightarrow \mathbb{R}$, we have

$$\frac{|f(x) - f(y)|}{|x - y|} \leq L$$

which means that a straight line joining any two points of $f(.)$ cannot have a slope whose absolute value is greater than L.

Example

$f(x) = x^{1/3}$ is not locally Lipschitz at $x = 0$ since $f'(x) = \frac{1}{3}x^{-2/3} \to \infty$ as $x \to 0$.

Existence and Uniqueness

Theorem 3.1 - Local Existence and Uniqueness

Let $f(t,x)$ be piecewise continuous in t and satisfy the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|$$

$\forall x, y \in B = \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}, \forall t \in [t_0, t_1]$. Then, there exists some $\delta > 0$ such that the state equation $\dot{x} = f(t,x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.

Theorem 3.2 - Global Existence and Uniqueness

Suppose that $f(t,x)$ is piecewise continuous in t and satisfies

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \quad \forall x, y \in \mathbb{R}^n, \forall t \in [t_0, t_1]$$

Then, the state equation $\dot{x} = f(t,x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_1]$.

Existence and Uniqueness

Lemma 3.2

If $f(t,x)$ and $\left[\frac{\partial f}{\partial x}\right]_{(t,x)}$ are continuous on $[a,b] \times D$ for some domain $D \subset \mathbb{R}^n$, then f is locally Lipschitz in x on $[a,b] \times D$.

Lemma 3.3

If $f(t,x)$ and $\left[\frac{\partial f}{\partial x}\right]_{(t,x)}$ are continuous on $[a,b] \times \mathbb{R}^n$, then f is globally Lipschitz in x on $[a,b] \times \mathbb{R}^n$ if and only if $\left[\frac{\partial f}{\partial x}\right]$ is uniformly bounded on $[a,b] \times \mathbb{R}^n$.

Examples

1.

$$\dot{x} = A(t)x + g(t) = f(t, x) \tag{1}$$

with $A(t)$, $g(t)$ piecewise continuous functions of t.

$$\begin{aligned} \|f(t,x) - f(t,y)\| &= \|A(t)\,x + g(t) - (A(t)\,y + g(t))\| \\ &= \|A(t)\,(x-y)\| \leq \|A(t)\|\,\|(x-y)\| \end{aligned}$$

Note that for any finite interval of time $[t_0, t_1]$, the elements of $A(t)$ are bounded. Thus $\|A(t)\| \leq a$ for any induced norm and

$$\|f(t,x) - f(t,y)\| \leq a\,\|(x-y)\|$$

Therefore, from Theorem 3.1 we conclude that (1) has a unique solution over $[t_0, t_1]$. Since t_1 can be arbitrarily large it follows that the system has a unique solution $\forall t \geq t_0$. There is no finite escape time.

Examples

2.

$$\dot{x} = -x^3 = f(x), \quad x \in \mathbb{R} \tag{2}$$

Is it globally Lipschitz?
No! From Lemma 3.3, $f(x)$ is continuous but the Jacobian $\frac{\partial f}{\partial x} = -3x^2$ is not globally bounded. Nevertheless, $\forall x(t_0) = x_0$, (2) has the unique solution

$$x(t) = \text{sgn}(x_0)\sqrt{\frac{x_0^2}{1 + 2x_0^2(t - t_0)}}$$

3.

$$\dot{x} = -x^2, \quad x(0) = -1 \tag{3}$$

From Lemma 3.2 we conclude that is locally Lipschitz in any compact subset of $\mathbb{R}$ because $f(x)$ and $\frac{\partial f}{\partial x}$ are continuos. Hence, there exists a unique solution over $[0, \delta]$ for some $\delta > 0$. In particular the solution is

$$x(t) = \frac{1}{t - 1}$$

and only exists over $[0, 1)$, i.e, there is finite escape time at $t = 1$!

Uniqueness and Existence

Theorem 3.3

Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq t_0$ and all $x \in D \subset \mathbb{R}^n$. Let W be a compact subset of D, $x_0 \in W$ and suppose it is known that every solution of

$$\dot{x} = f(t,x), \quad x(t_0) = x_0$$

lies entirely in W. Then, there is a unique solution that is defined for all $t \geq t_0$.

Proof.

By Theorem 3.1, there is a unique solution over $[t_0, t_0 + \delta]$. Let $[t_0, T)$ be its maximum interval of existence. We would like to show that $T = \infty$. Suppose that is not, i.e., T is finite. Then the solution must leave any compact subset of D. But this is a contradiction, becuase x never leaves the compact set W. Thus we can conclude that $T = \infty$. □

Example

Returning to the example

$$\dot{x} = -x^3$$

It is locally Lipschitz in $\mathbb{R}$. For any initial condition $x(0) = x_0 \in \mathbb{R}$, the solution cannot leave the compact set $W = \{x \in \mathbb{R} : |x| \leq x_0\}$ because for any instant of time

- if $x > 0$ then $\dot{x} < 0$
- if $x < 0$ then $\dot{x} > 0$

Thus, *without computing explicitly the solution* we can conclude from Theorem 3.3 that the system has a unique solution for all $t \geq 0$.

Continuous dependence on initial conditions and parameters

Consider the following nominal model

$$\dot{x} = f(t, x, \lambda_0) \tag{4}$$

where $\lambda_0 \in \mathbb{R}^p$ denotes the nominal vector of constant parameters of the model and $x \in \mathbb{R}^n$ is the state.

- Let $y(t)$ be a solution of (4) that starts at $y(t_0) = y_0$ and is defined on the interval $[t_0, t_1]$.
- Let $z(t)$ be a solution of $\dot{x} = f(t, x, \lambda)$ defined on $[t_0, t_1]$ with $z(t_0) = z_0$.

When does $z(t)$ remains close to $y(t)$?
Or in other words, is the solution continuous dependent on the initial condition and parameter λ? That is,

$$\forall_{\varepsilon>0}\exists_{\delta>0} : \|z_0 - y_0\| < \delta, \|\lambda - \lambda_0\| < \delta \Rightarrow \|z(t) - y(t)\| < \varepsilon, \forall_{t\in[t_0,t_1]}$$

Continuous dependence on initial conditions and parameters

Theorem 3.4

Let $f(t,x)$ be piecewise continuous in t and Lipschitz in x (with a Lipschits constant L) on $[t_0,t_1]\times W$, where $W\subset\mathbb{R}^n$ is an open connected set. Let $y(t)$ and $z(t)$ be solutions of

$$\dot{y} = f(t,y), \qquad y(t_0) = y_0$$
$$\dot{z} = f(t,z) + g(t,z), \qquad z(t_0) = z_0$$

such that $y(t), z(t) \in W,\ \forall t\in[t_0,t_1]$. Suppose that

$$\|g(t,z)\| \le \mu, \quad \forall (t,x)\in[t_0,t_1]\times W$$

for some $\mu > 0$. Then

$$\|y(t) - z(t)\| \le \|y_0 - z_0\| e^{L(t-t_0)} + \frac{\mu}{L}\left(e^{L(t-t_0)} - 1\right), \quad \forall t\in[t_0,t_1]$$

Proof

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau$$

$$z(t) = z_0 + \int_{t_0}^{t} [f(\tau, z(\tau)) + g(\tau, z(\tau))]\, d\tau$$

Subtracting and taking norms yields

$$\|y(t) - z(t)\| \leq \|y_0 - z_0\| + \int_{t_0}^{t} \|f(\tau, y(\tau)) - f(\tau, z(\tau))\|\, d\tau + \int_{t_0}^{t} \|g(\tau, z(\tau))\|\, d\tau$$

$$\leq \underbrace{\|y_0 - z_0\|}_{\gamma} + \int_{t_0}^{t} L\, \|y(\tau) - z(\tau)\|\, d\tau + \mu(t - t_0)$$

Applying the Gronwall-Bellman inequality, yields

$$\|y(t) - z(t)\| \leq \gamma + \mu(t - t_0) + \int_{t_0}^{t} L\left[\gamma + \mu(\tau - t_0)\right] e^{L(t-\tau)} d\tau$$

Integrating the right-hand side by parts ($\int uv' = uv - \int vu'$) we obtain

$$\begin{aligned}\|y(t) - z(t)\| &\leq \gamma + \mu(t - t_0) - \gamma - \mu(t - t_0) + \gamma e^{L(t-t_0)} + \int_{t_0}^{t} \mu e^{L(t-s)\, ds} \\ &= \gamma e^{L(t-t_0)} + \frac{\mu}{L}\left(e^{L(t-t_0)} - 1\right)\end{aligned}$$

Theorem 3.5 - Continuity of solutions

Let $f(t, x, \lambda)$ be continuous in (t, x, λ) and locally Lipschitz in x on $[t_0, t_1] \times D \times \{\|\lambda - \lambda_0\| \leq c\}$, where $D \subset \mathbb{R}^n$ is an open connected set. Let $y(t, \lambda_0)$ be a solution of

$$\dot{x} = f(t, x, \lambda_0),$$

with $y(t_0, \lambda_0) = y_0 \in D$. Suppose $y(t, \lambda_0)$ is defined and belongs to D for all $t \in [t_0, t_1]$. Then, given $\epsilon > 0$, there is $\delta > 0$ such that if

$$\|z_0 - y_0\| < \delta, \ \|\lambda - \lambda_0\| < \delta$$

then there is a unique solution $z(t, \lambda)$ of $\dot{x} = f(t, x, \lambda)$ defined on $[t_0, t_1]$, with $z(t_0, \lambda) = z_0$ such that $\|z(t, \lambda) - y(t, \lambda_0)\| < \epsilon$, $\forall t \in [t_0, t_1]$

Proof.

$$\dot{z} = f(t, z, \lambda_0) + \underbrace{f(t, z, \lambda) - f(t, z, \lambda_0)}_{g(t,z)}$$

By continuity $\forall \mu > 0, \ \exists \delta > 0$:

$$\|\lambda - \lambda_0\| < \delta \Rightarrow \|g(t, z)\| \leq \mu$$

Therefore, using Theorem 3.4 we conclude Theorem 3.5 by noticing that $\|y_0 - z_0\|$ and μ can be chosen arbitrarily small.

□

Comparison Principle

Quite often when we study the state equation $\dot{x} = f(t, x)$ we need to compute bounds on the solution $x(t)$. For that we have

- Gronwall-Bellman inequality
- The comparison Lemma $\rightarrow$ Compares the solution of the differential inequality $\dot{v}(t) \leq f(t, v(t))$ with the solution of $\dot{u}(t) = f(t, u)$. Moreover, $v(t)$ is not needed to be differentiable.

Definition

Upper right-hand derivative

$$D^+ v(t) = \lim \sup_{h \rightarrow 0^+} \frac{v(t+h) - v(t)}{h}$$

The following properties hold:

- if $v(t)$ is differentiable at t then $D^+ v(t) = \dot{v}(t)$
- if $\frac{1}{h} |v(t+h) - v(t)| \leq g(t, h)$ and $\lim_{h \rightarrow 0^+} g(t, h) = g_0(t)$, then $D^+ v(t) \leq g_0(t)$

Comparison Principle

Lemma 3.4 - Comparison Lemma

Let

$$\dot{u} = f(t, u), \quad u(t_0) = u_0,\ \mu \in \mathbb{R}$$

where $f(t, u)$ is continuous in t and locally Lipschitz in u, for all $t \geq 0$ and $u \in J \subset \mathbb{R}$. Let $[t_0, T)$ (T can be ∞) be the maximal interval of existence of the solution $u(t) \in J$. Let $v(t)$ be a continuous function that satisfies

$$D^+ v(t) \leq f(t, v), \quad v(t_0) \leq u_0$$

with $v(t) \in J$ for all $t \in [t_0, T)$. Then, $v(t) \leq u(t),\ \forall t \in [t_0, T)$

Example

Show that the solution of

$$\dot{x} = f(x) = -(1+x^2)x, \quad x(0) = a$$

is unique and defined for all $t \geq 0$.

Because $f(x)$ is locally Lipschitz it has a unique solution on $[0, t_1]$ for some $t_1 > 0$. Let $v(t) = x^2(t)$. Then

$$\dot{v}(t) \leq -2v(t), \quad v(0) = a^2$$

Let $u(t)$ be the solution of

$$\dot{u} = -2u,\ u(0) = a^2 \longrightarrow u(t) = a^2 e^{-2t}$$

Then, by comparison lemma the solution $x(t)$ is defined $\forall t \geq 0$ and satisfies

$$|x(t)| = \sqrt{v(t)} \leq |a|\, e^{-t},\ \forall t \geq 0$$

By Theorem 3.3 it follows that the solution is unique and defined for all $t \geq 0$.