An exponential observer for systems on $\mathrm{SE}(3)$ with implicit outputs

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Abstract This paper considers the state estimation problem of a class of systems described by implicit outputs and whose state lives in the special Euclidean group $\mathrm{SE}(3)$. This type of systems are motivated by applications in dynamic vision such as the estimation of the motion of a camera from a sequence of images. We propose an observer in the group of motion $\mathrm{SE}(3)$ and discuss conditions under which the linearized state estimation error converges exponentially fast. We also analyze the problem when the system is subject to disturbances and noises. We show that the estimate converges to a neighborhood of the real solution. The size of the neighborhood increases/decreases gracefully with the bound of the disturbance and noise.

1 Introduction

During the last few decades there has been an extensive study on the design of observers for nonlinear systems. In simple terms, an observer or estimator can be defined as a process that provides in real time the estimate of the state (or some

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function of it) of the plant from partial and possibly noisy measurements of the inputs and outputs, and inexact knowledge of the initial condition.

For linear systems evolving on n-dimensional vector spaces, state observer and filter designs employ the traditional Kalman filter [15] and Luenberger type observer [20]. In fact, it is well-known that the Kalman filter [15] is the optimal state estimation algorithm in a well defined sense [6].

For nonlinear systems, the extended Kalman filter is a widely used method for estimating the state. It is obtained by linearizing the nonlinear dynamics and the observation along the trajectory of the estimate. However, if there are substantial nonlinearities or the state lives in some special manifold, there are no guarantees that the state estimate will evolve in the same manifold and even that the estimate will converge to a neighborhood of the true one.

These problems are particularly relevant because they arise in many modern day applications such as the motion control of unmanned aerial vehicles, underwater vehicles, and autonomous robots (See e.g. [12], [5], [25], [3]). Other engineering applications that were studied in [9] are exothermic chemical reactor, a nonholonomic car, and a velocity-aided inertial navigation [7]. Typically, these applications require the design of robust nonlinear observers for systems evolving on Lie groups.

Motivated by the above considerations in [10], [11], [18], [17], [19], [16] a geometrical framework for the design of symmetric preserving observers on finite-dimensional Lie groups is described. In [8], it is shown that when the output map associated with a left-invariant dynamics on an arbitrary Lie group is right-left equivariant, then it is possible to build non-linear observers such that the error equation is autonomous.

In this paper, we consider left-invariant dynamical systems with implicit outputs, for which the results mentioned above do not apply. Systems of this kind typically arise in mobile robotic applications using dynamic vision such as the estimation of a motion of a camera from a sequence of images. In particular, in [2] and [4], the problem of estimating the position and orientation of a controlled rigid body using measurements from a monocular charged-coupled-device (CCD) camera attached to the vehicle is addressed. The reader is referred to [13], [14], [26] for several other examples of implicit output systems in the context of motion and shape estimation.

We propose an observer in the group of motion $\mathrm{SE}(3)$ and discuss conditions under which the linearized state estimation error converges exponentially fast. We also analyze the problem when the system is subject to disturbances and noises. We show that the estimate converges to a neighborhood of the real solution. The size of the neighborhood increases/decreases gracefully with the bound of the disturbance and noise.

The outline of the paper is as follows. Section 2 introduces the mathematical preliminaries and Section 3 formulates the state estimation problem. In Section 4 we propose a left-invariant dynamic observer for estimating the state of systems on $\mathrm{SE}(3)$ with implicit outputs, and determine under what conditions the state estimate converges exponentially to the true state. In Section 5 we analyze the robustness of the proposed observer in the presence of disturbance and noise. Concluding remarks are given in Section 6.

2 Mathematical Preliminaries

In this section we introduce notations and definitions used through out this paper. We denote the Euclidean norm in $\mathbb{R}^n$ by $\|\cdot\|$, and the identity matrix of size n by I_n. Given $A \in \mathbb{R}^{n\times n}$, we let $\det(A)$ and $\mathrm{Tr}(A)$ denote the determinant and the trace of the matrix A, respectively. We consider the scalar product of $A,B \in \mathbb{R}^{n\times n}$ as being defined by $\langle A,B\rangle \stackrel{\text{def}}{=} \mathrm{Tr}(A^{\mathrm{T}}B)$. The corresponding norm $\|A\| = \sqrt{\langle A,A\rangle}$ is the so-called Frobenius norm. Further, if the entries of $A \in \mathbb{R}^{n\times n}$ depend on t, and $A(t)$ is invertible for all t, from the identity $A^{-1}(t)A(t) = I_n$, one may deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}(A^{-1}(t))A(t) + A^{-1}(t)\frac{\mathrm{d}}{\mathrm{d}t}(A(t)) = 0. \tag{1}$$

The cross product of vectors $u, v \in \mathbb{R}^3$ is denoted by $u \times v$. For every $u \in \mathbb{R}^3$,

$$(u\times) = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

denotes the matrix representation of the linear map $v \mapsto u \times v$, $v \in \mathbb{R}^3$. It can be easily shown that, for every $u, v \in \mathbb{R}^3$, $\mathrm{Tr}((u\times)^{\mathrm{T}}(v\times)) = 2u^{\mathrm{T}}v$. Given a vector $u \in \mathbb{R}^3$, we denote by $\bar{u} \in \mathbb{R}^4$ its homogeneous coordinates, that is, $\bar{u} = \begin{bmatrix} u \\ 1 \end{bmatrix}$ [21].

The *special orthogonal* group in three-dimensions is denoted by $\mathrm{SO}(3) \stackrel{\text{def}}{=} \{R \in \mathbb{R}^{3\times 3} \colon R^{\mathrm{T}}R = I_3 \text{ and } \det(R) = +1\}$ and its Lie algebra, that is, the space of all skew-symmetric matrices by $\mathrm{so}(3) \stackrel{\text{def}}{=} \{(u\times) \in \mathbb{R}^{3\times 3} \colon u \in \mathbb{R}^3\}$.

The *special Euclidean* group is denoted by $\mathrm{SE}(3) \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} g_R & g_T \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \colon g_R \in \mathrm{SO}(3) \text{ and } g_T \in \mathbb{R}^3 \right\}$ and its Lie algebra is defined by $\mathrm{se}(3) \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} (\omega\times) & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \colon \omega,\, v \in \mathbb{R}^3 \right\}$.

For every $g = \begin{bmatrix} g_R & g_T \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3)$, we have $g^{-1} = \begin{bmatrix} g_R^{-1} & -g_R^{-1}g_T \\ 0 & 1 \end{bmatrix}$. Since $g^{-1}g = I_4$, we have $\dot{g} \stackrel{\text{def}}{=} \frac{\mathrm{d}g}{\mathrm{d}t} = g\left(-\frac{\mathrm{d}}{\mathrm{d}t}g^{-1}\right)g$. Thus, we can rewrite $\dot{g} = g\Omega$ where $\Omega \stackrel{\text{def}}{=} -\left(\frac{\mathrm{d}}{\mathrm{d}t}g^{-1}\right)g \in \mathrm{se}(3)$. We notice that in order to verify that $\Omega \in \mathrm{se}(3)$ it is sufficient to show the following:

i) $-\left(\frac{\mathrm{d}}{\mathrm{d}t}g^{-1}\right)g = g^{-1}\dot{g} = \begin{bmatrix} g_R^{-1} & -g_R^{-1}g_T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{g}_R & \dot{g}_T \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} g_R^{-1}\dot{g}_R & -g_R^{-1}\dot{g}_T \\ 0 & 0 \end{bmatrix}$,

ii) $(g_R^{-1}\dot{g}_R)^{\mathrm{T}} = (\dot{g}_R)^{\mathrm{T}}g_R = \left(\frac{\mathrm{d}}{\mathrm{d}t}g_R^{\mathrm{T}}\right)g_R = \left(\frac{\mathrm{d}}{\mathrm{d}t}g_R^{-1}\right)g_R = -g_R^{-1}\dot{g}_R$.

We next present a result that will be useful later in the paper.

Lemma 1. *Consider* $\xi = \begin{bmatrix} \xi_R & \xi_T \\ 0 & 0 \end{bmatrix} \in \mathrm{se}(3)$, *where* $\xi_R = (\overline{\xi}\times)$ *and* $\overline{\xi}, \xi_T \in \mathbb{R}^3$. *Then* $\|\xi\|^2 = 2\|\overline{\xi}\|^2 + \|\xi_T\|^2$.

Proof. A simple computation yields $\xi^{\mathrm{T}}\xi = \begin{bmatrix} \xi_R^{\mathrm{T}}\xi_R & \xi_R^{\mathrm{T}}\xi_T \\ \xi_T^{\mathrm{T}}\xi_R & \xi_T^{\mathrm{T}}\xi_T \end{bmatrix}$. Then by the definition $\|\xi\|^2 = \mathrm{Tr}\left(\xi^{\mathrm{T}}\xi\right) = \mathrm{Tr}(\xi_R^{\mathrm{T}}\xi_R) + \|\xi_T\|^2$. Now the result follows from $\mathrm{Tr}(\xi_R^{\mathrm{T}}\xi_R) = \mathrm{Tr}((\overline{\xi}\times)^{\mathrm{T}}(\overline{\xi}\times)) = 2\|\overline{\xi}\|^2$. $\square$

The Lie bracket of two matrices $A, B \in \mathbb{R}^{n\times n}$ is denoted by $[A,B]$ or, equivalently, $\mathrm{ad}_A B$, and is defined as the commutator $[A,B] = AB - BA$. Given $A, B \in \mathbb{R}^{n\times n}$, we denote $\mathrm{ad}_A^1 B = \mathrm{ad}_A B$ and $\mathrm{ad}_A^{k+1} B = \mathrm{ad}_A \mathrm{ad}_A^k B$ for every $k \in \mathbb{N}$.

3 Problem statement

Consider a left-invariant dynamical system evolving on SE(3), described by

$$\dot{g}(t) = g(t)\Omega(t), \quad g(0) = g_0, \tag{2}$$

where Ω takes values in $\mathrm{se}(3)$ and is assumed to be known for all $t \geq 0$.

Consider a set of given points $p_1, \ldots, p_N \in \mathbb{R}^3$, and let $y_j = [y_{j_1} \;\; y_{j_2} \;\; 1]^{\mathrm{T}} \in \mathbb{R}^3$, $j \in \mathscr{J}$ be the outputs of the dynamical system (2) given implicitly by

$$\alpha_j(t) y_j(t) = F(t)\Pi_0 g(t)\bar{p}_j, \tag{3}$$

where $\mathscr{J} \subseteq \{1, 2, \ldots, N\}$ is an index set that may depend on time, $\bar{p}_j \in \mathbb{R}^4$ is the homogeneous representation of p_j, the α_j's are unknown scalar continuous function of time satisfying $\alpha_j(t) > 0$ for every $t \geq 0$, $F \in \mathbb{R}^{3\times 3}$ is a known nonsingular matrix, and $\Pi_0 = \begin{bmatrix} I_3 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 4}$ is often referred to as the standard (or canonical) projection matrix [21]. We assume that the right-hand-side of (3) and $\Pi_0 g(t)\bar{p}_j$ are both bounded below and above, that is, for all $t \geq 0$,

$$m \leq \|F\Pi_0 g\bar{p}_j\|, \|\Pi_0 g\bar{p}_j\| \leq M \quad \text{with} \qquad 0 < m \leq M. \tag{4}$$

The problem addressed in this paper can be stated as follows.

Consider the continuous-time left-invariant dynamical system described by (2)-(3). *Let* $\hat{g} \in \mathrm{SE}(3)$ *be the estimate of the state* g *with a given initial estimate* $\hat{g}(0) = \hat{g}_0$. *Design a state observer for* (2)-(3) *that accepts as inputs the measured input* $\Omega(\tau)$ *and the output of the process* $y_j(\tau)$ *for every* $\tau \in [0,t)$, $j \in \mathscr{J}$, *and returns* $\hat{g}(t)$ *at time* t, *for every* $t \geq 0$. *The observer should satisfy some desired performance and robustness properties that will be mentioned later in the paper.*

Remark 1. System (2)–(3) arises for example when one needs to estimate the position and orientation of a robotic vehicle using measurements from an on-board monocular charged-coupled-device (CCD) camera. In that case, adopting the frontal pinhole camera model [21], the scalar α_j captures the unknown depth of a point p_j, and F is a matrix transformation that depends on the parameters of the camera such as the focal length, the scaling factors, and the center offsets. The assumption in (4) is very reasonable and only means that the image points are well defined in the sense that they live in some compact set. Notice that if for some point that assumption does not hold, then this only implies to take it out from the index set $\mathscr{J}$.

4 Observer design and convergence analysis

Consider the continuous-time left-invariant dynamical system (2)-(3). We propose the nonlinear observer

$$\dot{\hat{g}}(t) = \hat{g}(t)\Omega(t) + \zeta\,\Theta(\hat{g}(t), y(t))\,\hat{g}(t), \quad \hat{g}(0) = \hat{g}_0, \tag{5}$$

where $\hat{g} \in \mathrm{SE}(3)$ is the estimate of the state g, and $\Theta(\hat{g}, y) \in \mathrm{se}(3)$ is given by

$$\Theta(\hat{g}, y) \stackrel{\text{def}}{=} \begin{bmatrix} \Theta_R(\hat{g}, y) & \Theta_T(\hat{g}, y) \\ 0 & 0 \end{bmatrix}, \tag{6}$$

with

$$\Theta_R(\hat{g}, y) = \sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)} ((((\tilde{y}_j \times \Pi_0\hat{g}\bar{p}_j) \times \Pi_0\hat{g}\bar{p}_j) \times \Pi_0\hat{g}\bar{p}_j)\times), \tag{7}$$

$$\Theta_T(\hat{g}, y) = \sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)} ((-2\tilde{y}_j \times \Pi_0\hat{g}\bar{p}_j) \times \Pi_0\hat{g}\bar{p}_j), \tag{8}$$

$$\tilde{y}_j = F^{-1}\frac{y_j}{\|y_j\|}, \tag{9}$$

where

$$D(\hat{g}\bar{p}_j) \stackrel{\text{def}}{=} (\#\mathscr{J})\|\Pi_0\hat{g}\bar{p}_j\|^2(1 + \|\Pi_0\hat{g}\bar{p}_j\|), \tag{10}$$

$\#\mathscr{J}$ being the number of elements of $\mathscr{J}$, and $\zeta > 0$ is a tuning constant. Since $\alpha_j > 0$, the expressions (3) and (9) imply that

$$\tilde{y}_j = \frac{\Pi_0 g\bar{p}_j}{\|F\Pi_0 g\bar{p}_j\|}. \tag{11}$$

Remark 2. Notice that by defining $\hat{\Theta} \stackrel{\text{def}}{=} \hat{g}^{-1}\Theta\hat{g}$, system (5) can be rewritten as

$$\dot{\hat{g}} = \hat{g}(\Omega + \zeta\hat{\Theta}),$$

and, by a direct computation we can show that $\hat{\Theta} \in \mathrm{se}(3)$. Thus, like the dynamics of g in (2), also the dynamics of $\hat{g}$ is *left-invariant*. Moreover, if $\hat{g}(0) = g(0)$, then $\hat{\Theta} = 0$ for every $t \geq 0$, which means that the observer dynamics in that case is exactly the same as the original system.

Using the Lagrange identity for the cross product of vectors together with (11), the expression in (7) and (8) can be simplified respectively as

$$\Theta_R(\hat{g},y) = \sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)} \frac{\|\Pi_0\hat{g}\bar{p}_j\|^2}{\|F\Pi_0 g\bar{p}_j\|}((\Pi_0\hat{g}\bar{p}_j \times \Pi_0 g\bar{p}_j)\times), \tag{12}$$

$$\Theta_T(\hat{g},y) = \sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)} \frac{-2}{\|F\Pi_0 g\bar{p}_j\|}((\Pi_0 g\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j)\times \Pi_0\hat{g}\bar{p}_j). \tag{13}$$

Remark 3. Note that from (4), a lower bound for $D(\hat{g}\bar{p}_j)\|F\Pi_0 g\bar{p}_j\|$ is given by $m^2(m+1)m$, which implies that the observer is well defined.

4.1 The error dynamics

As in [10], we define the error $\eta(t) \stackrel{\text{def}}{=} \hat{g}(t)g^{-1}(t)$. Therefore, using (1), we may write

$$\dot{\eta} = \dot{\hat{g}}g^{-1} + \hat{g}\dot{g}^{-1} = \zeta\,\Theta(\hat{g},y)\eta, \quad \eta(0) = \hat{g}_0 g_0^{-1}, \tag{14}$$

where, taking into account that $g = \eta^{-1}\hat{g}$, $\Theta(\hat{g},y)$ can be rewritten as

$$\Theta(\hat{g},y) = \Theta(\eta) = \begin{bmatrix} \Theta_R(\eta) & \Theta_T(\eta) \\ 0 & 0 \end{bmatrix},$$

with

$$\Theta_R(\eta) = \sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)} \frac{\|\Pi_0\hat{g}\bar{p}_j\|^2}{\|F\Pi_0 g\bar{p}_j\|}((\Pi_0\hat{g}\bar{p}_j \times \Pi_0\eta^{-1}\hat{g}\bar{p}_j)\times), \tag{15}$$

$$\Theta_T(\eta) = \sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)} \frac{-2}{\|F\Pi_0 g\bar{p}_j\|}((\Pi_0\eta^{-1}\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j)\times \Pi_0\hat{g}\bar{p}_j). \tag{16}$$

Since a Lie group is a complex geometric object, it is a standard procedure to estimate results on a matrix Lie group G from results in the vector space which is its Lie algebra, here denoted by $\mathscr{L}$. We will adopt this procedure to analyze the error η and, later, prove convergence results. The Lie algebra $\mathscr{L}$ is the best linear approximation of G in the neighborhood of the identity I, and the exponential map exp, which sends elements in $\mathscr{L}$ to elements in G plays a crucial role in transferring data and results from one structure to the other. The exponential mapping is known to be bijective from a small neighborhood of $0 \in \mathscr{L}$ to a small neighborhood of the identity in G, and its inverse is denoted by log.

If η is sufficiently close to the identity, there is a representation $\eta = \exp(\varepsilon\xi)$, where $\varepsilon > 0$ and $\xi \in \mathrm{se}(3)$ satisfies $\|\xi\| = 1$. Since, $\exp(\varepsilon\xi) = I + \varepsilon\xi + O(\varepsilon^2)$, where $O(\varepsilon^2)$ represents the terms containing ε^k, for $k \geq 2$, for small ε, $I + \varepsilon\xi$ is a good approximation for η. In the rest of the paper, and for the sake of simplicity, we may use the alternative notation e^A instead of $\exp(A)$. We henceforth make the following assumption.

Assumption 4.1 *We assume that the error η is close enough to I_4, that is, $\eta \in \mathscr{N}_\varepsilon \stackrel{\text{def}}{=} \{v = \exp(\varepsilon\xi) \colon \xi \in \mathrm{se}(3)$ and $\|\xi\| = 1\}$, where $0 \leq \varepsilon < 1$.*

Remark 4. We may, without loss of generality assume that η is close to the identity. This is due to the fact that $x\mathscr{L} \sim \mathscr{L}$, for $x \in G$, is the best linear approximation of G in the neighborhood of x. So, if η is in the neighborhood of $x \in G$, then ηx^{-1} is close to the identity.

Using Lemma 1.7.3 of [22], which can be deduced from Lemma 3.4 in [24], we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\xi) = \left.\frac{u}{e^u - 1}\right|_{u = \mathrm{ad}_{\varepsilon\xi}} (\dot{\eta}\eta^{-1}),$$

where $\frac{u}{e^u - 1} = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m+1}(e^u - 1)^m$. Using (14), we have $\dot{\eta}\eta^{-1} = \zeta\Theta(\hat{g}, y)$ and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\xi) = \left.\frac{u}{e^u - 1}\right|_{u = \mathrm{ad}_{\varepsilon\xi}} (\zeta\Theta)$$

or, equivalently,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\xi) =& \zeta\Theta - \frac{1}{2}\mathrm{ad}_{\varepsilon\xi}\zeta\Theta - \frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{k!}\mathrm{ad}_{\varepsilon\xi}^k\zeta\Theta \\ &+ \left.\sum_{m=2}^{+\infty}\frac{(-1)^m}{m+1}(e^u - 1)^m\right|_{u = \mathrm{ad}_{\varepsilon\xi}} (\zeta\Theta). \end{aligned} \tag{17}$$

On the other hand

$$\exp(\varepsilon\xi) = I_4 + \varepsilon\xi + O(\varepsilon^2), \quad \exp(-\varepsilon\xi) = I_4 - \varepsilon\xi + O(\varepsilon^2)$$

and, using the fact that $\Theta(g,y) = \Theta(I_4) = 0$ and noticing that Θ is defined in the linear space of 4×4 real matrices, containing both SE(3) and se(3), we have

$$\Theta_R(\eta) = \sum_{j\in\mathscr{J}} -\frac{1}{D(\hat{g}\bar{p}_j)} \frac{\|\Pi_0\hat{g}\bar{p}_j\|^2((\Pi_0\hat{g}\bar{p}_j \times \Pi_0\varepsilon\xi\hat{g}\bar{p}_j)\times)}{\|F\Pi_0 g\bar{p}_j\|} + O(\varepsilon^2), \tag{18}$$

$$\Theta_T(\eta) = \sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)} \frac{2((\Pi_0\varepsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j)\times \Pi_0\hat{g}\bar{p}_j)}{\|F\Pi_0 g\bar{p}_j\|} + O(\varepsilon^2). \tag{19}$$

From (15) - (17) with $\Theta(I_4) = 0$, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\xi) = \zeta\bar{\Theta}(\varepsilon\xi) = \zeta \begin{bmatrix} \bar{\Theta}_R(\varepsilon\xi) & \bar{\Theta}_T(\varepsilon\xi) \\ 0 & 0 \end{bmatrix} + O(\varepsilon^2),$$

where

$$\bar{\Theta}_R(\varepsilon\xi) = \sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)} \frac{\|\Pi_0\hat{g}\bar{p}_j\|^2}{\|F\Pi_0 g\bar{p}_j\|}((\Pi_0\varepsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j)\times), \tag{20}$$

$$\bar{\Theta}_T(\varepsilon\xi) = \sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)} \frac{-2}{\|F\Pi_0 g\bar{p}_j\|}((\Pi_0\hat{g}\bar{p}_j \times \Pi_0\varepsilon\xi\hat{g}\bar{p}_j)\times \Pi_0\hat{g}\bar{p}_j). \tag{21}$$

Up to an approximation of the order ε^2, we obtain that $\varepsilon\xi$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\xi) = \zeta \begin{bmatrix} \bar{\Theta}_R(\varepsilon\xi) & \bar{\Theta}_T(\varepsilon\xi) \\ 0 & 0 \end{bmatrix}.$$

We have the following result.

Proposition 1. *Up to an approximation of the order ε^2, the following result holds.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon\xi\|^2 = -4\zeta \sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)} \frac{1}{\|F\Pi_0 g\bar{p}_j\|} \|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2. \tag{22}$$

Proof. From the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon\xi\|^2 = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\xi), (\varepsilon\xi) \right\rangle + \left\langle (\varepsilon\xi), \frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\xi) \right\rangle = 2\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\xi), (\varepsilon\xi) \right\rangle,$$

and

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\xi), (\varepsilon\xi) \right\rangle = \zeta \mathrm{Tr}\left(\left(\bar{\Theta}(\varepsilon\xi)\right)^{\mathrm{T}} (\varepsilon\xi) \right),$$

it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon\xi\|^2 = 2\zeta \mathrm{Tr}\left(\left(\bar{\Theta}(\varepsilon\xi)\right)^{\mathrm{T}}(\varepsilon\xi)\right)$. Using the fact that $\mathrm{Tr}\left((u_1\times)^{\mathrm{T}}(u_2\times)\right) = 2u_1^{\mathrm{T}}u_2$ for every $u_1, u_2 \in \mathbb{R}^3$, up to an approximation of order ε^3, we have

$$\mathrm{Tr}\left(\left(\bar{\Theta}(\varepsilon\xi)\right)^{\mathrm{T}}(\varepsilon\xi)\right) = 2\sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)}\frac{1}{\|F\Pi_0 g\bar{p}_j\|}\Big\{\|\Pi_0\hat{g}\bar{p}_j\|^2(\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}\varepsilon\bar{\xi} \\ -((\Pi_0\hat{g}\bar{p}_j\times\Pi_0\varepsilon\xi\hat{g}\bar{p}_j)\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}\varepsilon\xi_T\Big\}. \tag{23}$$

From the relation $(a\times b)^{\mathrm{T}}c = \det[a\ b\ c]$, where $[a\ b\ c]$ stays for the matrix whose first, second, and third columns are respectively the vectors $a, b, c \in \mathbb{R}^3$ and using the skew-symmetry of the determinant function $\det[\cdot\ \cdot\ \cdot]$, we obtain

$$\begin{aligned}
&\|\Pi_0\hat{g}\bar{p}_j\|^2(\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}\varepsilon\bar{\xi}\\
&\qquad = -(((\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j)\times\Pi_0 g\bar{p}_j)\times\Pi_0 g\bar{p}_j)^{\mathrm{T}}\varepsilon\bar{\xi}\\
&\qquad = -((\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j)\times\Pi_0 g\bar{p}_j)^{\mathrm{T}}(\Pi_0 g\bar{p}_j\times\varepsilon\bar{\xi})\\
&\qquad = -(\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}(\Pi_0 g\bar{p}_j\times(\Pi_0 g\bar{p}_j\times\varepsilon\bar{\xi}))\\
&\qquad = -(\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}((\varepsilon\bar{\xi}\times\Pi_0 g\bar{p}_j)\times\Pi_0 g\bar{p}_j)
\end{aligned}$$

and

$$\begin{aligned}
-((\Pi_0\hat{g}\bar{p}_j\times\Pi_0\varepsilon\xi\hat{g}\bar{p}_j)\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}\varepsilon\xi_T &= -(\Pi_0\hat{g}\bar{p}_j\times\Pi_0\varepsilon\xi\hat{g}\bar{p}_j)^{\mathrm{T}}(\Pi_0\hat{g}\bar{p}_j\times\varepsilon\xi_T)\\
&= -(\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}(\varepsilon\xi_T\times\Pi_0\hat{g}\bar{p}_j).
\end{aligned}$$

Therefore, $\|\Pi_0\hat{g}\bar{p}_j\|^2(\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}\varepsilon\bar{\xi} - ((\Pi_0\hat{g}\bar{p}_j\times\Pi_0\varepsilon\xi\hat{g}\bar{p}_j)\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}\varepsilon\xi_T = -(\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}((\varepsilon\bar{\xi}\times\Pi_0 g\bar{p}_j)\times\Pi_0 g\bar{p}_j) - (\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}(\varepsilon\xi_T\times\Pi_0\hat{g}\bar{p}_j)$. Note that the right-hand-side is $-(\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}\left((\varepsilon\bar{\xi}\times\Pi_0 g\bar{p}_j+\varepsilon\xi_T)\times\Pi_0\hat{g}\bar{p}_j\right) = -\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j\|^2$ by noting that, $\varepsilon\bar{\xi}\times\Pi_0\hat{g}\bar{p}_j+\varepsilon\xi_T = \Pi_0\varepsilon\xi\hat{g}\bar{p}_j$.

Hence (23) reduces to

$$\mathrm{Tr}\left(\left(\bar{\Theta}(\varepsilon\xi)\right)^{\mathrm{T}}(\varepsilon\xi)\right) = -2\sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)}\frac{1}{\|F\Pi_0 g\bar{p}_j\|}\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j\|^2,$$

and, consequently

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon\xi\|^2 = -4\zeta\sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)}\frac{1}{\|F\Pi_0 g\bar{p}_j\|}\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j\|^2.$$

□

4.2 Exponential convergence

In this section we show under suitable assumptions that the estimation error converges exponentially to zero as $t \to \infty$. Let M denote the upper bound for $\|F\Pi_0 g\bar{p}_j\|$ for all j.

We will recall Gronwall's Lemma [27, Ch. III, 1.1.3] that is required to prove our next result.

Lemma 2 (Gronwall inequality). *Let $g, h, y, \frac{\mathrm{d}y}{\mathrm{d}t}$ be locally integrable functions satisfying*

$$\frac{\mathrm{d}y}{\mathrm{d}t} \le gy + h \quad \textit{for } t \ge t_0. \tag{24}$$

Then, for all $t \ge t_0$,

$$y(t) \le y(t_0)\exp\left(\int_{t_0}^{t} g(\tau)\,\mathrm{d}\tau\right) + \int_{t_0}^{t} h(s)\exp\left(-\int_{t}^{s} g(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Our next result is as follows.

Theorem 1. *Let $\bar{T} \in [0, +\infty]$ and $\lambda > 0$ be such that*

$$\sum_{j\in\mathscr{J}} \frac{\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2}{(\#\mathscr{J})\|\Pi_0\hat{g}\bar{p}_j\|^2(1+\|\Pi_0\hat{g}\bar{p}_j\|)} \ge \lambda\|\varepsilon\xi\|^2$$

on the time interval $[0, \bar{T}[$. Then, for every $t \in [0, \bar{T}[$,

$$\|\varepsilon\xi(t)\|^2 \le \|\varepsilon\xi(0)\|^2 e^{-4\zeta\lambda M^{-1}t},$$

where M is an upper bound for $\|F\Pi_0 g\bar{p}_j\|$ for all j. In particular, if $\bar{T} = +\infty$, then $\|\varepsilon\xi(t)\|^2$ converges exponentially fast to zero as $t \to \infty$.

Proof. Under the hypothesis, (22) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon\xi\|^2 \le -4\zeta\lambda M^{-1}\|\varepsilon\xi\|^2, \tag{25}$$

for all $t \in [0, \bar{T}]$. The result follows from Gronwall's inequality (Lem. 2). □

Note that the rate of convergence can be improved by tuning $\zeta > 0$, that is, the rate of convergence increases with ζ. Next we prove the following result.

Theorem 2. *Let $\bar{T} \in [0, +\infty]$. Suppose there exists $T > 0$ such that, for every $t \ge 0$, with $t + T \le \bar{T}$,*

$$\frac{1}{T}\int_{t}^{t+T} \sum_{j\in\mathscr{J}} \frac{\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2}{(\#\mathscr{J})\|\Pi_0\hat{g}\bar{p}_j\|^2(1+\|\Pi_0\hat{g}\bar{p}_j\|)} \frac{1}{\|\varepsilon\xi\|^2}\mathrm{d}\tau \ge \lambda.$$

Then, for $n \in \mathbb{N}$ with $t \ge 0,\ t + nT \le \bar{T}$,

$$\|\varepsilon\xi(t+nT)\|^2 \leq \|\varepsilon\xi(t)\|^2 e^{-4\zeta\lambda M^{-1}nT}.$$

In particular, if $\bar{T} = +\infty$, then $\|\varepsilon\xi(t)\|^2$ exponentially fast to zero as $t \to \infty$.

Proof. Multiplying both the sides of (22) by $(T\|\varepsilon\xi\|^2)^{-1}$, we have

$$\frac{1}{T}\frac{1}{\|\varepsilon\xi\|^2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon\xi\|^2 = \frac{1}{T}\frac{1}{\|\varepsilon\xi\|^2}\sum_{j\in\mathscr{J}}\frac{-4\zeta}{D(\hat{g}\bar{p}_j)}\frac{1}{\|F\Pi_0 g\bar{p}_j\|}\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2,$$

or,

$$\frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}t}\log(\|\varepsilon\xi\|^2) \leq \frac{1}{T}\frac{1}{\|\varepsilon\xi\|^2}\sum_{j\in\mathscr{J}}\frac{-4\zeta M^{-1}}{D(\hat{g}\bar{p}_j)}\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2.$$

Since the statement is trivial for $n = 0$, we consider the case $n \geq 1$. Integrating on the interval $[t+(n-1)T, t+nT]$, we obtain

$$\int_{t+(n-1)T}^{t+nT}\frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}t}\log(\|\varepsilon\xi\|^2)\mathrm{d}\tau \\ \leq \int_{t+(n-1)T}^{t+nT}\frac{1}{T}\frac{1}{\|\varepsilon\xi\|^2}\sum_{j\in\mathscr{J}}\frac{-4\zeta M^{-1}}{D(\hat{g}\bar{p}_j)}\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2\mathrm{d}\tau.$$

Note that

$$\int_{t+(n-1)T}^{t+nT}\frac{\mathrm{d}}{\mathrm{d}t}\log(\|\varepsilon\xi(\tau)\|^2)\mathrm{d}\tau = \log(\|\varepsilon\xi(t+nT)\|^2) - \log(\|\varepsilon\xi(t+(n-1)T)\|^2),$$

and the properties of logarithm imply that

$$\int_{t+(n-1)T}^{t+nT}\frac{\mathrm{d}}{\mathrm{d}t}\log(\|\varepsilon\xi(\tau)\|^2)\mathrm{d}\tau = \log\left(\frac{\|\varepsilon\xi(t+nT)\|^2}{\|\varepsilon\xi(t+(n-1)T)\|^2}\right).$$

Hence using the assumption, we have $\frac{1}{T}\log\left(\frac{\|\varepsilon\xi(t+nT)\|^2}{\|\varepsilon\xi(t+(n-1)T)\|^2}\right) \leq -4\zeta M^{-1}\lambda$ or, equivalently,

$$\frac{\|\varepsilon\xi(t+nT)\|^2}{\|\varepsilon\xi(t+(n-1)T)\|^2} \leq e^{-4\zeta M^{-1}\lambda T},$$

from which we derive $\frac{\|\varepsilon\xi(t+nT)\|^2}{\|\varepsilon\xi(t)\|^2} \leq e^{-4\zeta M^{-1}\lambda nT}$.

Finally, if $\bar{T} = +\infty$, we have $\|\varepsilon\xi(t)\|^2 \leq \max_{s\in[0,T]}\|\varepsilon\xi(s)\|^2\, e^{-4\zeta M^{-1}\lambda[t/T]}$, where $[t/T]$ denotes the largest natural number contained in the quotient t/T, that is, $[t/T] \leq t/T < [t/T]+1$. $[t/T] \geq 0$ is a non-negative integer number. $\square$

Remark 5. Theorem 1 may be seen as the "limit" of Theorem 2 when T goes to 0.

5 Robustness analysis of the observer

In this section we investigate the effect of disturbance and noise on the estimation error. We now consider the process model (2)-(3) subjected to disturbances and noise as follows:

$$\dot{g}(t) = g(t)\left(\Omega(t)+w(t)\right), \quad g(0) = g_0, \tag{26}$$

$$y_j(t) = \tilde{y}_j(t)+v_j(t), \tag{27}$$

where $w \in \mathrm{se}(3)$ is the disturbance, $\tilde{y}_j = [\tilde{y}_{j1}\ \tilde{y}_{j2}\ \ 1]^{\mathrm{T}} \in \mathbb{R}^3$ is the real output defined implicitly by $\alpha_j\tilde{y}_j = FHg\bar{p}_j$ with $0 < \kappa \leq \alpha_j$, $y_j = [y_{j1}\ y_{j2}\ \ 1]^{\mathrm{T}} \in \mathbb{R}^3$ is the measured output with noise $v_j = [v_{j1}\ v_{j2}\ \ 0]^{\mathrm{T}} \in \mathbb{R}^3$. Further, the disturbance and noise signals are assumed to be deterministic but unknown. Note that (27) is equivalent to $y_j = \alpha_j^{-1}\left(F\Pi_0 g\bar{p}_j+\alpha_j v_j\right)$. Define $M_p \stackrel{\text{def}}{=} \sup\limits_{\substack{t\in[0,t_1]\\ j\in\mathscr{J}}} \|F\Pi_0 g\bar{p}_j+\alpha_j v_j\|$.

Let $|F^{-1}|$ denotes a bound for the functional norm of $F^{-1}(t)$ defined by $|F^{-1}(t)| \stackrel{\text{def}}{=} \sup\{F^{-1}(t)u\colon u \in \mathbb{R}^3 \text{and } \|u\| = 1\}$, that is, we assume $F^{-1}(t)$ is bounded in the time interval $[0,t_1]$ we are considering the estimator in. Define $M_v \stackrel{\text{def}}{=} \sup\limits_{\substack{t\in[0,t_1]\\ j\in\mathscr{J}}} \|v_j(t)\|$ and $M_w \stackrel{\text{def}}{=} \sup\limits_{t\in[0,t_1]} \|w(t)\|$, that is, M_v and M_w respectively denote the upper bounds for the noise $\|v_j\|$ and disturbance $\|w\|$, we suppose to exist, in the same time interval $[0,t_1]$.

We consider the same observer claimed in (5), which can be rewritten as

$$\dot{\hat{g}}(t) = \hat{g}(t)\Omega(t)+\zeta\Theta\left(\hat{g}(t),y(t)\right)\hat{g}(t), \quad \hat{g}(0) = \hat{g}_0, \tag{28}$$

where $\Theta(\hat{g},y)$ is given by

$$\Theta(\hat{g},y) = \bar{\Theta}(\hat{g},y)+\tilde{\Theta}(\hat{g},y),$$

with

$$\bar{\Theta}(\hat{g},y) = \begin{bmatrix}\bar{\Theta}_R(\hat{g},y) & \bar{\Theta}_T(\hat{g},y)\\ 0 & 0\end{bmatrix} \text{ and } \tilde{\Theta}(\hat{g},y) = \begin{bmatrix}\tilde{\Theta}_R(\hat{g},y) & \tilde{\Theta}_T(\hat{g},y)\\ 0 & 0\end{bmatrix},$$

where

$$\bar{\Theta}_R(\hat{g},y) = \sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)} \frac{\|\Pi_0\hat{g}\bar{p}_j\|^2((\Pi_0\hat{g}\bar{p}_j \times \Pi_0 g\bar{p}_j)\times)}{\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|},$$
$$\bar{\Theta}_T(\hat{g},y) = \sum_{j\in\mathscr{J}} \frac{-2}{D(\hat{g}\bar{p}_j)} \frac{((\Pi_0 g\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j)\times \Pi_0\hat{g}\bar{p}_j)}{\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|},$$
$$\tilde{\Theta}_R(\hat{g},y) = \sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)} \frac{\|\Pi_0\hat{g}\bar{p}_j\|^2((\Pi_0\hat{g}\bar{p}_j \times F^{-1}\alpha_j v_j)\times)}{\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|},$$
$$\tilde{\Theta}_T(\hat{g},y) = \sum_{j\in\mathscr{J}} \frac{-2}{D(\hat{g}\bar{p}_j)} \frac{((F^{-1}\alpha_j v_j \times \Pi_0\hat{g}\bar{p}_j)\times \Pi_0\hat{g}\bar{p}_j)}{\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|}.$$

Note that both $\bar{\Theta}$ and $\tilde{\Theta}$ depend on noise v_j. Again, we define the error $\eta(t) \stackrel{\text{def}}{=} \hat{g}(t)g^{-1}(t)$. Therefore, using (1) yields

$$\dot{\eta} = \dot{\hat{g}}g^{-1} + \hat{g}\dot{g}^{-1} = \zeta\Theta(\eta)\eta - \hat{g}w\hat{g}^{-1}\eta, \quad \eta(0) = \hat{g}_0 g_0^{-1}, \tag{29}$$

where, by using $g = \eta^{-1}\hat{g}$ we can rewrite $\Theta(\eta)$ as

$$\Theta(\eta) = \bar{\Theta}(\eta) + \tilde{\Theta} = \begin{bmatrix} \bar{\Theta}_R(\eta) & \bar{\Theta}_T(\eta) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \tilde{\Theta}_R & \tilde{\Theta}_T \\ 0 & 0 \end{bmatrix},$$

with

$$\bar{\Theta}_R(\eta) = \sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)} \frac{\|\Pi_0\hat{g}\bar{p}_j\|^2((\Pi_0\hat{g}\bar{p}_j \times \Pi_0\eta^{-1}\hat{g}\bar{p}_j)\times)}{\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|}, \tag{30}$$
$$\bar{\Theta}_T(\eta) = \sum_{j\in\mathscr{J}} \frac{-2}{D(\hat{g}\bar{p}_j)} \frac{((\Pi_0\eta^{-1}\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j)\times \Pi_0\hat{g}\bar{p}_j)}{\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|}, \tag{31}$$
$$\tilde{\Theta}_R = \sum_{j\in\mathscr{J}} \frac{1}{D(\hat{g}\bar{p}_j)} \frac{\|\Pi_0\hat{g}\bar{p}_j\|^2((\Pi_0\hat{g}\bar{p}_j \times F^{-1}\alpha_j v_j)\times)}{\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|}, \tag{32}$$
$$\tilde{\Theta}_T = \sum_{j\in\mathscr{J}} \frac{-2}{D(\hat{g}\bar{p}_j)} \frac{((F^{-1}\alpha_j v_j \times \Pi_0\hat{g}\bar{p}_j)\times \Pi_0\hat{g}\bar{p}_j)}{\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|}. \tag{33}$$

Remark 6. Note that $\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|$ is equal to $\alpha_j\|\alpha_j^{-1}F\Pi_0 g\bar{p}_j + v_j\| = \alpha_j\|y_j\| \geq \alpha_j \geq \kappa$ and from (4), it follows that $m^2(m+1)\kappa$ is a lower bound for $D(\hat{g}\bar{p}_j)\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|$. From this we conclude that the observer is well defined.

Note that $\eta = \exp(\varepsilon\xi) = I_4 + \varepsilon\xi + O(\varepsilon^2)$. Using Lemma 1.7.3 of [22], we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\xi) = \dot{\eta}\eta^{-1} - \frac{1}{2}[\varepsilon\xi, \dot{\eta}\eta^{-1}] + O(\varepsilon^2).$$

From (29), we have $\dot{\eta}\eta^{-1} = \zeta(\bar{\Theta}(\eta) + \tilde{\Theta}) - \hat{g}w\hat{g}^{-1}$ and hence the above equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\xi) = \zeta\bar{\Theta}(\varepsilon\xi) + \zeta\tilde{\Theta} - \hat{g}w\hat{g}^{-1} - \frac{1}{2}[\varepsilon\xi,\, \zeta\tilde{\Theta} - \hat{g}w\hat{g}^{-1}] + O(\varepsilon^2).$$

Up to an approximation of the order ε^2, we have that $\varepsilon\xi$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\xi) = \zeta\bar{\Theta}(\varepsilon\xi) + \zeta\tilde{\Theta} - \hat{g}w\hat{g}^{-1} - \frac{1}{2}[\varepsilon\xi,\, \zeta\tilde{\Theta} - \hat{g}w\hat{g}^{-1}], \tag{34}$$

and, multiplying by $\varepsilon\xi$ yields,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon\xi\|^2 =& 2\langle\zeta\bar{\Theta}(\varepsilon\xi),\, \varepsilon\xi\rangle + 2\langle\zeta\tilde{\Theta},\, \varepsilon\xi\rangle - 2\langle\hat{g}w\hat{g}^{-1},\, \varepsilon\xi\rangle \\ &- \langle[\varepsilon\xi,\, \zeta\tilde{\Theta} - \hat{g}w\hat{g}^{-1}],\, \varepsilon\xi\rangle.\end{aligned} \tag{35}$$

To estimate a bound for the variation $\frac{\mathrm{d}}{\mathrm{dt}}\|\varepsilon\xi\|^2$, we may start by estimate a bound for the individual terms on the right-hand-side of (35), which are given by the following result.

Proposition 2. *The following statements hold.*

i) $\langle\bar{\Theta}(\varepsilon\xi), \varepsilon\xi\rangle = -\sum_{j\in\mathscr{J}} \frac{2}{D(\hat{g}\bar{p}_j)} \frac{\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2}{\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|}$.

ii) $\left\langle\tilde{\Theta}, \varepsilon\xi\right\rangle \leq \sum_{j\in\mathscr{J}} \frac{2|F^{-1}|M_v}{(\#\mathscr{J})}\|\varepsilon\xi\|$.

iii) $\langle\hat{g}w\hat{g}^{-1},\, \varepsilon\xi\rangle \leq \|w\|\|\varepsilon\xi\|$.

iv) $\langle[\varepsilon\xi,\, \zeta\tilde{\Theta} - \hat{g}w\hat{g}^{-1}],\, \varepsilon\xi\rangle \leq \left(2\zeta\sum_{j\in\mathscr{J}} \frac{|F^{-1}|M_v}{(\#\mathscr{J})(1+\|\Pi_0\hat{g}\bar{p}_j\|)} + M_w\right)\|\varepsilon\xi\|^2$.

Proof. In the following $\begin{bmatrix}\xi_R & \xi_T\\ 0 & 0\end{bmatrix} \stackrel{\text{def}}{=} \xi$ with $(\bar{\xi}\times) \stackrel{\text{def}}{=} \xi_R$.

i) The result follows from the noise free case, proceeding as in the proof of Proposition 1.

ii) First, note that

$$(\tilde{\Theta})^{\mathrm{T}}\varepsilon\xi = \begin{bmatrix}\tilde{\Theta}_R & \tilde{\Theta}_T\\ 0 & 0\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\varepsilon\xi_R & \varepsilon\xi_T\\ 0 & 0\end{bmatrix} = \begin{bmatrix}\left(\tilde{\Theta}_R\right)^{\mathrm{T}}\varepsilon\xi_R & \left(\tilde{\Theta}_R\right)^{\mathrm{T}}\varepsilon\xi_T\\ \left(\tilde{\Theta}_T\right)^{\mathrm{T}}\varepsilon\xi_R & \left(\tilde{\Theta}_T\right)^{\mathrm{T}}\varepsilon\xi_T\end{bmatrix}.$$

Then $\left\langle\tilde{\Theta}, \varepsilon\xi\right\rangle = \mathrm{Tr}\left(\tilde{\Theta}^{\mathrm{T}}\varepsilon\xi\right) = \mathrm{Tr}\left(\left(\tilde{\Theta}_R\right)^{\mathrm{T}}\varepsilon\xi_R\right) + \left(\tilde{\Theta}_T\right)^{\mathrm{T}}\varepsilon\xi_T$. Now, we have

$$\mathrm{Tr}\left(\left(\tilde{\Theta}_R\right)^{\mathrm{T}}\varepsilon\xi_R\right) = \sum_{j\in\mathscr{J}} \frac{2}{D(\hat{g}\bar{p}_j)} \frac{\|\Pi_0\hat{g}\bar{p}_j\|^2(\Pi_0\hat{g}\bar{p}_j \times F^{-1}\alpha_j v_j)^{\mathrm{T}}\varepsilon\overline{\xi}}{\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|}, \quad \text{and}$$

$$(\tilde{\Theta}_T)^{\mathrm{T}}\varepsilon\xi_T = \sum_{j\in\mathscr{J}} \frac{-2\alpha_j}{D(\hat{g}\bar{p}_j)} \frac{((F^{-1}v_j \times \Pi_0\hat{g}\bar{p}_j)\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}\varepsilon\xi_T}{\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|}.$$

Proceeding as in the proof of Proposition 1, we can arrive to

$$\langle\tilde{\Theta},\varepsilon\xi\rangle = \sum_{j\in\mathscr{J}} \frac{2\alpha_j}{D(\hat{g}\bar{p}_j)} \frac{(F^{-1}v_j\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}(\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j)}{\|F\Pi_0 g\bar{p}_j+\alpha_j v_j\|},$$

$$\langle\tilde{\Theta},\varepsilon\xi\rangle \le \sum_{j\in\mathscr{J}} \frac{2\alpha_j}{D(\hat{g}\bar{p}_j)} \frac{\|F^{-1}v_j\times\Pi_0\hat{g}\bar{p}_j\|\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j\|}{\|F\Pi_0 g\bar{p}_j+\alpha_j v_j\|}.$$

Since $\|u_1\times u_2\|\le\|u_1\|\|u_2\|$ for every $u_1,u_2\in\mathbb{R}^3$, we have

$$\langle\tilde{\Theta},\varepsilon\xi\rangle \le \sum_{j\in\mathscr{J}} \frac{2}{(\#\mathscr{J})(1+\|\Pi_0\hat{g}\bar{p}_j\|)}\left(\frac{\alpha_j}{\|F\Pi_0 g\bar{p}_j+\alpha_j v_j\|}\right)\|F^{-1}v_j\|\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\|.$$

Further, note that $\|y_j\|^{-1} = \dfrac{\alpha_j}{\|F\Pi_0 g\bar{p}_j+\alpha_j v_j\|} \le 1$ and $\|F^{-1}v_j\|\le|F^{-1}|M_v$. Hence

$$\langle\tilde{\Theta},\varepsilon\xi\rangle \le \sum_{j\in\mathscr{J}} \frac{2|F^{-1}|M_v}{(\#\mathscr{J})(1+\|\Pi_0\hat{g}\bar{p}_j\|)}\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\|.$$

Recall that $\Pi_0\varepsilon\xi\hat{g}\bar{p}_j = \varepsilon\bar{\xi}\times\Pi_0\hat{g}\bar{p}_j+\varepsilon\xi_T$ and by triangle inequality it follows that $\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\|\le\|\varepsilon\bar{\xi}\times\Pi_0\hat{g}\bar{p}_j\|+\|\varepsilon\xi_T\|$. In other words, $\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\|\le\|\varepsilon\xi\|(1+\|\Pi_0\hat{g}\bar{p}_j\|)$ by noting that $\|\varepsilon\bar{\xi}\times\Pi_0\hat{g}\bar{p}_j\|\le\|\varepsilon\bar{\xi}\|\|\Pi_0\hat{g}\bar{p}_j\|$, $\|\varepsilon\bar{\xi}\|\le\|\varepsilon\xi\|$, and $\|\varepsilon\xi_T\|\le\|\varepsilon\xi\|$. Hence

$$\langle\tilde{\Theta},\varepsilon\xi\rangle \le \sum_{j\in\mathscr{J}} \frac{2|F^{-1}|M_v}{(\#\mathscr{J})}\|\varepsilon\xi\|.$$

iii) First note that, $\langle\hat{g}w\hat{g}^{-1},\varepsilon\xi\rangle\le\|\hat{g}w\hat{g}^{-1}\|\|\varepsilon\xi\|$. By the definition, we have $\|\hat{g}w\hat{g}^{-1}\| = \sqrt{\mathrm{Tr}(\hat{g}w^{\mathrm{T}}\hat{g}^{-1}\hat{g}w\hat{g}^{-1})}$. Since $\hat{g}^{-1}\hat{g}=I_4$, we have $\|\hat{g}w\hat{g}^{-1}\| = \sqrt{\mathrm{Tr}(\hat{g}w^{\mathrm{T}}w\hat{g}^{-1})}$. Recall that, the trace of a matrix is invariant under similarity transformation, that is, $\mathrm{Tr}(BAB^{-1})=\mathrm{Tr}(A)$ for every $A\in\mathbb{R}^{n\times n}$ [23, Ch. V, 7]. Thus, we conclude that $\|\hat{g}^{-1}w\hat{g}\| = \sqrt{\mathrm{Tr}(w^{\mathrm{T}}w)} = \|w\|$. Hence $\langle\hat{g}w\hat{g}^{-1},\varepsilon\xi\rangle\le\|w\|\|\varepsilon\xi\|$.

iv) For simplicity, we define $Z \overset{\mathrm{def}}{=} \begin{bmatrix} Z_R & Z_T \\ 0 & 0\end{bmatrix} \overset{\mathrm{def}}{=} \zeta\tilde{\Theta} - \hat{g}w\hat{g}^{-1}$. We find $\langle[\varepsilon\xi,Z],\varepsilon\xi\rangle = \mathrm{Tr}(\varepsilon\xi^{\mathrm{T}}(Z\varepsilon\xi-\varepsilon\xi Z))$. Note that, for every $A,B,C\in\mathrm{se}(3)$, we have

$$A^{\mathrm{T}}BC = \begin{bmatrix} A_R^{\mathrm{T}} & 0 \\ A_T^{\mathrm{T}} & 0\end{bmatrix}\begin{bmatrix} B_RC_R & B_RC_T \\ 0 & 0\end{bmatrix} = \begin{bmatrix} A_R^{\mathrm{T}}B_RC_R & A_R^{\mathrm{T}}B_RC_T \\ A_T^{\mathrm{T}}B_RC_R & A_T^{\mathrm{T}}B_RC_T\end{bmatrix}$$

and $\mathrm{Tr}(A^{\mathrm{T}}BC) = -\mathrm{Tr}(A_RB_RC_R) + A_T^{\mathrm{T}}B_RC_T$. Hence $\langle[\varepsilon\xi,Z],\varepsilon\xi\rangle = \big(-\mathrm{Tr}(\varepsilon\xi_RZ_R\varepsilon\xi_R)+(\varepsilon\xi_T)^{\mathrm{T}}Z_R\varepsilon\xi_T\big) - \big(-\mathrm{Tr}(\varepsilon\xi_R\varepsilon\xi_RZ_R)+(\varepsilon\xi_T)^{\mathrm{T}}\varepsilon\xi_RZ_T\big)$. It is easy to check that $(\varepsilon\xi_RZ_R\varepsilon\xi_R)^{\mathrm{T}} = -(\varepsilon\xi_RZ_R\varepsilon\xi_R)$, and hence $\varepsilon\xi_RZ_R\varepsilon\xi_R$ is skew-symmetric, which implies that its trace is zero. On the other hand, the term $(\varepsilon\xi_T)^{\mathrm{T}}Z_R\varepsilon\xi_T = (\varepsilon\xi_T)^{\mathrm{T}}(\bar{z}\times\varepsilon\xi_T)$ vanishes as well, where $(\bar{z}\times) \stackrel{\text{def}}{=} Z_R$. The term $\mathrm{Tr}(\varepsilon\xi_R\varepsilon\xi_R Z_R)$ vanishes because for positive semi-definite matrices A and B, we have $0 \le \mathrm{Tr}(AB) \le \mathrm{Tr}(A)\mathrm{Tr}(B)$ [1, pg. 329], so that

$$0 \le \mathrm{Tr}(-\varepsilon\xi_R\varepsilon\xi_R Z_R) \le \mathrm{Tr}(-\varepsilon\xi_R\varepsilon\xi_R)\mathrm{Tr}(Z_R) = 0.$$

Notice that, for a given vector $u \in \mathbb{R}^3$, $u^{\mathrm{T}}(-\varepsilon\xi_R)\varepsilon\xi_R u = (\varepsilon\xi_R u)^{\mathrm{T}}(\varepsilon\xi_R u) = \|\varepsilon\xi_R u\|^2 \ge 0$ and $u^{\mathrm{T}}Z_R u = 0$. Therefore $\langle[\varepsilon\xi, Z], \varepsilon\xi\rangle = -(\varepsilon\xi_T)^{\mathrm{T}}\varepsilon\xi_R Z_T = Z_T^{\mathrm{T}}(\varepsilon\overline{\xi}\times\varepsilon\xi_T)$.

Note the following facts:

a) $\|y_j\|^{-1} = \dfrac{\alpha_j}{\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|} \le 1.$

b) Using (33) together with a), we have

$$\zeta\tilde{\Theta}_T^{\mathrm{T}}(\varepsilon\overline{\xi}\times\varepsilon\xi_T) \le 2\zeta\sum_{j\in\mathscr{J}}\left|\frac{((F^{-1}v_j\times\Pi_0\hat{g}\bar{p}_j)\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}(\varepsilon\overline{\xi}\times\varepsilon\xi_T)}{D(\hat{g}\bar{p}_j)}\right|.$$

From $|((F^{-1}v_j\times\Pi_0\hat{g}\bar{p}_j)\times\Pi_0\hat{g}\bar{p}_j)^{\mathrm{T}}(\varepsilon\overline{\xi}\times\varepsilon\xi_T)| \le |F^{-1}|M_v\|\Pi_0\hat{g}\bar{p}_j\|^2\|\varepsilon\overline{\xi}\times\varepsilon\xi_T\|$ we obtain

$$\zeta\tilde{\Theta}_T^{\mathrm{T}}(\varepsilon\overline{\xi}\times\varepsilon\xi_T) \le 2\zeta\sum_{j\in\mathscr{J}}\frac{|F^{-1}|M_v\|\varepsilon\xi\|^2}{(\#\mathscr{J})(1+\|\Pi_0 \hat{g}\bar{p}_j\|)}.$$

c) It can be easily shown that $(\hat{g}w\hat{g}^{-1})_T^{\mathrm{T}}(\varepsilon\overline{\xi}\times\varepsilon\xi_T) \le \|w\|\|\varepsilon\xi\|^2$ or, equivalently, $(\hat{g}w\hat{g}^{-1})_T^{\mathrm{T}}(\varepsilon\overline{\xi}\times\varepsilon\xi_T) \le M_w\|\varepsilon\xi\|^2$.

Using b) and c) above, we conclude that

$$\langle[\varepsilon\xi, \zeta\tilde{\Theta} - \hat{g}w\hat{g}^{-1}], \varepsilon\xi\rangle \le \left(2\zeta\sum_{j\in\mathscr{J}}\frac{|F^{-1}|M_v}{(\#\mathscr{J})(1+\|\Pi_0\hat{g}\bar{p}_j\|)} + M_w\right)\|\varepsilon\xi\|^2.$$

□

The following result follows immediately from Proposition 2.

Proposition 3. *The following statement holds.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon\xi\|^2 \le -\sum_{j\in\mathscr{J}}\frac{4\zeta}{D(\hat{g}\bar{p}_j)}\frac{\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j\times\Pi_0\hat{g}\bar{p}_j\|^2}{\|F\Pi_0 g\bar{p}_j+\alpha_j v_j\|} + M_w(2+\|\varepsilon\xi\|)\|\varepsilon\xi\|$$
$$+2\zeta\sum_{j\in\mathscr{J}}\frac{|F^{-1}|M_v\|}{(\#\mathscr{J})}\left(2+\frac{\|\varepsilon\xi\|}{1+\|\Pi_0\hat{g}\bar{p}_j\|}\right)\|\varepsilon\xi\|.$$

Proof. The result follows by using the bounds given by Proposition 2 in the expression (35). □

The following result follows immediately, by recalling that $\varepsilon < 1$ from Assumption 4.1.

Proposition 4. *We have the estimate*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon\xi\|^2 \leq -\sum_{j\in\mathscr{J}} \frac{4\zeta}{D(\hat{g}\bar{p}_j)} \frac{\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2}{\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|} + 6\zeta|F^{-1}|M_v + 3M_w. \tag{36}$$

By (4), $\|F\Pi_0 g\bar{p}_j\|$ is bounded above by M and, by the definition $\alpha_j \leq \|F\Pi_0 g\bar{p}_j\|$. Then each $\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|$ is bounded above by $M_p = M(1+M_v)$.

Next, we derive the noisy version of Theorem 1.

Theorem 3. *Let $\bar{T} \in [0, +\infty]$ and $\lambda > 0$ be such that*

i) $\|\varepsilon\xi(0)\| < 1$,

ii) $\displaystyle\sum_{j\in\mathscr{J}} \frac{\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2}{(\#\mathscr{J})\|\Pi_0\hat{g}\bar{p}_j\|^2(1+\|\Pi_0\hat{g}\bar{p}_j\|)} \geq \lambda\|\varepsilon\xi\|^2$, *and*

iii) $\displaystyle\frac{6\zeta|F^{-1}|M_v + 3M_w}{4\zeta\lambda M_p^{-1}} < 1$.

are satisfied on the time interval $[0, \bar{T})$. Then, for every $t \in [0, \bar{T}[$,

$$\|\varepsilon\xi(t)\|^2 \leq \|\varepsilon\xi(0)\|^2 e^{-4\zeta\lambda M_p^{-1}t} + \frac{6\zeta|F^{-1}|M_v + 3M_w}{4\zeta\lambda M_p^{-1}}\left(1 - e^{-4\zeta\lambda M_p^{-1}t}\right).$$

In particular, if $\bar{T} = +\infty$, then for every constant $\rho > 0$ there exists $t_\rho \geq 0$ such that

$$\|\varepsilon\xi(t)\|^2 < \frac{6\zeta|F^{-1}|M_v + 3M_w}{4\zeta\lambda M_p^{-1}} + \rho$$

for all $t \geq t_\rho$.

Proof. Define $M_{v,w} \overset{\text{def}}{=} 6\zeta|F^{-1}|M_v + 3M_w$ and $\tilde{\lambda} \overset{\text{def}}{=} 4\zeta\lambda M_p^{-1}$. Since $\|\varepsilon\xi(0)\| < 1$ there exists small enough $t_0 \in (0, \bar{T}]$ such that $\|\varepsilon\xi(t)\| < 1$ for all $t \in [0, t_0)$. From Proposition 4, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon\xi\|^2 \leq -\tilde{\lambda}\|\varepsilon\xi\|^2 + M_{v,w},$$

for all $t \in [0, t_0]$. From Gronwall's inequality (24) we have

$$\|\varepsilon\xi(t)\|^2 \leq \|\varepsilon\xi(0)\|^2 e^{-\tilde{\lambda}t} + \int_0^t M_{v,w} \exp\left(-\int_t^s -\tilde{\lambda}\,\mathrm{d}\tau\right)\mathrm{d}s. \tag{37}$$

Note that $\exp\left(-\int_t^s -\tilde{\lambda}\,\mathrm{d}\tau\right) = e^{\tilde{\lambda}(s-t)}$. Hence

$$\begin{aligned}\|\varepsilon\xi(t)\|^2 &\leq \|\varepsilon\xi(0)\|^2 e^{-\tilde{\lambda}t} + \int_0^t M_{v,w} e^{\tilde{\lambda}(s-t)}\,\mathrm{d}s\\ &\leq \|\varepsilon\xi(0)\|^2 e^{-\tilde{\lambda}t} + M_{v,w}\tilde{\lambda}^{-1}(1-e^{-\tilde{\lambda}t}) \end{aligned}\tag{38}$$

for all $t \in [0, t_0)$.Therefore we see that $\|\varepsilon\xi\|$ is non-increasing in $[0, t_0)$ if we have that

$$\|\varepsilon\xi(0)\|^2 e^{-\tilde{\lambda}t} + M_{v,w}\tilde{\lambda}^{-1}(1-e^{-\tilde{\lambda}t}) \leq \|\varepsilon\xi(0)\|^2,$$

that is, if $\|\varepsilon\xi(0)\|^2 \geq M_{v,w}\tilde{\lambda}^{-1}$. Since at time t_0, we have $\|\varepsilon\xi(t_0)\|^2 < 1$ we may repeat the argument for a suitable interval $[t_0, t_0+t_1]$ for some $t_1 > 0$ with $t_0+t_1 \leq \bar{T}$. Therefore, if $\|\varepsilon\xi(t_0)\|^2 \geq M_{v,w}\tilde{\lambda}^{-1}$, then $\|\varepsilon\xi(t)\|^2$ decreases in $[t_0, t_0+t_1]$. Repeating again successively the argument, we see that the sequence of instants of time $s_m \stackrel{\text{def}}{=} \sum_{i=0}^{m} t_i$ must "reach" the instant $\bar{T}$, otherwise by the definition we must have $\|\varepsilon\xi(s)\| = 1$ at $s = \lim_{m\to+\infty} s_m \leq \bar{T}$ that is impossible because $\|\varepsilon\xi(s_m)\|^2 \leq \|\varepsilon\xi(0)\|^2 < 1$ for all $m \in \mathbb{N}$. So, in particular $\|\varepsilon\xi\|^2 \leq \|\varepsilon\xi(0)\|^2 < 1$ in $[0, \bar{T}[$. Coming back to the beginning of this proof we may then suppose that $t_0 = \bar{T}$ and so, estimate (37) holds for all $t \in [0, \bar{T}[$. In the case $\bar{T} = +\infty$, from (37), we conclude that for any given constant $\rho > 0$, we may find $t_\rho \geq 0$ such that $\|\varepsilon\xi(t)\|^2 < M_{v,w}\tilde{\lambda}^{-1} + \rho$ for all $t \geq t_\rho$. □

We also have the noisy version of Theorem 2.

Theorem 4. *Let $\bar{T} \in]0, +\infty]$. Suppose there exist positive constants T, λ such that,*

i) $\|\varepsilon\xi(0)\|^2 \leq \dfrac{(6\zeta\,|F^{-1}|M_v + 3M_w)T}{1-e^{-4\zeta\lambda M_p^{-1}T}}$,

ii) $\dfrac{1}{T}\displaystyle\int_t^T \sum_{j\in\mathscr{J}} \frac{\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2}{(\#\mathscr{J})\|\Pi_0\hat{g}\bar{p}_j\|^2(1+\|\Pi_0\hat{g}\bar{p}_j\|)}\,\mathrm{d}\tau \geq \lambda$ *for every* $0 \leq t, t+T < \bar{T}$, *and*

iii) $\dfrac{(6\zeta\,|F^{-1}|M_v + 3M_w)T(2-e^{-4\zeta\lambda M_p^{-1}T})}{1-e^{-4\zeta\lambda M_p^{-1}T}} < 1$,

are satisfied. Then for all $s \in [0, \bar{T})$,

$$\|\varepsilon\xi(s)\|^2 \leq \frac{(6\zeta\,|F^{-1}|M_v + 3M_w)T(2-e^{-4\zeta\lambda M_p^{-1}T})}{1-e^{-4\zeta\lambda M_p^{-1}T}}.$$

Proof. Define $M_{v,w} \stackrel{\text{def}}{=} 6\zeta|F^{-1}|M_v + 3M_w$ and $\tilde{\lambda} \stackrel{\text{def}}{=} 4\zeta\lambda M_p^{-1}$. Suppose that $\|\varepsilon\xi(t)\|^2 - M_p(t-s) > 0$ for all $t \in [s, s+T]$. From estimate (36), we may derive

$$\begin{aligned}&\frac{1}{T}\frac{1}{\|\varepsilon\xi(t)\|^2 - M_{v,w}(t-s)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varepsilon\xi(t)\|^2 - M_{v,w}(t-s)\right)\\ &\leq -\frac{1}{T}\sum_{j\in\mathscr{J}}\frac{4}{D(\hat{g}\bar{p}_j)}\frac{\|\Pi_0\varepsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2}{\|F\Pi_0 g\bar{p}_j + v_j\|(\|\varepsilon\xi(t)\|^2 - M_p(t-s))}.\end{aligned}$$

Integrating on $[s, s+T]$, we arrive to

$$\log\left(\frac{\|\varepsilon\xi(s+T)\|^2 - M_{v,w}T)}{\|\varepsilon\xi(s)\|^2}\right) \leq -\tilde{\lambda}T,$$

that is, $\|\varepsilon\xi(s+T)\|^2 \leq e^{-\tilde{\lambda}T}\|\varepsilon\xi(s)\|^2 + M_{v,w}T$, and so $\|\varepsilon\xi(s+T)\|^2 \leq \|\varepsilon\xi(s)\|^2$ if

$$\|\varepsilon\xi(s)\|^2 \geq \frac{M_{v,w}T}{e^{-\tilde{\lambda}T}}. \tag{39}$$

On the other hand notice that from estimate (36), we have $\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon\xi\|^2 \leq M_{v,w}$. Since $\|\varepsilon\xi(0)\|^2 \leq \frac{M_{v,w}T}{1-e^{-\tilde{\lambda}T}}$, we have that

$$\|\varepsilon\xi(t)\|^2 \leq \frac{M_{v,w}T}{1-e^{-\tilde{\lambda}T}} + M_{v,w}t = \frac{M_{v,w}T + (1-e^{-\tilde{\lambda}T})M_{v,w}t}{1-e^{-\tilde{\lambda}T}} \tag{40}$$

for all $t \in [0, T]$. Now we notice that to have, for some time $\tau \geq T$,

$$\|\varepsilon\xi(\tau)\|^2 = \frac{M_{v,w}T(2-e^{-\tilde{\lambda}T})}{1-e^{-\tilde{\lambda}T}} \tag{41}$$

at some time $\tau > 0$ we need to have $\|\varepsilon\xi(t)\|^2 \geq \frac{M_{v,w}T}{1-e^{-\tilde{\lambda}T}}$ in the interval $[\tau - T, \tau]$, because if for some $t \in [\tau - T, \tau]$ we have $\|\varepsilon\xi(t)\|^2 < \frac{M_{v,w}T}{1-e^{-\tilde{\lambda}T}}$ then necessarily

$$\begin{aligned}\|\varepsilon\xi(\tau)\|^2 <& \frac{M_{v,w}T}{1-e^{-\tilde{\lambda}T}} + M_{v,w}(\tau - t)\\ \leq& \frac{M_{v,w}T}{1-e^{-\tilde{\lambda}T}} = \frac{M_{v,w}T(2-e^{-\tilde{\lambda}T})}{1-e^{-\tilde{\lambda}T}}\end{aligned}$$

which contradicts (41). On the other side, if $\|\varepsilon\xi(t)\|^2 \geq \frac{M_{v,w}T}{1-e^{-\tilde{\lambda}T}}$ in the interval $[\tau - T, \tau]$, then since $\frac{M_{v,w}T}{1-e^{-\tilde{\lambda}T}} \geq M_{v,w}T$ and, using estimate (39), we have that $\|\varepsilon\xi(\tau)\|^2 \geq \|\varepsilon\xi(\tau - T)\|^2$ so, for every $\tau > T$, $\|\varepsilon\xi(\tau)\|^2 = \frac{M_{v,w}T(2-e^{-\tilde{\lambda}T})}{1-e^{-\tilde{\lambda}T}}$ only if

$$\|\varepsilon\xi(\tau - T)\|^2 = \frac{M_{v,w}T(2-e^{-\tilde{\lambda}T})}{1-e^{-\tilde{\lambda}T}}. \tag{42}$$

Now from estimate (40) we have

$$\begin{aligned}\|\varepsilon\xi(t)\|^2 \leq& \frac{M_{v,w}T + (1-e^{-\tilde{\lambda}T})M_{v,w}t}{1-e^{-\tilde{\lambda}T}}\\ <& \frac{M_{v,w}T(2-e^{-\tilde{\lambda}T})}{1-e^{-\tilde{\lambda}T}}\end{aligned}$$

for all $t \in [0,T[$. Therefore, from (42), we have $\|\varepsilon\xi(t)\|^2 < \frac{M_{v,w}T(2-e^{-\tilde{\lambda}T})}{1-e^{-\tilde{\lambda}T}}$ for all $t > 0$. □

Remark 7. Theorem 3 may be "almost" seen as the limit of Theorem 4 as T goes to 0. We say almost because we need to impose that the square of the norm of the initial error is smaller than $\frac{M_{v,w}T}{1-e^{-\tilde{\lambda}T}} < \frac{M_{v,w}T(2-e^{-\tilde{\lambda}T})}{1-e^{-\tilde{\lambda}T}}$, that is, $\|\varepsilon\xi(0)\|^2 \leq \frac{M_{v,w}T}{1-e^{-\tilde{\lambda}T}} < 1$, while in Theorem 3 it was enough to impose that $\|\varepsilon\xi(0)\|^2 < 1$. Notice that $\frac{M_{v,w}T(2-e^{-\tilde{\lambda}T})}{1-e^{-\tilde{\lambda}T}}$ goes to $\frac{M_{v,w}}{\tilde{\lambda}}$ as T goes to 0.

6 Conclusions

This paper provides a nonlinear observer design structure for a left-invariant dynamical system evolving on the three-dimensional special Euclidean group with measurements given by implicit functions. The observer is simple in construction. Under suitable assumptions, we show that the linearized state estimation error converges exponentially fast to the true state. Furthermore, we show that if the dynamical system is subject to disturbance and noise, the estimator converges to an open neighborhood of the true value of the state. The size of the neighborhood increases/decreases gracefully with the bound of the disturbance and noise.

Acknowledgements This work was supported in part by projects DENO / FCT-PT (PTDC/EEA-ACR/67020/2006), Co3-AUVs (EU FP7 under grant agreement No. 231378), FCT (ISR/IST pluri-anual funding) through the PIDDAC Program funds, and the CMU-Portugal program.

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