Motorcycle Modeling for High-Performance Maneuvering

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MAXIMUM VELOCITY PROFILE ESTIMATION

odern sport motorcycles possess an impressive combination of power and agility and are capable of a truly broad range of maneuvers. What factors limit the performance of these machines? Racing provides an ideal testing ground for understanding the limits of performance since the race pilot (rider) is operating the motorcycle close to its limits. Furthermore, the best race pilots play a crucial role in the development of prototype motorcycles.

Producing a high-performance motorcycle prototype entails extensive engineering before skilled riders can evaluate the performance and handling qualities of the physical system. In the engineering stage, virtual prototyping can play an important role. The goal is to build a mathematical model (the virtual prototype) that captures significant aspects of the physical dynamics to facilitate performance analysis and assess design tradeoffs.

State-of-the-art motorcycle models, such as those described in [1] and [2], consist of interconnected rigid bodies together with suspension and other flexible components, supplemented by sophisticated tire [3] and engine models. The accuracy of the predicted behavior depends not only on effective conceptual modeling but also on the use of realistic parameter values. Frame flexibility, tire-road contact geometry, and tire shear force play central roles in motorcycle modeling. Models are used to explore the dynamic properties of motorcycles, including linearization to study vibration modes under constant-speed and constant-turn-radius conditions [4]. Multibody codes and related tools are finding increased use among designers and engineers interested in improving handling qualities, performance, and safety. The automotive and motorcycle industries use virtual prototyping tools to reduce the number of physical tests that need to be performed during the design process.



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Simulation studies determine whether the motorcycle can perform maneuvering tasks such as path following with a specified velocity profile. Although open-loop simulation can be used to analyze automobile handling, a virtual motorcycle rider is needed to stabilize the roll mode. Without feedback to provide appropriate steering and throttle/brake inputs, the simulated motorcycle usually falls quickly.

In [5] and [6], a virtual rider based on a simplified motorcycle model [7], [8] is applied to a commercial multibody code motorcycle model. The control strategy is based on a receding horizon scheme that uses preview information on the path to be followed. The development and use of a virtual rider is described in [9].

This article focuses on high-performance maneuvering required in racing competitions. We are motivated by the observation that the best race pilots seek to understand the performance limits of their motorcycles and then exploit this knowledge in determining a race line (a path within the track) that allows them to traverse the circuit in minimum time. The skills of these riders have been honed through years of experience. Their keen senses are important to the development team, providing feedback for refining the motorcycle design and the particular bike setup for a given race. Finally, through repeated practice on the given track, race pilots develop a sense of the best race line for that track and, for each turn, determine precise braking points for switching from maximum acceleration to maximum braking.

Simulation models can be used to explore the maximum performance and minimum lap time problem for ground vehicles. A common approach involves posing the problem as a classical minimum-time optimal control problem [10]. Strategies for approximating the solution to the minimum-time problem for cars and motorcycles have been extensively developed. For instance, the quasi-static strategies developed in [11] for approximating the minimum lap time performance for race cars have been used since the 1950s. Although transient behavior is neglected, the approach of [11] allows the use of detailed race car models. A more direct attack on the minimum-time problem for race car performance is described in [12], where an optimal control problem involving a seven-degree-of-freedom (7DOF) race car model is discretized using a parallel shooting method. The resulting nonlinear program is solved using a sequential quadratic programming algorithm. This work produces both the race line and the velocity profile, dealing with the dynamic behavior of the race car model in a more complete manner. We also note that [12] provides a comprehensive review of maximum performance research for cars.

Due mainly to instability issues, the exploration of maximum performance for motorcycle models is more recent. In [13], optimal control techniques are used to define and assess a notion of motorcycle maneuverability. The cost function uses penalty functions to address constraints such as the width of the road. Using a combination of penalties, [13] produces plausible approximate race lines. A more direct attack on the maximum performance and minimum lap time problem for motorcycles is presented in [14], which also uses penalty functions to handle inequality constraints. The optimal solution is found by solving a discretized two-point boundary value problem expressing the first-order optimality conditions. Symbolic software is used to develop routines for evaluating the derivatives that occur in the boundary value problem. This strategy is used to compute an optimal trajectory for a complete race track.

One way to reduce the complexity of the maximum performance problem for motorcycles is to seek the optimal velocity profile for a given fixed path. Removing from consideration the selection of the race line, we can focus directly on how various constraints limit the performance of a motorcycle. We pursue this approach here.

We begin by reviewing motorcycle features, including engine power, tire forces, and wheelie and stoppie maneuvers, that lead to performance limitations. Next we develop an algorithm for finding the minimum-time trajectory for a constrained 1DOF vehicle model. This model is a point-mass vehicle constrained to follow a given path and subject to accelerations representative of more general vehicles. Examination of this idealized situation clarifies the dynamic features needed to find the minimum-time velocity profile. A detailed nonholonomic motorcycle model capturing gross vehicle motion and the associated contact forces is presented. We then develop a quasisteady-state technique for approximating the velocity constraint and acceleration limits that are in play for the vehicle at each location along the desired path. With this information in hand, an approximately optimal velocity profile is constructed and used to build an approximate motorcycle trajectory. Finally, we show by example that a trajectory produced in this fashion can be used as a reference trajectory for exploring the aggressive trajectory space of a comprehensive multibody motorcycle model developed using commercial software.

MODELING FOR MAXIMUM VELOCITY MANEUVERING

A racing motorcycle is capable of extremely aggressive maneuvers. During a race competition, the goal of the rider is to complete a lap in minimum time. At each point on the track, the motorcycle is subject to physical constraints that limit the available acceleration and deceleration. The most important constraint is due to the tires. Since the lateral and longitudinal forces that a tire can produce are coupled, a large lateral force greatly reduces the available longitudinal force. Additional constraints are mainly due to the engine, aerodynamics, and mass distribution.

Consider first the case in which the motorcycle is moving along a straight line. No lateral force is needed in this

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case, and the longitudinal acceleration is limited at low speed by the maximum longitudinal force that the rear tire can produce. At high speed, due to the presence of increased aerodynamic drag, the longitudinal acceleration limit is determined instead by the maximum engine torque. In contrast, during a turn, the lateral acceleration is limited by the maximum lateral force that the tires can produce. The available engine torque is not normally a limiting factor, however, since the available longitudinal tire force is greatly reduced due to the coupling of lateral and longitudinal tire forces.

The transition from straight running to cornering must be approached with care. As the radius of curvature of a turn decreases, the required lateral tire force increases. Furthermore, the required lateral acceleration (and corresponding lateral tire forces) is proportional to the square of the velocity. Consequently, it is easy to enter a turn with so much speed that the lateral force required for turning far exceeds what the tires can produce. In that case, the rider must modify the desired trajectory to avoid losing control of the motorcycle.

Tire Modeling

To better describe the longitudinal and lateral tire force coupling, we briefly describe how tire forces are modeled using Pacejka's magic formula [3]. The magic formula is a set of equations relating load, slip ratio, slip angle, and camber angle, denoted by F_z , κ , α , and γ , respectively, to the longitudinal force, side force, and aligning moment. These equations use a clever composition of trigonometric functions to provide a family of parameterized functions for fitting empirical tire data. The original formula developed for car tires has become standard in that context. The extension to motorcycle tires necessitates substantial changes to accommodate the different roles of sideslip and camber forces in the two cases [4], [15].

Tire forces and moments are produced through a combination of geometry and slip. The camber angle γ is the angle between the wheel plane and the line perpendicular to the road surface. Due to the shape of a motorcycle tire, nonzero camber results in a lateral force called camber thrust. Additional tire forces and moments are produced by slip between the tire and road surface. Roughly speaking, slip occurs when the velocity vector of the contact point between tire and road is different from the velocity vector determined by the speed and heading of the wheel. The slip angle α is the angle between the wheel's actual direction of travel and the direction toward which it is pointing, while the slip ratio κ provides a nondimensional description of the relative motion between the tire and the road surface [3]. The slip ratio is nonzero when the tire's rotational speed is greater or less than the free-rolling speed.

Magic formula parameter values for a given tire are used to calculate the steady-state force and moment system for realistic operating conditions. Additional features for modeling dynamic effects are developed [3] but are not discussed here. In the magic formula scheme, the cases of pure longitudinal slip and pure lateral slip are treated separately and then combined using loss functions that characterize the reduction of forces in combined slip.

In pure longitudinal slip, the slip angle α and camber angle γ are set to zero so that the tire does not generate any lateral force. The longitudinal force F_{x0} in pure slip is then a function of the slip ratio κ and the normal load F_z . This function is given by

$$F_{x0}(\kappa, F_z) = D_x \sin[C_x \arctan(B_x \kappa - E_x (B_x \kappa - \arctan(B_x \kappa)))],$$

where the coefficient functions $B_x = B_x(F_z)$, $D_x = D_x(F_z)$, and $E_x = E_x(F_z, \text{sgn}(\kappa))$, as well as the constant C_x , shape the response. A typical plot under constant load is depicted in Figure 1. Note that the longitudinal force increases with increasing slip ratio κ up to a maximum longitudinal force followed by a significant drop. When the tire is forced to work beyond the peak, the rider experiences a sudden loss of grip as the slip dynamics transition from a stable region with positive slope to an unstable region. Physically, the tire spins up rapidly under power as the shear force decreases under increasing slip ratio while the engine torque remains nearly constant.

In pure lateral slip, the slip ratio κ is set to zero so that the tire does not generate any longitudinal force. The lateral force in pure lateral slip, which is a function of the slip angle α , the camber angle γ , and the normal load F_z , has the form



FIGURE 1 Longitudinal tire force in pure longitudinal slip. For a given load F_z , with zero slip angle and zero camber, the longitudinal tire force is a function of only the slip ratio κ . For small values of κ , the curve is nearly linear. The longitudinal force is limited, however, resulting in a significant and sudden loss of grip upon passing the peak (at $\kappa \simeq 0.08$, in this example).

$$F_{y0}(\alpha, \gamma, F_z) = D_y \sin[C_y \arctan(B_y \alpha - E_y \\ \times (B_y \alpha - \arctan(B_y \alpha))) \\ + C_\gamma \arctan(B_\gamma \gamma - E_\gamma \\ \times (B_\gamma \gamma - \arctan(B_\gamma \gamma)))],$$

where C_y , C_γ , and E_γ are constant and $B_y = B_y(F_z, \gamma)$, $D_y = D_y(F_z, \gamma)$, $E_y = E_y(\gamma, \text{sgn}(\alpha))$, and $B_\gamma = B_\gamma(F_z, \gamma)$. A typical plot for the lateral force as a function of slip angle α for a given load and zero slip ratio κ is shown in Figure 2.



FIGURE 2 Lateral tire force in pure lateral slip. For constant load F_z , with zero slip ratio and zero camber, the lateral tire force is a symmetric and strictly increasing function of the slip angle α . For small slip angles, the lateral tire force characteristic is nearly linear.



FIGURE 3 The friction ellipse. The envelope of longitudinal and lateral tire forces, obtained by varying the slip ratio κ and slip angle α , resembles an ellipse, from which the name derives. The shape and position of this ellipse depend on the load and camber angle.

The longitudinal and lateral tire forces under combined slip are given by [4]

 $F_x = F_{x0}(\kappa, F_z) G_{x\alpha}(\alpha, \kappa, F_z)$

and

$$F_{y} = F_{y0}(\alpha, \gamma, F_z) G_{y\kappa}(\alpha, \kappa, \gamma, F_z)$$

where $G_{x\alpha}(\cdot)$ describes the loss of longitudinal force due to sideslip and $G_{y\kappa}(\cdot)$ describes the loss of lateral force due to longitudinal slip. For further details, see [3].

Longitudinal and lateral forces for a given normal load F_z are illustrated in Figure 3. The envelope of these curves, representing the maximum available traction and cornering forces, is called the friction ellipse. The shape and position of the friction ellipse change according to the normal load and camber angle.

Engine and Drivetrain Modeling

The engine supplies the torque needed for controlling the speed of the vehicle. In a modern motorcycle, the engine torque is transmitted through a chain or a driveshaft to the rear wheel. The interaction between the rear wheel and the ground produces a shear force causing the vehicle to move.

It is common practice to measure the steady-state engine torque on a test bench. This measurement is performed by setting the throttle valve to a fixed position and then modulating the load torque to achieve a desired engine speed. In this manner, one obtains the steady-state

> torque for several load points, that is, for several combinations of throttle opening position and engine speed. The steady-state torque map is used for simulating the engine in handling and performance analysis. A more sophisticated engine model is the mean value engine model (MVEM), which describes the development of measured engine variables (or states) such as revolutions per minute and intake manifold pressure on time scales longer than an engine cycle [16], [17]. For control purposes, the MVEM attempts to provide accuracy while ensuring fast computation time.

> Figure 4 provides an example of the engine torque curve. The curve shown, which is plotted versus the engine speed, refers to a wideopen throttle condition. Engine torque is transmitted to the rear wheel through a set of gears and a chain. A simple mathematical description of the drivetrain takes into account inertial properties of the crankshaft, the main and counter shafts, and the rear wheel. A graphical representation of a modern motorcycle drivetrain is shown in Figure 5.

The load seen by the engine depends on factors such as the selected gear and the slip ratio. The longitudinal dynam-

ics of a straight-running vehicle with a rigid suspension and other simplifications can be written as

$$m\dot{v}_x = F_x^r - F_A,\tag{1}$$

where *m* is the vehicle mass, v_x is the longitudinal velocity, F_x^r is the rear-wheel longitudinal force, and F_A is the aerodynamic drag. The inertia of the gears forming the drivetrain can be lumped into the equivalent gear-dependent rearwheel dynamics model

$$I_{eq}\dot{\omega}_{rw} = \tau_e \ \rho_{e,rw} - r_{rw} \ F_x^r, \qquad (2)$$

where r_{rw} is the rear-wheel radius, τ_e is the engine torque, I_{eq} is the gear-dependent equivalent rear-wheel inertia, and the gear ratio $\rho_{e,rw}$ is the gear-dependent ratio of the engine rotation speed to the rear-wheel rotation speed. Equations (1) and (2) constitute a planar dynamical system with states v_x and ω_{rw} and input τ_e . We can also use (v_x, κ_r) as state coordinates, where $\kappa_r = (\omega_{rw} r_{rw} - v_x)/v_x$ denotes the rear-wheel slip ratio. Differentiating the slip ratio with respect to time yields

$$\dot{\kappa}_{r} = -\left[\frac{1+\kappa_{r}}{mv_{x}} + \frac{r_{rw}^{2}}{I_{eq}v_{x}}\right]F_{x}^{r}(\kappa_{r}) + \frac{1+\kappa_{r}}{mv_{x}}F_{A}(v_{x}) + \frac{r_{rw}\rho_{e,rw}}{I_{eq}v_{x}}\tau_{e},$$
(3)

where $F_x^r = F_x^r(\kappa_r)$ is the slip-dependent rear-tire force and $F_A = F_A(v_x)$ is the velocity-dependent aerodynamic drag force. Within the normal working region, the slope of the longitudinal-force-versus-slip curve is large so that the slip dynamics (3) are stable and fast relative to the rigid body velocity mode. That is, for slowly varying τ_e , $\dot{\kappa}_r$ goes to zero quickly, and thus κ_r behaves like a static function of v_x and τ_e , that is,

$$\kappa_r \approx \bar{\kappa}_r(v_x, \tau_e)$$

where $\bar{\kappa}_r(\cdot, \cdot)$ is the implicit function defined by setting $\dot{\kappa}_r = 0$ in (3). The use of a static relation to approximate a relatively fast dynamic subsystem can be analyzed using singular perturbation theory [18]. Under the condition $\dot{\kappa}_r = 0$, \dot{v}_x and $\dot{\omega}_r$ satisfy the static relationship

$$(1 + \bar{\kappa}_r(v_x, \tau_e))\dot{v}_x = \dot{\omega}_r r_{rw}.$$
(4)

Combining (4) with (1) and (2), we obtain the instantaneous slip system governed by



FIGURE 4 Engine torque curve. The engine torque increases with engine speed until reaching the maximum torque value. This measured curve corresponds to a wide-open throttle for a sport motorcycle.



FIGURE 5 Modern motorcycle drivetrain. The crankshaft is connected through the primary drive to the gearbox. The main shaft and countershaft are connected through the pair of gears selected by the rider using the gearshift pedal. The secondary drive connects the gearbox to the rear wheel.

$$\left(m + \frac{I_{\text{eq}}}{r_{rw}^2} (1 + \bar{\kappa}_r(v_x, \tau_e))\right) \dot{v}_x = \frac{\rho_{e, rw}}{r_{rw}} \tau_e - F_A(v_x) \,.$$

Taking the slip ratio as nearly constant, $\kappa_r \approx \hat{\kappa}_r$ gives the useful quasi-steady-state approximation

$$m_{\rm eq}\dot{v}_x = F_{\rm eq} - F_A\,,\tag{5}$$

where the equivalent force

$$F_{\rm eq} = \frac{\rho_{e,\,nv}}{r_{nv}}\tau_e$$

depends on the selected gear, while the equivalent mass

$$m_{\rm eq} = m + \frac{l_{\rm eq}}{r_{\rm rw}^2} (1 + \hat{\kappa}_r)$$

depends on both the selected gear and the quasi-steadystate slip ratio \hat{k}_r .

Figure 6 provides an example of how the engine torque in Figure 4 is transformed to the rearwheel torque, which is proportional to the equivalent force according to $\tau_{rw} = \rho_{e,rw} \tau_e$ as the gear setting ranges from first to sixth. The envelope of those curves, given by the red line in Figure 6, represents maximum wheel torque versus wheel speed.

Brake Modeling

Since acceleration at high speed is limited by available engine torque, one might expect a similar deceleration limitation due to the brakes. In reality, this case does not exist since modern brake disks, pads, and hydraulics are dimen-



sioned so that brakes have virtually no capacity problem. On a racing motorcycle, the rider can always apply sufficient brake force to lock the wheels. In high-fidelity simulations, brakes are typically modeled as torques applied to the front and rear wheels.

Wheelie and Stoppie

Skilled riders like to show off by performing tricks such as the wheelie (see Figure 7) and the stoppie (see Figure 8) in which the front and rear wheels, respectively, come off the ground. Upon passing the apex (roughly, the point of maximum curvature of a turn), the racing rider opens the throttle to accelerate out of the turn. The apex of a corner is the place where the chosen race line touches the inside edge of the track. Since modern motorcycles possess motors with considerable power and torque, this exit acceleration can result in the front wheel lifting off the ground. This phenomenon is often observed during professional races such as the SuperBike and MotoGP championships. Race wheelies, however, are typically short-lived since a sustained wheelie requires precise throttle action to precisely control the pitch motion of the motorcycle. A similar phenomenon, the stoppie, can occur during hard braking when entering a turn. In this situation, if too much torque is applied to the front wheel, the rear wheel can lift off the ground.

COMPUTING THE OPTIMAL VELOCITY PROFILE FOR A POINT-MASS MOTORCYCLE

We now describe the basis for estimating the maximum velocity profile for a given path. The algorithm is first discussed for a point-mass vehicle and then extended to a simplified motorcycle model.

Consider a smooth curve in the plane, which we wish to traverse in minimum time. Here, we view the motorcycle as a point mass moving along the curve with velocity v. Using a moving frame, the accelerations seen by the point motorcycle are given as a tangential or longitudinal acceleration \dot{v} and a perpendicular or lateral acceleration σv^2 . The lateral acceleration depends on the instantaneous curvature $\sigma = \pm 1/R$, where R is the radius of the osculating circle that is second-order tangent to the curve at the current location. As viewed from above, the sign of σ is positive when the curve is turning right and negative when the curve is turning left. The curvature is zero at points where the curve is straight, that is, $R = \infty$.

The physics of the point motorcycle are thus described by

$$m\dot{v} = f_{\text{long}},$$
 (6)

$$m\sigma v^2 = f_{\text{lat}},\tag{7}$$

FIGURE 6 Equivalent engine torque at the rear wheel. The effect of the gearbox is to modify the torque transmitted to the rear wheel so that engine power is optimized for different vehicle speeds. The red curve, which depicts the available rear wheel torque, is the envelope of the equivalent engine torque curves corresponding to first through sixth gears for a sport motorcycle.

where the applied force (f_{long} , f_{lat}) is an idealization of the force provided by the motorcycle tires interacting with the road surface. As such, the force is required to lie in the friction ellipse given by



FIGURE 7 The wheelie maneuver. MotoGP contender Loris Capirossi pops a shortlived wheelie on his Ducati Desmosedici under maximum acceleration at the exit of a turn. Racing pilots (riders) possess the high level of skill required to finesse the front wheel back onto the ground without upset. (Photo courtesy of Michael Troutman, DMT Imaging.)

$$\left(\frac{f_{\rm long}}{f_{\rm long}}\right)^2 + \left(\frac{f_{\rm lat}}{f_{\rm lat}^{\rm max}}\right)^2 \le 1,\tag{8}$$

where $f_{\text{long}}^{\text{max}}$ and $f_{\text{lat}}^{\text{max}}$ are the maximum longitudinal and lateral forces, respectively. Consistent with our curvature definition, we see that f_{long} and f_{lat} act in the forward and right directions, respectively.

Our goal is to find the velocity profile v as a function of the arc length s to traverse a given curve in minimum time. Normally, we view the velocity v as a function of time, not arc length. To indicate that arc length rather than time is the independent variable, we use a bar to indicate that a quantity is an explicit function of s. Thus, a curvature profile is given as $\bar{\sigma}(s)$ for $s \in [s_0, s_1]$, whereas the corresponding curvature trajectory σ as a function of t is given by $\sigma(t) = \bar{\sigma}(s(t))$, where $s(\cdot)$ is an arc-length trajectory. Restated, our goal is to find a velocity profile $\bar{v}(\cdot)$ that minimizes the time that it takes to traverse the curve given by a curvature profile $\bar{\sigma}(\cdot)$.

Note that the shape of a curve is determined by its curvature profile $\bar{\sigma}(\cdot)$. Indeed, let $(\bar{x}(\cdot), \bar{y}(\cdot))$ be a smooth arclength-parameterized curve, where the *y*-axis is oriented 90° in the clockwise direction from the *x*-axis, for example, *x* pointing north and *y* pointing east. Differentiating with respect to *s* (denoted by '), the orientation of the unit length tangent vector can be specified by a smoothly varying heading angle $\bar{\psi}(\cdot)$. That is,

$$\begin{pmatrix} \bar{x'}(s)\\ \bar{y'}(s) \end{pmatrix} = \begin{pmatrix} \cos\bar{\psi}(s)\\ \sin\bar{\psi}(s) \end{pmatrix}.$$
 (9)

Differentiating (9), we obtain the arc-length acceleration vector

$$\begin{pmatrix} \bar{x''}(s)\\ \bar{y''}(s) \end{pmatrix} = \bar{\psi}'(s) \begin{pmatrix} -\sin\bar{\psi}(s)\\ \cos\bar{\psi}(s) \end{pmatrix},$$

which is perpendicular to the tangent vector. Fitting the osculating circle to the curve, we obtain

$$\bar{\psi}'(s) = \bar{\sigma}(s) \tag{10}$$

so that the curvature is also the rate at which the curve changes direction with respect to arc length. Integrating (9) and (10) from an initial position $(\bar{x}(0), \bar{y}(0))$ and heading $\bar{\psi}(0)$ shows that there is a three-dimensional family of curves with the same shape. We assume that the curvature profile $\bar{\sigma}(\cdot)$ is continuously differentiable so that $(\bar{x}(\cdot), \bar{y}(\cdot))$ is a C^3 curve, providing a five-dimensional profile $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{\psi}(\cdot), \bar{\sigma}(\cdot), \bar{\sigma}'(\cdot))$ that prescribes a portion of the desired vehicle behavior.

The motion of the point motorcycle can be described using either a velocity trajectory $v(\cdot)$ or a velocity profile $\bar{v}(\cdot)$ since each is uniquely determined by the other when the velocity is strictly positive. This dependence follows from the fact that $\dot{s}(t) = \bar{v}(s(t)) = v(t)$ yields an arc-length trajectory $t \mapsto s(t)$ that is strictly monotone increasing and hence invertible.

We find it useful to work in the spatial domain with the arc length *s* as the independent variable rather than in the time domain. To this end, using $\dot{v}(t) = \bar{v}'(s(t)) \bar{v}(s(t))$ and $\bar{a}(s) = \bar{f}_{long}(s)/m$, we write the longitudinal dynamics (6) as

$$\bar{v}'(s) \,\bar{v}(s) = \bar{a}(s)$$

or, suppressing the independent variable s,

$$\bar{v}' = \bar{a}/\bar{v}.\tag{11}$$

FIGURE 8 The stoppie maneuver. The 2005 MotoGP champion Valentino Rossi pulls a brief stoppie on his Yamaha YZR-M1 under hard braking during the approach to a tight turn. Maximum deceleration is achieved by applying as much front brake as possible without lifting the rear wheel too far off the ground. Great skill is needed to maintain balance and heading under maximal braking. (Photo courtesy of Michael Troutman, DMT Imaging.)



Combining (7) and (8), it follows that the input acceleration is constrained by

$$\bar{a}_{\min}(s,\bar{v}) \le \bar{a}(s) \le \bar{a}_{\max}(s,\bar{v}), \qquad (12)$$

where, in this case,

$$\bar{a}_{\max}(s,\bar{v}) = +\frac{f_{\log g}^{\max}}{m} \sqrt{1 - \left(\frac{\bar{\sigma}(s)\bar{v}^2}{f_{\log}^{\max}/m}\right)^2}$$

and

$$\bar{a}_{\min}(s,\bar{v}) = -\frac{f_{\log g}^{\max}}{m} \sqrt{1 - \left(\frac{\bar{\sigma}(s)\bar{v}^2}{f_{\text{lat}}^{\max}/m}\right)^2}.$$

A dynamic velocity profile satisfying (11) must also satisfy

$$\bar{v}(s) \le \bar{v}_M(s) \tag{13}$$

for all *s*, where $\bar{v}_M(\cdot)$ is the maximum velocity profile corresponding to the curvature profile $\bar{\sigma}(\cdot)$. From (7) and (8), we see that the maximum velocity at *s* is given by

$$\bar{v}_M(s) = \sqrt{\frac{f_{\text{lat}}^{\text{max}}/m}{|\bar{\sigma}(s)|}}$$

with $\bar{v}_M(s) = +\infty$ (an extended value) whenever $\bar{\sigma}(s) = 0$ to represent the absence of a limit on velocity.

We say that $\bar{v}(\cdot)$ is a feasible velocity profile if $\bar{v}(\cdot)$ satisfies the differential equation (11) and the constraints (12) and (13) on the domain of definition of $\bar{\sigma}(\cdot)$. If $\bar{v}(\cdot)$ is feasible and s_c satisfies $\bar{v}(s_c) = \bar{v}_M(s_c)$, then s_c must be a stationary point of $\bar{v}_M(\cdot)$ since $\bar{v}'(s_c) = 0$ (all available force is used for lateral acceleration) and if $\bar{v}'_M(s_c)$ is not zero then



FIGURE 9 An example test track curve. This *x*-*y* plane curve test track, which is followed in clockwise fashion, involves three right turns followed by one left turn.

 $\bar{v}(s) > \bar{v}_M(s)$ for some *s* near s_c violating (13). Note that, since $\bar{\sigma}(\cdot)$ is continuously differentiable, $\bar{v}'_M(s)$ is defined at all *s* such that $\bar{v}_M(s)$ is finite. In practice, contact points normally occur at local minimizers of $\bar{v}_M(\cdot)$. A local minimum of $\bar{v}_M(\cdot)$ corresponds to local maximum of $|\bar{\sigma}(\cdot)|$, which roughly corresponds to the apex of a turn. A contact point s_c can also occur at a local maximum of $\bar{v}_M(\cdot)$, especially if $v_M(s_c)$ is also a local minimum, that is, $\bar{v}_M(\cdot)$ is constant on a neighborhood of s_c . A curvature profile exhibiting this possibility can be easily constructed. While the question remains open, it appears that it may be possible to construct a smooth $\bar{v}_M(\cdot)$, allowing a contact point s_c that is an isolated local maximizer of $\bar{v}_M(\cdot)$.

An optimal velocity profile maximizes the velocity at each point along the path while remaining feasible. This property implies, as occurs in various time optimal problems, that every value of the optimal applied longitudinal force is either the maximum or minimum allowed by (12) and (13), so that the point-mass motorcycle is always either accelerating or braking as much as possible.

To this end, consider the problem of traversing, in clockwise fashion, the curve depicted in Figure 9 whose curvature profile is shown in Figure 10. Clearly, the optimal velocity profile must touch $\bar{v}_M(\cdot)$ at at least one point since otherwise it would be possible to find a faster velocity profile. Figure 11 depicts the maximum velocity profile $\bar{v}_M(\cdot)$ for the path in Figure 9 together with the optimal velocity profile $\bar{v}_{opt}(\cdot)$. As noted above, each location s_c such that $\bar{v}_{opt}(s_c) = \bar{v}_M(s_c)$ satisfies the necessary condition $\bar{v}'(s_c) = 0$. Also, when a turn is sufficiently isolated (for example, turns one and two), the approach involves maximum braking, while the departure involves maximum acceleration. This observation provides a strategy for determining the optimal velocity profile.



FIGURE 10 Curvature profile of the test track curve in Figure 9. The curvature is shown as a function of arc length. The direction of the turns is easily determined by the sign of the curvature (positive for right turns), indicating three right turns followed by one left turn.

Suppose that the number of connected regions of local minimizers for $\bar{v}_M(\cdot)$ is finite, with each minimum region defining a turn. For each turn, we compute a locally optimal velocity profile as follows. Starting at a local minimizer, we integrate forward with

$$\bar{v}' = \bar{a}_{\max}(s, \bar{v})/\bar{v} \tag{14}$$

and backward with

$$\bar{v}' = \bar{a}_{\min}(s,\bar{v})/\bar{v} \tag{15}$$

until the maximum velocity constraint (13) is violated. The optimal velocity profile $\bar{v}_{opt}(\cdot)$ is then given by the minimum of the local profiles (see Figure 11). Switches from maximum acceleration to maximum braking occur at locations where the locally optimal velocity profile that is globally optimal changes.

In our example, switches from maximum acceleration to maximum braking occur at approximately 315 and 548 m. Despite the fact that the track has four well-defined turns, we see in Figure 11 that only three locally optimal velocity profiles are used to determine the optimal velocity profile. In this example, it turns out that the locally optimal velocity profile for turn four includes a portion of the locally optimal velocity profile for turn three so that the turn-three profile is not needed (or is redundant). Indeed, as we integrate (15) backward from the turn-four minimum region of $\bar{v}_M(\cdot)$ (starting at, say, 755 m), the dynamic velocity profile converges to the turn-three minimum region of $\bar{v}_M(\cdot)$ within a finite distance (at approximately 656 m). We then continue until the maximum velocity constraint is violated at approximately 496 m. The turn-four locally optimal velocity profile thus covers both the turn-three and turn-four cases.

This last feature points to an interesting technical detail. The fact that there is a finite-distance convergence to the turn-three minimum region implies that the vector field at the point of convergence cannot be Lipschitz, and indeed it is not. Furthermore, at that point we do not have the uniqueness properties that are ensured for a locally Lipschitz vector field. Nonuniqueness is reflected by the fact that the trajectory can reach the constraint curve at many different points depending on where the trajectory starts off of the constraint curve. The important point for our purposes is that the curve obtained by integrating (15) backward is the unique curve satisfying the given initial condition. Nonuniqueness would be an issue if we needed to begin at a local minimizer of $\bar{v}_M(\cdot)$ and integrate (15) forward. Similar remarks apply to (14) with reversed directions.

Suppose now that the point motorcycle is subjected to accelerations of a more general nature, for example, variable aerodynamic drag. In this case, the form of the constrained dynamic system is unchanged, satisfying (11), (12), and (13). That is, a feasible velocity profile satisfies



FIGURE 11 Optimal velocity profile. The optimal velocity $\bar{v}_{opt}(\cdot)$ (solid) is shown together with the maximum velocity constraint $\bar{v}_M(\cdot)$ (dashed). The optimal solution touches the maximum velocity profile only at points that locally minimize the maximum velocity profile $\bar{v}_M(\cdot)$. Different colors are used to depict different locally optimal velocity profiles. Switches from maximum acceleration to maximum deceleration occur at approximately 315 m and 548 m. The transitions from maximum deceleration to maximum acceleration are, in contrast, smooth.

$$\bar{v}' = \bar{a}/\bar{v}$$

where the available acceleration is constrained according to

$$\bar{a}_{\min}(s, \bar{v}) \leq \bar{a}(s) \leq \bar{a}_{\max}(s, \bar{v})$$

and the velocity state must satisfy the maximum velocity constraint

$$\bar{v}(s) \leq \bar{v}_M(s)$$
.

As noted above, the maximum velocity profile $\bar{v}_M(\cdot)$ is continuously differentiable at all points of finite value. The acceleration constraints are continuous in s and \bar{v} , $\bar{v} \mapsto \bar{a}_{\min}(s, \bar{v})$ is nondecreasing, $\bar{v} \mapsto \bar{a}_{\max}(s, \bar{v})$ is nonincreasing, and $\bar{v} < \bar{v}_M(s)$ implies that $\bar{a}_{\max}(s, \bar{v}) > \bar{a}_{\min}(s, \bar{v})$. We are interested in minimizing

$$J(\bar{v}(\cdot)) = \int_{s_0}^{s_1} \frac{ds}{\bar{v}(s)}$$
(16)

subject to the dynamics (11) and the constraints (12) and (13). The cost $J(\bar{v}(\cdot))$, defined in (16), is simply the time that it takes to go from s_0 to s_1 using the velocity profile $\bar{v}(\cdot)$.

Now, it is possible to ride the velocity constraint in regions where

$$\bar{a}_{\min}(s, \bar{v}_M(s)) \le \bar{v}'_M(s) \bar{v}_M(s) \le \bar{a}_{\max}(s, \bar{v}_M(s))$$

by choosing $\bar{a}(s) = \bar{v}'_M(s)\bar{v}_M(s)$. The condition $\bar{v}'_M(s) = 0$ above is a special case. To obtain the set of locally optimal velocity profiles, we begin in each constraint-riding region and integrate backward using $\bar{a} = \bar{a}_{\min}(s, \bar{v})$ and forward using $\bar{a} = \bar{a}_{\max}(s, \bar{v})$ until the maximum velocity constraint is violated. The optimal velocity profile $\bar{v}_{opt}(\cdot)$ is then given by the minimum of the local profiles.

To be concrete, with aerodynamic drag, the acceleration constraints are offset according to

$$\bar{a}_{\max}(s,\bar{v}) = +\frac{f_{\log g}^{\max}}{m} \sqrt{1 - \left(\frac{\bar{\sigma}(s)\bar{v}^2}{f_{\rm lat}^{\max}/m}\right)^2 - b_{\min}\bar{v}^2}$$

and

$$\bar{a}_{\min}(s,\bar{v}) = -\frac{f_{\log g}^{\max}}{m} \sqrt{1 - \left(\frac{\bar{\sigma}(s)\bar{v}^2}{f_{\log}^{\max}/m}\right)^2} - b_{\max}\bar{v}^2$$

where b_{\min} and b_{\max} model the minimum and maximum available drag (for example, $b_{\min} = \rho c_D A/2$, see below). The variable drag coefficient $b \in [b_{\min}, b_{\max}]$ models the extent to which the rider can modulate the drag force through body posture. We thus see that b_{\min} corresponds to the streamlined stance used during acceleration and on high-speed straights, whereas b_{\max} corresponds to a higher drag upright stance used when braking during the approach to a turn. Note that, in this case, a local minimum of $\bar{v}_M(\cdot)$ is no longer a possible contact point for $\bar{v}_{opt}(\cdot)$ and $\bar{v}_M(\cdot)$.



FIGURE 12 Optimal velocity profile with variable aerodynamic drag. The optimal velocity $\bar{v}_{opt}(\cdot)$ (solid), when aerodynamic drag is present, is shown together with the velocity constraint $\bar{v}_M(\cdot)$ (dashed). The loss of symmetry is due to the presence of the aerodynamic drag force. Also, points of contact between the optimal velocity profile and the maximum velocity profile no longer occur at local minimizers of $\bar{v}_M(\cdot)$. As before, different colors are used to depict different locally optimal velocity profiles. Switches from maximum acceleration to maximum deceleration occur at approximately 327 m, 553 m, and 676 m.

Figure 12 shows the optimal velocity profile when variable aerodynamic drag is included. Note the loss of the symmetry that is present when there is no aerodynamic drag. Also, since the presence of aerodynamic drag allows for greater deceleration, the switching points occur further down the track at approximately 327, 553, and 676 m. In this case, all four locally optimal velocity profiles are used to determine the optimal velocity profile. Surprisingly, for the drag parameters we use, the time to traverse the 900-m course is slightly shorter than in the no-drag case. Figure 13 compares the optimal velocity and acceleration trajectories resulting with and without aerodynamic drag.

When the number of contact regions is finite, the number of acceleration switches is also finite so that $\bar{v}'_{opt}(\cdot)$, hence $\dot{v}_{opt}(\cdot)$, is piecewise continuous. In the above examples, the no-drag case possesses two points of discontinuity, whereas the case with aero drag possesses three, as can be seen in figures 11 and 12.

COMPUTING THE OPTIMAL VELOCITY PROFILE FOR A NONHOLONOMIC MOTORCYCLE WITH CONSTRAINTS

The algorithm for computing the optimal velocity profile for a point-mass motorcycle provides a framework for estimating optimal velocity profiles for more comprehensive motorcycle models. More comprehensive models possess additional states (including roll angle and rate) and are subject to more complicated constraints than the simple point-mass motorcycle above. We thus need to manage this additional complexity to evaluate the pointwise (in space *s*) maximum velocity as well as the minimum and maximum acceleration functions. The basic strategy is to use a quasi-steady-state approach where the basic idea is to compute quantities as if some of the system states were in steady state.

We now proceed to develop a constrained nonholonomic motorcycle model that can be used to estimate the performance of more realistic motorcycle models, which include, for instance, suspension and tire models. Our nonholonomic motorcycle model is based on the bicycle model developed in [7] and [8]; see also [19].

The nonholonomic model can be viewed as a planar body that moves along the ground with leaning. The generalized coordinates describing the position and attitude of the vehicle (see Figure 14) are the coordinates (x, y) of the point of contact of the rear wheel, the roll (or lean) angle φ , and the yaw (or heading) angle ψ , while δ represents the effective steer angle of the front wheel as measured in the *x*-*y* plane. For the sake of consistency, the inertial *x*-*y*-*z* coordinate system is taken as northeast-down with angles oriented by the right-hand rule so that, for example, the heading angle ψ is measured from the north (around the inertial *z*-axis) with east being $+\pi/2$ rad. The roll angle φ is positive when the motorcycle is leaned right (right-hand rule around the body-



FIGURE 13 Changes in the optimal velocity profile due to aerodynamic drag: (a) the longitudinal velocity (upper) and the corresponding acceleration trajectory (lower) for the point-mass vehicle when no aerodynamic drag is present and (b) the longitudinal velocity (upper) and acceleration (lower) for the point-mass vehicle subject to the aerodynamic drag. Periods of acceleration and braking are shown in green and red, respectively.

fixed *x*-axis). This nomenclature is consistent with that used in aerospace applications and adopted for ground vehicle dynamics [20]. As in [7] and [8], we use simplified kinematics in which the wheel contact points are located within a fixed body plane without changing the wheelbase (as if the steering axis is perpendicular to the body-fixed *x*-axis and the wheels have zero radius). All mass (with magnitude *m*) is assumed to be concentrated at the center of mass, located at height *h* (when the vehicle is vertical) and distance *b* forward of the rear-wheel contact point. The acceleration of gravity is denoted by *g*, and *p* is the wheelbase.

The motion of the motorcycle is constrained in a nonholonomic fashion so that the motion of each wheel contact point is allowed only in the direction that the wheel is pointing. Thus, for a constant effective steer angle δ , the motorcycle follows a circle of curvature $\sigma = \tan \delta/p$ and signed radius $R = 1/\sigma$; see Figure 14. This relation between δ and σ also holds instantaneously and maps $\delta \in (-\pi/2, \pi/2)$ to $\sigma \in (-\infty, \infty)$ in an invertible fashion. We can thus use σ in place of δ in the dynamics. Note that the effective steer angle δ is not the angle of rotation of the steering shaft of the motorcycle.

The kinematics of planar motion are given by

$$\dot{x} = v_x \cos \psi, \qquad \dot{y} = v_x \sin \psi,$$

$$\dot{v}_x = u_1, \qquad \dot{\psi} = v_x \sigma, \qquad \dot{\sigma} = u_2, \qquad (17)$$

where $u_1 = \dot{v}_x$ and $u_2 = \dot{\sigma}$ are the longitudinal and lateral controls, respectively. These kinematics are the same as those of a nonholonomic car. The roll dynamics of the nonholonomic motorcycle

$$h\ddot{\varphi} = g\sin\varphi - (\sigma v_x^2 + b\ddot{\psi} - h\dot{\psi}^2\sin\varphi)\cos\varphi \qquad (18)$$

are nearly those of an inverted pendulum with a lateral acceleration applied at the pivot point. Indeed, the coefficient of $\cos \varphi$ contains the lateral acceleration $\sigma v_x^2 + b\ddot{\psi}$ seen at the pivot as well as a centripetal acceleration term $h\dot{\psi}^2 \sin \varphi$ resulting from the rotation about the pivot with angular velocity $\dot{\psi}$ and offset $h \sin \varphi$.



FIGURE 14 The nonholonomic motorcycle model. This model describes the motion of a rigid plane whose thin and massless wheels are constrained to not slide sideways. The effective steering angle δ determines the instantaneous radius of curvature *R*. The center of mass is located at the ride height *h* (when the vehicle is vertical) and a distance *b* forward of the rear wheel contact point. The roll angle φ shown in the figure is positive.

Equation (18) can also be derived by considering the motion of a system consisting of a point-mass inverted pendulum attached to a fictitious planar rigid body. The rigid body is fully actuated so that it can be made to follow any position-orientation trajectory despite dynamic interactions with the unactuated pendulum. Let (x_c, y_c) and ψ be the position and orientation of the center of mass of the rigid body. Attach the pendulum at the center of mass with the axis of rotation aligned with the body *x*-axis (the heading reference) and let φ be the roll angle of the pendulum with respect to the inverted position, positive around the body *x*-axis. Let m_c and I_c denote the mass and inertia of the rigid body, respectively, let m_p denote the kinetic and potential energies and the associated Lagrangian



FIGURE 15 Approximation of the roll trajectory. The quasi-steadystate roll trajectory (dashed) approximates the dynamic roll trajectory (solid) satisfying the equation of motion (18).



FIGURE 16 Interaction forces between the tires and ground. These forces are applied at the contact points of the front and rear wheels. The longitudinal forces F_x^f and F_x^r as well as the lateral forces F_y^f and F_v^r lie in the ground plane.

L = T - V. The system motion satisfies

$$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi}$$

= $- m_p h[(h\dot{\psi}^2 \sin \varphi - (\ddot{y}_c \cos \psi - \ddot{x}_c \sin \psi)) \cos \varphi$
+ $g \sin \varphi - h\ddot{\varphi}],$ (19)

where $\ddot{y}_c \cos \psi - \ddot{x}_c \sin \psi$ is the lateral acceleration at the pivot. Expressing the lateral acceleration in terms of v_x , σ , and $\ddot{\psi}$ as above yields (18). Note that (19) is independent of the inertial properties chosen for the fictitious planar rigid body and, furthermore, the dynamic constraint (19) is also independent of the mass m_p of the point-mass pendulum.

Equations (17) and (18) together with

$$\psi = \dot{v}_x \,\sigma + v_x \dot{\sigma} = u_1 \sigma + v_x u_2$$

constitute the nonholonomic motorcycle dynamics. This set of equations provides an idealized motorcycle with direct control over the longitudinal dynamics through $\dot{v}_x = u_1$ and the lateral dynamics through $\dot{\sigma} = u_2$.

The algorithm for computing the optimal velocity profile described in the previous section can be viewed as providing a quasi-steady-state trajectory for the nonholonomic motorcycle (17), (18) subject to acceleration constraints that are independent of the roll angle φ . At each point along the desired path, the algorithm provides the optimal v_{opt} and \dot{v}_{opt} with corresponding σ_{opt} and $\dot{\sigma}_{opt}$, defining the control and a portion of the state. The remainder of the quasisteady-state state is found by trimming the system so that the derivatives of some of the states are instantaneously fixed at desired values. In this case, we require that $\dot{v}_x = \dot{v}_{opt}$, $\dot{\sigma} = \dot{\sigma}_{opt}$, and $\ddot{\varphi} = 0$ with $v_x = v_{opt}$ and $\sigma = \sigma_{opt}$.

Due to the decoupled nature of the nonholonomic motorcycle, the first two conditions are already satisfied; to satisfy the third, we compute the angle φ_{qs} such that $\ddot{\varphi} = 0$ in (18). It is clear that the quasi-steady-state roll trajectory $\varphi_{qs}(\cdot)$ obtained in this fashion is not a dynamic trajectory since it does not satisfy (18) unless it is constant. Nevertheless, $\varphi_{qs}(\cdot)$ provides a reasonable estimate of the dynamic trajectory $\varphi(\cdot)$ satisfying (18) as the example presented in Figure 15 suggests. The roll trajectory $\varphi(\cdot)$ and its quasi-steady-state approximation $\varphi_{qs}(\cdot)$ correspond to the optimal velocity profile with aerodynamic drag presented above. Upon further examination, the dynamic roll trajectory appears to be a filtered version of the quasi-steady-state roll trajectory. This relationship can, in fact, be made precise [21], [22]. Indeed, it can be shown that the dynamic roll trajectory is approximated by $\mathcal{A}[\varphi_{qs}(\cdot)]$, where \mathcal{A} is a noncausal lowpass filter that depends on the required lateral acceleration trajectory. The linear time-varying filter $\mathcal{A}[\cdot]$ approximates a noncausal linear time-invariant (LTI) filter with impulse response $h(t) = (\alpha/2) e^{-\alpha|t|}$ and frequency response $\hat{h}(\omega) = \alpha^2/(\alpha^2 + \omega^2)$, where the characteristic value (or natural frequency) α is of the form $\sqrt{g_{\rm eff}/h}$ with an effective

gravity g_{eff} that is stronger in turns. In particular, the dynamic roll trajectory is close to the quasi-steady-state roll trajectory when the variation of the quasi-steady-state roll trajectory is slow relative to this lowpass filtering.

To determine whether more realistic constraints involving tire forces, wheelie, and so on are satisfied when a maneuver is performed by the nonholonomic motorcycle, we develop a method for evaluating the interaction forces between the vehicle and the ground. To this end, note that the nonholonomic motorcycle is a planar rigid body whose motion is constrained by forces at the front- and rear-wheel contact points (see Figure 16). One approach is thus to write the dynamics of the full six-degree-of-freedom (6DOF) rigid body subject to the forces shown in Figure 16 so that the forces F_z^r and F_z^f are forces of constraint determined by constraining the contact points to be in the *x-y* plane while F_y^r and F_y^f are the forces needed to enforce the nonholonomic constraints.

Another approach for determining the interaction forces is to derive the dynamics of a four-degree-of-freedom (4DOF) rigid body with contact points constrained to the horizontal plane (holonomic constraints). The lateral tire forces are then determined by enforcing the nonholonomic constraints, whereas the normal forces are estimated using a forcemoment balance. We will describe this strategy in more detail.

The configuration of the planar rigid body system can be parameterized by the set of generalized coordinates $q = (x, y, \varphi, \psi)^T$, where (x, y) is the position of the idealized rear contact point and φ and ψ are once again the roll and yaw (or heading) angles for the vehicle. The kinetic and potential energies of the system have the form $\tilde{T}(q, \dot{q}) = \dot{q}^T \tilde{M}(q) \dot{q}/2$ and $\tilde{V}(q)$, respectively. Applying the Lagrangian formalism, the system dynamics have the form

$$\widetilde{M}(q)\ddot{q} + \widetilde{C}(q,\dot{q}) + \widetilde{G}(q) = \widetilde{f},$$

where *f* is a vector of generalized forces [23]. Exploiting symmetry and adding the influence of the external forces F_x^f , F_y^r , F_x^r , F_y^f , and F_A , we write the equations of motion with the translational velocity (\dot{x}, \dot{y}) expressed as (v_x, x_y) in the vehicle body frame as

$$M(\varphi) \begin{pmatrix} \dot{v}_{x} \\ \dot{v}_{y} \\ \ddot{\varphi} \\ \ddot{\psi} \end{pmatrix} + C(\varphi, v_{x}, v_{y}, \dot{\varphi}, \dot{\psi}) + G(\varphi)$$
$$= \begin{pmatrix} F_{x}^{f} c_{\delta} - F_{y}^{f} s_{\delta} + F_{x}^{r} - F_{A} \\ F_{x}^{f} s_{\delta} + F_{y}^{f} c_{\delta} + F_{y}^{r} \\ 0 \\ p(F_{x}^{f} s_{\delta} + F_{y}^{f} c_{\delta}) \end{pmatrix}, \qquad (20)$$

where $M(\varphi)$ is a roll-dependent mass matrix (or inertia tensor) and $C(\cdot)$ and $G(\cdot)$ are vectors of Coriolis and gravity terms, respectively. The effective steering angle δ affects the direction of application of the front-wheel forces F_x^f and F_y^f with the necessary rotation indicated using $c_{\delta} = \cos \delta$ and $s_{\delta} = \sin \delta$. The term F_A is the aerodynamic drag, commonly expressed as $F_A = (1/2)\rho c_D A v_x^2$, where ρ is the air density, c_D is the drag coefficient, and A is a reference drag area. Imposing the nonholonomic conditions

$$\dot{\psi} = v_x \sigma, \quad v_y = 0, \quad \delta = \arctan p\sigma,$$
 (21)

the third equation of (20), representing the roll dynamics, reduces to (18) as expected.

To complete the model, we compute (or, rather, estimate) the normal forces F_z^f and F_z^r by a vertical force balance and a pitch-axis moment balance given by

$$0 = F_z^r + F_z^J + mg,$$

$$0 = \left(F_x^f c_\delta - F_y^f s_\delta + F_x^r\right) hc_{\varphi}$$
(22)

$$+ F_A h_{cp} c_{\varphi} + F_z^r b - F_z^f a, \qquad (23)$$

where a = p - b is the horizontal distance from the center of mass to the front contact point. This expression is not exact since it neglects some inertial contributions (which can be computed using the 6DOF approach) but is sufficient for the sake of presentation, providing a reasonable approximation and indicating the general structure. Equations (22) and (23) show the influence of the longitudinal forces on the balance of the vehicle. A large acceleration due to large F_x^r increases the load on the rear wheel while reducing the load on the front. Similarly, the effect of hard braking corresponding to F'_x large but negative is to increase the front normal force while reducing the normal force on the rear. This phenomenon is known as load transfer. Due to load transfer, under high-performance race conditions, the longitudinal force redundancy is resolved using $F_x^r \ge 0$ and $F_x^j \le 0$. We neglect the usual rolling resistance as being negligible compared with acceleration and braking forces. Furthermore, modern racing motorcycles possess a clutch mechanism that greatly reduces the transmission of engine braking torque to the rear wheel.

The steady-state trajectories of the nonholonomic motorcycle with forces modeled by (20)-(23) consist of constantspeed circles parameterized by (v_x, σ) over a range of longitudinal velocity $v_x \ge v_{\min} > 0$ and curvature σ . To trim the motorcycle at a constant operating condition, we proceed as follows. First, note that since the 4DOF model (20) is equivalent to the nonholonomic motorcycle model (17), (18) under the nonholonomic constraints (21), we obtain the steady-state roll angle φ_{ss} by solving (18) for φ with the specified (v_x, σ) and $\ddot{\varphi} = 0 = \ddot{\psi}$. For $h\sigma \ll 1$, $\varphi_{ss} \approx \tan^{-1} \sigma v_x^2/g$. Now, with nonzero aerodynamic drag, the system is dissipative so that, driving from the rear wheel, we set $F_x^J = 0$ and look for $F_x^r > 0$. The first, second, and fourth equations of (20) with $\dot{v}_x = \dot{v}_y = \ddot{\psi} = 0$ and $v_x, \sigma, \varphi, \dot{\psi}, \delta$ set to trim values can now be solved for F_x^r , F_y^r , and F_y^t . Finally, we obtain F_z^r and F_z^j from (22) and (23). Ranging over v_x and σ , we can thus build up the equilibrium manifold of constant operating conditions for the nonholonomic motorcycle.

For nonzero fixed curvature σ , the overall required lateral tire force increases as the velocity v_x increases, roughly as the square of the velocity. It follows that the friction ellipse tire-force constraint shown in Figure 17 limits the achievable longitudinal velocity v_x whenever $\sigma \neq 0$. Thus,

Observable differences occur during the transition from acceleration to braking due to suspension dynamics that are neglected in the quasi-steady-state computation.

as shown above for the point-mass motorcycle, the maximum velocity is a function of $|\sigma|$ and can be easily computed numerically. This computation is somewhat more complicated since the size of each friction ellipse depends on the normal forces. In this fashion, we can compute the maximum velocity profile $\bar{v}_M(\cdot)$ corresponding to the desired path curvature profile $\bar{\sigma}(\cdot)$.

To define the maximum and minimum acceleration functions $\bar{a}_{max}(s, \bar{v}_x)$ and $\bar{a}_{min}(s, \bar{v}_x)$, we take a quasisteady-state approach. For a specific location *s* with curvature $\sigma = \bar{\sigma}(s)$ and velocity $v_x < \bar{v}_M(s)$, consider the maximum acceleration case. We begin with the steadystate condition for (v_x, σ) . Then we increase the rear longitudinal force F_x^r while freezing the roll angle at its steady-state value. The second and fourth equations of (20) together with (22) and (23) allow us to solve for the forces F_y^r , F_y^f , F_z^r , and F_z^f as affine functions of F_x^r . The maximum acceleration is obtained when the friction ellipse constraint is reached or when the longitudinal



FIGURE 17 Top view of the motorcycle model with a graphical representation of the tire force limits. Longitudinal and lateral tire forces acting on the front and rear wheels (light blue) are constrained to remain within the corresponding friction ellipses (red).

force corresponding to the maximum engine torque is reached. Note that the wheelie constraint $-F_z^f \ge 0$ is also easily accommodated.

The minimum acceleration function is computed in a similar fashion, first reducing F_x^r to zero and then

increasing the braking force $-F_x^I$ while checking the tire and stoppie constraints. As previously mentioned, the maximum braking force is not a real limitation for modern racing motorcycles, since the rider can always apply enough braking torque to lock the front wheel.

The computation of the optimal velocity profile is accomplished in much the same way as for the point mass. Since the various functions are implemented numerically rather than with closed-form expressions, we use for-

ward Euler integration. As before, the integration is performed with respect to the arc length *s* rather than time. A finite number of points s_1, \ldots, s_N on the track are chosen. The algorithm then has the following form:

- 1) At the current position s_k , the velocity is $\bar{v} = \bar{v}(s_k)$ and the curvature is $\bar{\sigma} = \bar{\sigma}(s_k)$.
- 2) Trim the vehicle at constant velocity \bar{v} and curvature $\bar{\sigma}$ obtaining the steady-state roll angle $\bar{\varphi}_{ss}$ together with the interaction forces between the tires and ground.
- 3) Holding φ at $\bar{\varphi}_{ss}$, let $\bar{F}_{x}^{r}(s_{k})$ be the largest longitudinal rear force F_{x}^{r} satisfying the
 - a) tire constraints: the longitudinal and lateral forces of the front and rear wheels remain within the respective friction ellipses;
 - b) engine constraint: the longitudinal force to be produced by the rear tire can be generated by the engine; and
 - c) wheelie constraint: normal load on the front tire has to be greater than or equal to zero.
- 4) Define $\bar{a}_{\max}(s_k, \bar{v}) := \dot{v}_T$, where \dot{v}_T is the longitudinal acceleration corresponding to the maximum lateral force $\bar{F}_x^r(s_k)$. Propagate the velocity profile using

 $\bar{v}(s_{k+1}) = \bar{v}(s_k) + (s_{k+1} - s_k) \cdot \bar{a}_{\max}(s_k, \bar{v}(s_k)) / \bar{v}(s_k).$

- If the maximum velocity curve is reached (or exceeded), proceed to Step 6. Otherwise, set k = k + 1 and go to Step 1.
- 6) Find the next local minimum of the maximum velocity curve, occurring at the location s_j. Set v(s_i) = v(s_i) = v(s_i). Also, set k = j for future use.
- 7) At the current position s_j , the velocity is $\bar{v} = \bar{v}(s_j)$, and the curvature is $\bar{\sigma} = \bar{\sigma}(s_j)$.
- 8) Trim the vehicle at constant velocity \bar{v} and curvature $\bar{\sigma}$, obtaining the steady-state roll angle $\bar{\varphi}_{ss}$ together with the interaction forces between tires and ground.
- Holding φ at φ
 _{ss}, let -F
 _x^f(s_j) be the largest decelerating longitudinal front force -F
 _x^f satisfying the

- a) tire constraints: the longitudinal and lateral forces of the front and rear wheels remain within their respective friction ellipses
- b) stoppie constraint: normal load on the rear tire must be nonnegative.
- 10) Define $\bar{a}_{\min}(s_j, \bar{v}) := \dot{v}_T$, where \dot{v}_T is the longitudinal acceleration corresponding to the decelerating lateral force $\bar{F}_x^f(s_j)$. Propagate the velocity profile backwards using $\bar{v}(s_{j-1}) = \bar{v}(s_j) (s_j s_{j-1}) \cdot \bar{a}_{\min}(s_j, \bar{v}(s_j))/\bar{v}(s_j)$, saving for comparison all previously computed values of $\bar{v}(s_{j-1})$.
- 11) If the presently determined velocity $\bar{v}(s_{j-1})$ exceeds a previously computed velocity (signifying intersection), take $\bar{v}(s_{j-1})$ to be the minimum of the two and proceed to Step 1 (using the *k* set in Step 6). Otherwise, set j = j 1 and go to Step 7.

Note that we have omitted the obvious exits or jumps that would be taken whenever we run off the end of the track in the forward or reverse directions. As for the point-mass motorcycle, the optimal velocity profile (estimate) is given by the minimum of the local optimal velocity profiles.

APPLICATION TO A REALISTIC MOTORCYCLE MODEL

We now show how the above techniques can be used as part of a strategy for producing more realistic, highperformance motorcycle trajectories. We have at hand a high-fidelity motorcycle model that includes suspension components, sophisticated tire models, and detailed drivetrain models. Furthermore, we have a virtual rider in the form of a feedback control system capable of driving the motorcycle model along approximate trajectories that are sufficiently close to being feasible. Given a desired path, we are interested in finding trajectories of the highfidelity motorcycle model that are high performance, operating close to the performance limits.



FIGURE 18 Comparison between quasi-steady-state and dynamic simulation along a chicane maneuver. Part (a) shows the velocity profile computed with the quasi-steady-state method (red) as well as by dynamic simulation with the virtual rider (dashed blue), (b) shows the acceleration profiles obtained with the quasi-steady-state method (red) and simulation (dashed blue), where the spikes in the simulated acceleration at around 130 m and elsewhere are due to gear shifting, (c) is the roll angle profile obtained with the quasi-steadystate method (red) and simulation (dashed blue), where the roll angle reaches approximately 45° providing a lateral acceleration of approximately 1 g, and (d) is the required curvature profile given as the input to the quasi-steady-state method and as produced by the simulated multibody motorcycle (dashed blue).



FIGURE 19 Quasi-steady-state and dynamic tire forces. Parts (a) and (b) are the longitudinal tire forces computed with the quasi-steadystate method (red) and those obtained by simulation with the virtual rider (dashed blue), (c) and (d) show the quasi-steady-state (red) and dynamic (dashed blue) lateral tire forces; and (e) and (f), for the same maneuver, show the normal forces computed with the quasi-steadystate method (red) and obtained by dynamic simulation (dashed blue).

The overall strategy is as follows. First we build up a corresponding nonholonomic motorcycle model together with appropriate constraints. To this end, we select geometric and inertia parameters for the nonholonomic motorcycle model to provide a nominal match of the corresponding properties of the high-fidelity motorcycle model. For instance, the height h can be chosen to correspond to an approximate average position of the center of mass under the expected maneuvering. Details of the drive train and engine are examined to produce a maximum wheel-torque curve such as that depicted in Figure 6. The tire model parameters are examined and abstracted to produce friction ellipse limits.

With a constrained nonholonomic motorcycle model in hand, we are now ready to construct approximate trajectories that can be evaluated with the high-fidelity motorcycle model. Given a desired path and an initial velocity, we use the algorithm presented in the previous section to construct a velocity profile that is feasible for the noholonomic motorcycle and that makes use of the maximum and minimum acceleration functions developed using the quasi-steady-state approach. This velocity profile is then used to obtain a dynamic trajectory for the nonholonomic motorcycle, checking that the corresponding constraints are satisfied. Finally, this simplified trajectory is used as a reference trajectory for the high-fidelity model driven by the virtual rider.

For the sake of illustration, consider a track composed of a simple chicane, which is a quick succession of sharp opposite-direction turns (usually two turns) that must be negotiated at relatively slow speed. The curvature of the chicane used in our example is shown in Figure 18, where

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a change of sign in σ can be seen. In the same figure, we report the velocity profile computed with the quasisteady-state method as well as the velocity profile followed by a multibody motorcycle model driven by the virtual rider (dashed blue line). The lateral acceleration reaches 1 g at the apex of the turns with a corresponding roll angle of 45°. Despite the simplicity of the model used for the quasi-steady-state computation, there is good agreement between the roll angles obtained during dynamic simulation and the quasi-steady-state method. The roll angles are shown in Figure 18 together with the longitudinal acceleration profiles.

When using the nonholonomic motorcycle model, and especially when using the quasi-steady-state method, it is possible to transition instantaneously from acceleration to braking. In contrast, when working with a model that includes a suspension, it takes time to transition from an acceleration posture in which the rear suspension is more compressed to a deceleration posture in which the front suspension is more compressed. During this load-transfer transient, the normal force at the front tire may be such that the available braking force is temporarily less than the ideal force predicted by the quasisteady-state method. Race pilots use a combination of suspension setup and rider skill to manage such load-transfer effects.

The interaction forces between tires and ground predicted by the quasi-steady-state method and obtained through a dynamic simulation are shown in Figure 19. As in the case of Figure 18, despite the simplicity of the model, there is a good agreement between predicted and simulated forces. Observable differences occur during the transition from acceleration to braking due to suspension dynamics that are neglected in the quasi-steady-state computation.

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