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Pedro Cruz and Pedro Batista

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A solution for the attitude determination of three-vehicle heterogeneous formations

Pedro Cruz[a], Pedro Batista[a,b]

[a]Instituto Superior Técnico, Universidade de Lisboa, Lisbon, Portugal
[b]Institute for Systems and Robotics, Laboratory for Robotics and Engineering Systems, Portugal

Abstract

This paper proposes a solution to a new attitude determination problem for a three-vehicle formation, where there are restrictions on the detection of other vehicles. Vision-based sensors are considered, which measure line-of-sight (LOS) vectors between different vehicles and inertial vectors that can vary according to the vehicle. There is a constraint on the LOS vectors which are only measured in relation to a chief vehicle. Moreover, each vehicle can measure only one inertial vector. The solution for the different attitude relations is devised geometrically, making use of a multi-stage process. First, two candidates for the relative attitude of each branch are determined. Then, these relative attitude candidates are used to compute inertial attitude candidates for the chief vehicle. The comparison between these results disambiguates the problem, which in general has a unique solution. There are some degenerate solutions, which are also determined. Finally, simulations are carried out considering noise in the sensors. The results are coherent with state-of-the-art approaches to similar problems.

Keywords: Attitude determination, heterogeneous formation, vision-based navigation, rotation matrix.

1. Introduction

Attitude determination consists in computing the relation between the orientation of two different frames. Every vehicle needs some kind of knowledge of their attitude, so that navigation is possible. Thus, there is a lot of interest in this field. In contrast with attitude estimation, where filtering techniques are used, attitude determination obtains the information available at a given moment [1]. The problem of finding the attitude appeared with the development of spacecraft technology. One of the first and most recognised methods is the Tri-Axial Attitude Determination (TRIAD) algorithm [2], which uses two unit vectors, represented in a body and a reference frame, to obtain the relation between those two frames. More vector measurements were considered in the renowned Wahba's problem [3], which defines a cost function related to this optimization problem. A widely known and employed solution that minimizes this function is the Quaternion Estimator (QUEST) algorithm [4], which makes use of the quaternion representation of a rotation matrix. There are more recent approaches to solve the Wahbas's problem, one example is the fast linear quaternion attitude estimator (FLAE), which can reduce the computational time while having the similar accuracy to other methods, as described in [5].

This paper deals with deterministic scenarios, where there is a minimal set of measurements, such as in the TRIAD. These problems, usually, consider arc length and dihedral angle measurements. Therefore, three deterministic scenarios are possible: two directions as in the TRIAD, a direction and an arc length or three arc lengths [6]. In space it is common to rely on star trackers or sun sensors to get these measurements, whereas in unmanned aerial vehicles (UAVs) or other aerial platforms, the use of the Global Positioning System (GPS) data is more common. However, there are benefits in adopting systems that are independent from the GPS, because of its vulnerabilities to interferences and the precision issues or unavailability [7] in some environments as, for example, the interplanetary space. Therefore, we assume vision-based navigation sensor systems, that allow, for example, the measurement of LOS unit vectors between vehicles [8] or the measurement of inertial references from camera-based sensors [9]. Other systems that can also give an inertial reference are, for example, magnetometers [10].

A formation is a group of vehicles that cooperate towards a common overall goal. For a smooth, safe, and successful operation, these vehicles require both relative and inertial navigation. Vehicle formations have many applications, which include ground [11], underwater [12], aerial [13] and space scenarios [14]. The solutions developed for the determination of the inertial attitude can often be employed to find the relative attitude among the vehicles in a formation. However, constraints in the vector measurements require different solutions to be employed. The case

Email addresses: pedro.f.cruz@tecnico.ulisboa.pt (Pedro Cruz), pbatista@isr.tecnico.ulisboa.pt (Pedro Batista)

considered in [15] proposes a two-vehicle formation, where each vehicle detects the other while both detect the same common object. The reference presents both the deterministic solution for the problem and the corresponding covariance analysis. In [16], a three-vehicle formation, with LOS vectors among all the vehicles is analysed. Again, both the deterministic solution for the problem and the covariance analysis are given.

This paper considers a heterogeneous formation, that is, the sensor set can vary from vehicle to vehicle. Thus, we can have a different quantity of sensors in each vehicle and/or their quality can be distinct according to their importance. This can impact not only cost, but also weight, which are key drivers in the space industry. In the case addressed in this paper, the number of vehicles detected by the sensors varies with the vehicle, which can be a result of the environment and/or equipment constraints. As a consequence, each vehicle plays a different role in the attitude determination, since some vehicles get more information than others. The applications of this problem include situations that physically constrain the visibility between different vehicles, such as long formations or an environment with opaque obstacles. For example, long formations are useful for the application of interferometry techniques in space, which benefits, in terms of resolution, from having a wide distribution [17].

The main contributions of this work are the proposal of an attitude determination problem in a heterogeneous formation and the respective solution. This framework is different from others in the literature reviewed and cannot be solved using those methods alone. Simulations are shown to assess the performance of the proposed attitude determination system.

This paper is structured as follows: first, a section with the problem statement, where the notation used throughout the paper is defined and the problem is described. Then, in the following section, the solution for the problem is developed, considering the determination of the different candidates and their comparison. Moreover, in the same section, the degenerate solutions of the problem are taken into account. Finally, the simulations that were conducted are described, along with the noise model used, and the results are presented.

2. Problem statement

2.1. Notation and definitions

Throughout this document scalars are represented in regular typeface, whereas vectors and matrices are represented in bold, the latter in capital case. Reference frames are represented in calligraphic typeface and between brackets, such as $\{\mathcal{I}\}$. Body-fixed frames are numbered and represented by the letter $\mathcal{B}$, for example, $\{\mathcal{B}_1\}$. The symbol $\mathbf{0}$ represents the null vector or matrix and $\mathbf{I}$ represents the identity matrix with the appropriate dimensions.

The set of unit vectors in $\mathbb{R}^3$ is denoted by $S(2) := \left\{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\right\}$. The special orthogonal group of dimension 3, which describes proper rotations, is denoted by

$$SO(3) := \left\{\mathbf{X} \in \mathbb{R}^{3\times 3} : \mathbf{X}\mathbf{X}^T = \mathbf{X}^T\mathbf{X} = \mathbf{I} \ \wedge \ \det(\mathbf{X}) = 1\right\} .$$

The skew-symmetric matrix parametrized by $\mathbf{x} \in \mathbb{R}^3$, which encodes the cross product between $\mathbf{x}$ and another vector, is denoted by

$$\mathbf{S}(\mathbf{x}) := \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \ \mathbf{x} = [x_1, x_2, x_3] .$$

The rotation matrix in $SO(3)$ that transforms a given vector expressed in the reference frame $\{\mathcal{A}\}$ to the reference frame $\{\mathcal{C}\}$ is denoted by $\mathbf{R}_A^C$. If any of the reference frames is a body-fixed frame, the number that identifies the body is used instead, *i.e.*, the rotation from $\{\mathcal{B}_i\}$ to $\{\mathcal{B}_j\}$, $i, j \in \mathbb{N}$, is denoted by $\mathbf{R}_i^j$. When multiple candidates exist for the same quantity a letter subscript outside parentheses identifies the respective candidate. For example, $\left(\mathbf{R}_i^j\right)_A$ and $\left(\mathbf{R}_i^j\right)_B$ are two candidates for $\mathbf{R}_i^j$.

The rotation matrix of an angle $\theta \in \mathbb{R}$ around the axis described by the unit vector $\mathbf{x} \in \mathrm{S}(2)$ is denoted by $\mathbf{R}(\theta, \mathbf{x})$, which can be written as [1]

$$\mathbf{R}(\theta, \mathbf{x}) := \cos(\theta)\,\mathbf{I}_{\mathbf{3}\times\mathbf{3}} + (1 - \cos(\theta))\mathbf{x}\mathbf{x}^T - \sin(\theta)\,\mathbf{S}(x) \ . \tag{1}$$

The four-quadrant inverse tangent function is denoted by $\mathrm{atan2}\,(b, a)$, with $a, b \in \mathbb{R}$.

2.2. Problem definition

Consider a formation composed by three vehicles, in which each of the vehicles has its own body-fixed frame, $\{\mathcal{B}_1\}$, $\{\mathcal{B}_2\}$, and $\{\mathcal{B}_3\}$. Consider as well an inertial reference frame, $\{\mathcal{I}\}$. In the proposed framework, there are two kinds of measurements: one is a LOS vector that points to the position of another vehicle and the other is an inertial vector, for example, a known inertial direction. The LOS vector is only known in the body-fixed frame of the vehicle that measures it, while the inertial vector is available both in the body-fixed frame, where it is measured, and the inertial frame, since the sensor has knowledge about the inertial reference that it is measuring. It is worth mentioning that all measurement vectors are considered as unit vectors, so they only give information about the direction that the sensor detects, as it is common in attitude determination problems.

In this formation, the main constraint is that only one of the vehicles can measure LOS vectors to both the other vehicles, whereas the other two can only measure a single LOS vector relative to this first vehicle. Meaning, for example, that these two are too far from each other for the sensors to measure the LOS, or that their sensors are not oriented in such a way that they can get both LOS.

Furthermore, every vehicle has a sensor that measures one inertial vector. The vehicle that can measure LOS to the other two is identified as vehicle 1 and will sometimes be referred as chief. The other vehicles are often called deputies and are identified as vehicles 2 and 3, respectively. The geometry of the scenario is represented in Fig. 1.

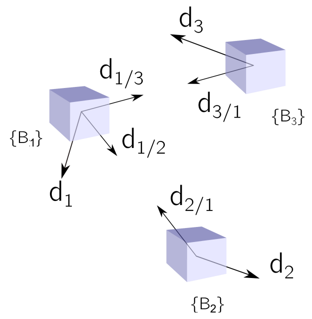

Figure 1: Three-vehicle heterogeneous formation

In the figure and throughout this document, the LOS measurements are denoted by $\mathbf{d}_{i/j} \in S(2)$, $i, j = 1, 2, 3$, $i \neq j$, which represents the unit vector from the i-th to the j-th vehicles expressed in the i-th vehicle body-fixed frame, $\{\mathcal{B}_i\}$. The inertial vector measurements are denoted by $\mathbf{d}_i \in S(2)$, $i = 1, 2, 3$, which represents the inertial vector measured by the i-th platform and expressed in its own body-fixed frame.

If a vector is expressed in a different frame, a superscript is used, for both the LOS vectors and the inertial vectors, to specify the frame where the vector is described. For example, ${}^{j}\mathbf{d}_{i/j}$, $i, j = 1, 2, 3$, $i \neq j$, is the LOS from the i-th to the j-th vehicle expressed in $\{\mathcal{B}_j\}$ and ${}^{I}\mathbf{d}_i$, $i = 1, 2, 3$, is the inertial vector of the i-th vehicle, expressed in the inertial frame, $\{\mathcal{I}\}$.

The problem that is here considered is that of determining all the rotation matrices, both relative ($\mathbf{R}_2^1$, $\mathbf{R}_3^1$, $\mathbf{R}_3^2$) and inertial ($\mathbf{R}_1^I$, $\mathbf{R}_2^I$, $\mathbf{R}_3^I$), using the measurement vectors that were described, as well as the references ${}^{I}\mathbf{d}_1$, ${}^{I}\mathbf{d}_2$, and ${}^{I}\mathbf{d}_3$.

Remark. In [16], the relative attitude between two vehicles is determined using two LOS vectors between them and one LOS between each of them and a third common body or vehicle. In short, in [16] there are LOS vectors between all three platforms. In the problem addressed in this paper this is not, in general, the case, since there are no LOS vectors between two of the vehicles, the so-called deputies. Thus, the solution described in [16] does not apply to the problem at hand. Moreover, in this paper the inertial attitude is also computed for all three platforms using a single inertial measurement in each vehicle. Interestingly enough, the problem addressed in [16] can be seen as a particular case of the relative attitude determination problem addressed in this paper, when all the relevant vectors are co-planar.

3. Attitude determination

The proposed solution is divided into different stages. The first stage consists in computing the candidates $\left(\mathbf{R}_2^1\right)_A$ and $\left(\mathbf{R}_2^1\right)_B$, from the relation between the chief vehicle and vehicle 2, followed by the computation of $\left(\mathbf{R}_3^1\right)_C$ and $\left(\mathbf{R}_3^1\right)_D$ from the relation between the chief vehicle and vehicle 3. Next, the previous candidates generate, respectively, $\left(\mathbf{R}_1^I\right)_A$, $\left(\mathbf{R}_1^I\right)_B$, $\left(\mathbf{R}_1^I\right)_C$, and $\left(\mathbf{R}_1^I\right)_D$, using the TRIAD algorithm. By construction two of the latter candidates are identical, because the rotation $\mathbf{R}_1^I$ is the same whether we compute it using $\mathbf{R}_2^1$ or $\mathbf{R}_3^1$. Hence, we compare $\left(\mathbf{R}_1^I\right)_A$ with $\left(\mathbf{R}_1^I\right)_C$ and $\left(\mathbf{R}_1^I\right)_D$, and we compare $\left(\mathbf{R}_1^I\right)_B$ with $\left(\mathbf{R}_1^I\right)_C$ and $\left(\mathbf{R}_1^I\right)_D$. From the comparisons we find two identical candidates for $\mathbf{R}_1^I$, which together with the associated candidates for $\mathbf{R}_2^1$ and $\mathbf{R}_3^1$ correspond to the solution. The disambiguation process just described is depicted in Fig. 2. Finally, we determine $\mathbf{R}_3^2$, $\mathbf{R}_2^I$ and $\mathbf{R}_3^I$ resorting to the product between the rotations previously computed, thus completing the attitude solution set.

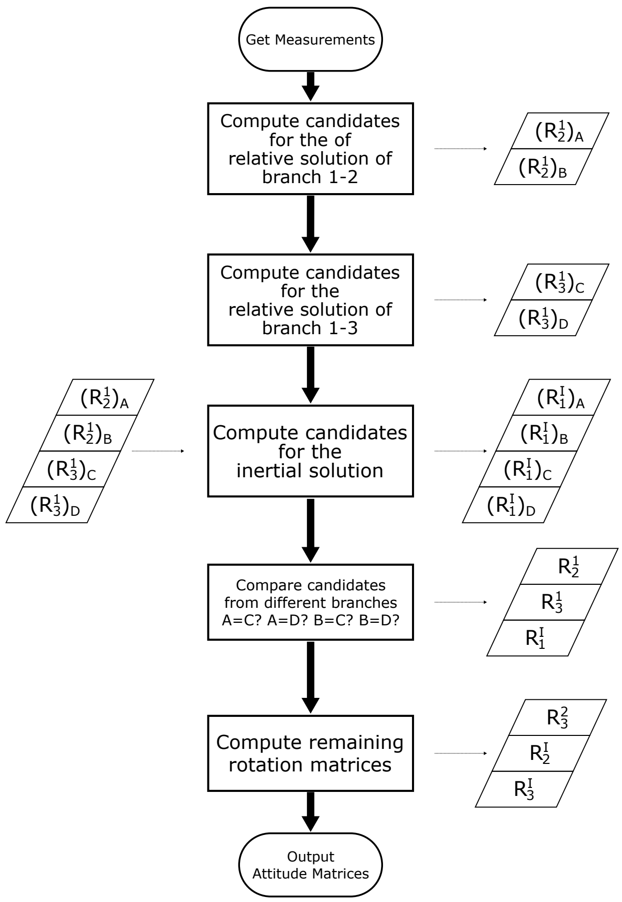

Figure 2: Algorithm flowchart

3.1. Relative attitude solution

The formation is first considered as two branches, each corresponding to two vehicles: the chief and each of the deputies. Each branch is used to find the corresponding candidates for the relative attitude between the vehicles of the branch.

3.1.1. Solution for $\mathbf{R}_2^1$

In this section, the relation between vehicle 1 and vehicle 2 is considered. Two constraints of the problem are taken into account to determine the candidates for $\mathbf{R}_2^1$. The first relation translates the anti-parallel association between the LOS vectors of the two vehicles, because they are both measuring opposing directions, as given by

$$-\mathbf{d}_{1/2} = \mathbf{R}_2^1 \mathbf{d}_{2/1} \ . \tag{2}$$

The second relation is associated with the conservation of the angle between two vectors, regardless of the coordinate frame, provided that it is the same for both. Consequently,

$${}^I\mathbf{d}_1^T \, {}^I\mathbf{d}_2 = \mathbf{d}_1^T \mathbf{R}_2^1 \mathbf{d}_2 \ . \tag{3}$$

The problem of finding $\mathbf{R}_2^1$ is addressed by decomposing this rotation matrix into a product of two rotations, each parametrized by an angle and an axis, which are determined using (2) and (3). This decomposition is possible because the product of two proper rotation matrices is still a proper rotation matrix. This rotation decomposition is defined as

$$\mathbf{R}_2^1 := \mathbf{R}\left(\theta_2, \mathbf{n}_2\right) \mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \ . \tag{4}$$

The proposed solution considers $\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)$ as a rotation that verifies (2). There are infinite possibilities for the parameters of this rotation matrix, because the relation leaves one degree of freedom undetermined. Then, $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right)$ must be such that both (2) and (3) are verified by the product $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right) \mathbf{R}\left(\theta_1, \mathbf{n}_1\right)$.

Since there are infinite possibilities for θ_1 and $\mathbf{n}_1$ that allow the respective rotation to verify (2), a choice is made. The following lemma addresses this choice.

Lemma 1. *Suppose that* $\mathbf{d}_1 \neq \pm \mathbf{d}_{1/2}$. *Then, the transformation* $\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)$ *that verifies*

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \mathbf{d}_{2/1} = -\mathbf{d}_{1/2} \tag{5}$$

is defined as

$$\theta_1 := \pi \tag{6}$$

and

$$\begin{cases} \mathbf{n}_1 := \frac{\mathbf{d}_{2/1} - \mathbf{d}_{1/2}}{\|\mathbf{d}_{2/1} - \mathbf{d}_{1/2}\|} & , \text{ for } \mathbf{d}_{2/1} \neq \mathbf{d}_{1/2} \\ \mathbf{n}_1 := \frac{\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1}{\|\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\|} & , \text{ for } \mathbf{d}_{2/1} = \mathbf{d}_{1/2} \end{cases} \ . \tag{7}$$

Proof. The proof follows by direct computation for both cases. Firstly, suppose that $\mathbf{d}_{2/1} \neq \mathbf{d}_{1/2}$ and take the left side of (5), considering the parameters from (6) and (7), as given by

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \mathbf{d}_{2/1} = \mathbf{R}\left(\pi, \frac{\mathbf{d}_{2/1} - \mathbf{d}_{1/2}}{\|\mathbf{d}_{2/1} - \mathbf{d}_{1/2}\|}\right) \mathbf{d}_{2/1} \ .$$

Then, expand it using (1), which, considering the value of the trigonometric functions, gives

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \mathbf{d}_{2/1} = \left[-\mathbf{I} + 2\frac{\left(\mathbf{d}_{2/1} - \mathbf{d}_{1/2}\right)\left(\mathbf{d}_{2/1} - \mathbf{d}_{1/2}\right)^T}{\|\mathbf{d}_{2/1} - \mathbf{d}_{1/2}\|^2}\right] \mathbf{d}_{2/1} \ .$$

Using the distributive property and rearranging, it follows that

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \mathbf{d}_{2/1} = -\mathbf{d}_{2/1} + 2\frac{\left(\mathbf{d}_{2/1}^T \mathbf{d}_{2/1} - \mathbf{d}_{1/2}^T \mathbf{d}_{2/1}\right)}{\|\mathbf{d}_{2/1} - \mathbf{d}_{1/2}\|^2}\left(\mathbf{d}_{2/1} - \mathbf{d}_{1/2}\right) \ . \tag{8}$$

Afterwards, (8) is rewritten as

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \mathbf{d}_{2/1} = -\mathbf{d}_{2/1} + \frac{2\left(1 - \mathbf{d}_{1/2}^T \mathbf{d}_{2/1}\right)}{\|\mathbf{d}_{2/1} - \mathbf{d}_{1/2}\|^2}\left(\mathbf{d}_{2/1} - \mathbf{d}_{1/2}\right) \ .$$

Notice that

$$\|\mathbf{d}_{2/1} - \mathbf{d}_{1/2}\|^2 = 2\left(1 - \mathbf{d}_{1/2}^T \mathbf{d}_{2/1}\right) \ ,$$

hence,

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \mathbf{d}_{2/1} = -\mathbf{d}_{1/2} \ .$$

Thus, the first part of the proof is concluded. Now, suppose instead that $\mathbf{d}_{2/1} = \mathbf{d}_{1/2}$, which means that

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \mathbf{d}_{2/1} = \mathbf{R}\left(\pi, \frac{\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1}{\|\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\|}\right) \mathbf{d}_{2/1} \ . \tag{9}$$

Then, using (1) on the right side of (9) gives

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \mathbf{d}_{2/1} = \left(-\mathbf{I} + 2\frac{\left(\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\right)\left(\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\right)^T}{\|\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\|^2}\right) \mathbf{d}_{2/1} \ . \tag{10}$$

Next, from the condition $\mathbf{d}_{2/1} = \mathbf{d}_{1/2}$ and after some rearrangements, it follows that (10) is rewritten as

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \mathbf{d}_{2/1} = -\mathbf{d}_{1/2} - 2\frac{\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1 \mathbf{d}_1^T \mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_{1/2}}{\|\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\|^2} \ . \tag{11}$$

Finally, noticing that $\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_{1/2} = \mathbf{0}$, (11) is given as

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \mathbf{d}_{2/1} = -\mathbf{d}_{1/2} \ ,$$

thus concluding the proof. □

The choice for θ_1 and $\mathbf{n}_1$, which is expressed in Lemma 1, influences the parameters of $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right)$. Firstly, the axis $\mathbf{n}_2$ must be one that ensures that $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right) \mathbf{R}\left(\theta_1, \mathbf{n}_1\right)$ verifies (2). Furthermore, the angle θ_2 must be such that (3) is verified by $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right) \mathbf{R}\left(\theta_1, \mathbf{n}_1\right)$. Thus, both (2) and (3) are verified by this product. The next lemma gives the parameters of $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right)$.

Lemma 2. *Consider* $\mathbf{R}(\theta_1, \mathbf{n}_1)$ *parametrized by* (6) *and* (7). *Assume also that*

$$\mathbf{d}_1 \neq \pm \mathbf{d}_{1/2}$$

and

$$\mathbf{d}_2 \neq \pm \mathbf{d}_{2/1} \ .$$

Then, the transformation $\mathbf{R}(\theta_2, \mathbf{n}_2)$ *that verifies*

$$\mathbf{R}(\theta_2, \mathbf{n}_2)\, \mathbf{R}(\theta_1, \mathbf{n}_1)\, \mathbf{d}_{2/1} = -\mathbf{d}_{1/2}$$

and

$${}^{I}\mathbf{d}_1^T\, {}^{I}\mathbf{d}_2 = \mathbf{d}_1^T \mathbf{R}(\theta_2, \mathbf{n}_2)\, \mathbf{R}(\theta_1, \mathbf{n}_1)\, \mathbf{d}_2 \ ,$$

is described by

$$\theta_2 := \mathit{atan2}(a_{s_{12}}, a_{c_{12}}) \pm \arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right) \quad (12)$$

and

$$\mathbf{n}_2 := -\mathbf{d}_{1/2} \ , \quad (13)$$

with

$$\begin{cases} a_{p_{12}} & := \mathbf{d}_1^T (\mathbf{d}_{1/2})(\mathbf{d}_{1/2})^T \mathbf{d}_2^* - {}^{I}\mathbf{d}_1^T\, {}^{I}\mathbf{d}_2 \\ a_{c_{12}} & := \mathbf{d}_1^T \mathbf{S}\left(\mathbf{d}_{1/2}\right)^2 \mathbf{d}_2^* \\ a_{s_{12}} & := \mathbf{d}_1^T \mathbf{S}\left(-\mathbf{d}_{1/2}\right) \mathbf{d}_2^* \end{cases} ,$$

where

$$\mathbf{d}_2^* := \mathbf{R}(\theta_1, \mathbf{n}_1)\, \mathbf{d}_2 \ . \quad (14)$$

Proof. For any rotation angle, the rotation of a vector around that very same vector does not change this vector, and therefore $\mathbf{n}_2$ is limited to $\left(\pm \mathbf{d}_{1/2}\right)$, since (2) and (5) are both verified. The sign choice of the axis only affects the signal of θ_2, so the negative sign, *i.e.*, $\mathbf{n}_2 = -\mathbf{d}_{1/2}$, was chosen for coherence with the sign of (2). Now, the proof for the value of θ_2 is made by direct computation. Firstly, consider the alternative rotation representation obtained from substituting $\mathbf{S}(\mathbf{x})^2 = -\mathbf{I} + \mathbf{x}\mathbf{x}^T$ in (1), which gives

$$\mathbf{R}(\theta, \mathbf{x}) = \mathbf{x}\mathbf{x}^T - \cos(\theta)\mathbf{S}(\mathbf{x})^2 - \sin(\theta)\mathbf{S}(\mathbf{x}) \ . \quad (15)$$

Then, considering that $\mathbf{R}(\theta_1, \mathbf{n}_1)$ is as given in Lemma 1, substitute $\mathbf{R}_2^1 = \mathbf{R}\left(\theta_2, -\mathbf{d}_{1/2}\right) \mathbf{R}(\theta_1, \mathbf{n}_1)$ in (3), which yields

$$\mathbf{d}_1^T \mathbf{R}\left(\theta_2, -\mathbf{d}_{1/2}\right) \mathbf{R}(\theta_1, \mathbf{n}_1)\, \mathbf{d}_2 = {}^{I}\mathbf{d}_1^T\, {}^{I}\mathbf{d}_2 \ ,$$

which, applying the alternative definition from (15) and also the notation from (14), is given as

$$\mathbf{d}_1^T \left[(-\mathbf{d}_{1/2})(-\mathbf{d}_{1/2})^T - \cos(\theta_2)\mathbf{S}\left(-\mathbf{d}_{1/2}\right)^2 - \sin(\theta_2)\mathbf{S}\left(-\mathbf{d}_{1/2}\right)\right] \mathbf{d}_2^* = {}^{I}\mathbf{d}_1^T\, {}^{I}\mathbf{d}_2 \ . \quad (16)$$

Making use of the coefficients defined in (2), it follows that (16) is a trigonometric equation, which is expressed as

$$a_{p_{12}} = a_{c_{12}} \cos(\theta_2) + a_{s_{12}} \sin(\theta_2) \ .$$

The solution of this equation is given in Lemma 6 (see Appendix A) and results in (12). □

The combination of Lemmas 1 and 2 concludes the partial solution, as stated in the following corollary.

Corollary 1. *Assume that*

$$\mathbf{d}_1 \neq \pm \mathbf{d}_{1/2}$$

and

$$\mathbf{d}_2 \neq \pm \mathbf{d}_{2/1} \ .$$

Then, $\mathbf{R}_2^1$ *is given by*

$$\mathbf{R}_2^1 = \mathbf{R}(\theta_2, \mathbf{n}_2)\, \mathbf{R}(\theta_1, \mathbf{n}_1) \ , \quad (17)$$

with θ_1, $\mathbf{n_1}$, θ_2, *and* $\mathbf{n_2}$ *respectively given by* (6), (7), (12), *and* (13).

Proof. This result follows directly from Lemmas 1 and 2. □

Remark. Notice that there are, in general, two solutions for $\mathbf{R}(\theta_2, \mathbf{n}_2)$, which result in two candidates, $\left(\mathbf{R}_2^1\right)_A$ and $\left(\mathbf{R}_2^1\right)_B$, that satisfy both (2) and (3), when $\mathbf{d}_1 \neq \pm \mathbf{d}_{1/2}$ and $\mathbf{d}_2 \neq \pm \mathbf{d}_{2/1}$.

Interestingly enough, both possible solutions are related to each other. This relation is addressed in the next corollary.

Corollary 2. *In the conditions of Corollary 1, recall the angle* θ_2 *defined by* (12) *and, without loss of generality, consider that* $(\theta_2)_A$ *is the angle associated with* $\left(\mathbf{R}_2^1\right)_A$ *and* $(\theta_2)_B$ *is the angle associated with* $\left(\mathbf{R}_2^1\right)_B$, *given by*

$$(\theta_2)_A := \mathit{atan2}(a_{s_{12}}, a_{c_{12}}) + \arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right) \quad (18)$$

and

$$(\theta_2)_B := \mathit{atan2}(a_{s_{12}}, a_{c_{12}}) - \arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right) \ . \quad (19)$$

Then, the relation between $\left(\mathbf{R}_2^1\right)_A$ *and* $\left(\mathbf{R}_2^1\right)_B$ *is expressed as*

$$\left(\mathbf{R}_2^1\right)_A = \mathbf{R}\left(2\arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right), -\mathbf{d}_{1/2}\right) \left(\mathbf{R}_2^1\right)_B \ . \quad (20)$$

Proof. First, for visual ease, define

$$\theta_{12} := 2\arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right) \ .$$

Therefore, comparing (18) and (19) allows to write

$$(\theta_2)_A = (\theta_2)_B + \theta_{12} \ . \quad (21)$$

Taking into consideration the decomposition (4) and the solution (17), then $\left(\mathbf{R}_2^1\right)_A$ and $\left(\mathbf{R}_2^1\right)_B$ are, respectively, expressed as

$$\left(\mathbf{R}_2^1\right)_A = \mathbf{R}\left(\left(\theta_2\right)_A, -\mathbf{d}_{1/2}\right)\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \tag{22}$$

and

$$\left(\mathbf{R}_2^1\right)_B = \mathbf{R}\left(\left(\theta_2\right)_B, -\mathbf{d}_{1/2}\right)\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \ . \tag{23}$$

Substituting (21) in (22) gives

$$\left(\mathbf{R}_2^1\right)_A = \mathbf{R}\left(\left(\theta_2\right)_B + \theta_{12}, -\mathbf{d}_{1/2}\right)\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \ . \tag{24}$$

Then, (24) can be rewritten as

$$\left(\mathbf{R}_2^1\right)_A = \mathbf{R}\left(\theta_{12}, -\mathbf{d}_{1/2}\right)\mathbf{R}\left(\left(\theta_2\right)_B, -\mathbf{d}_{1/2}\right)\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \ . \tag{25}$$

Substituting (23) in (25) readily yields (20). □

Conveniently, the two cases where the solution in Corollary 1 does not work are degeneracies of the problem. These degenerate cases have infinite solutions, as it is shown in the next lemma.

Lemma 3. *If*

$$\mathbf{d}_1 = \pm\mathbf{d}_{1/2} \tag{26a}$$

or

$$\mathbf{d}_2 = \pm\mathbf{d}_{2/1} \ , \tag{26b}$$

then there are infinite solutions for $\mathbf{R}_2^1$*.*

Proof. Fix $\mathbf{R}_2^1$, such that both (2) and (3) are verified, and define

$$\mathbf{A}\left(\theta\right) := \mathbf{R}\left(\theta, -\mathbf{d}_{1/2}\right)\mathbf{R}_2^1 \ ,$$

where $\theta \in \mathbb{R}$ is arbitrary. It will be shown by direct computation that this arbitrary rotation satisfies both (2) and (3), if either (26a) or (26b) hold. Indeed,

$$\mathbf{A}\left(\theta\right)\mathbf{d}_{2/1} = \mathbf{R}\left(\theta, -\mathbf{d}_{1/2}\right)\mathbf{R}_2^1\mathbf{d}_{2/1} \ ,$$

which, using (2), simplifies to

$$\mathbf{A}\left(\theta\right)\mathbf{d}_{2/1} = -\mathbf{R}\left(\theta, -\mathbf{d}_{1/2}\right)\mathbf{d}_{1/2} \ ,$$

or, equivalently,

$$\mathbf{A}\left(\theta\right)\mathbf{d}_{2/1} = -\mathbf{d}_{1/2} \ . \tag{27}$$

Thus, there is a rotation with an arbitrary angle that keeps the result of (2) unchanged. Up until this point, none of the assumptions of this lemma were used, which means that (2) can be satisfied by an infinite number of rotations, regardless of the situation. Next, compute

$$\mathbf{d}_1^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = \mathbf{d}_1^T\mathbf{R}\left(\theta, -\mathbf{d}_{1/2}\right)\mathbf{R}_2^1\mathbf{d}_2 \ . \tag{28}$$

Without loss of generality, the first assumption is used with the positive sign, *i.e.*, $\mathbf{d}_1 = \mathbf{d}_{1/2}$, which applied to (28) yields

$$\mathbf{d}_1^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = \mathbf{d}_{1/2}^T\mathbf{R}\left(\theta, -\mathbf{d}_{1/2}\right)\mathbf{R}_2^1\mathbf{d}_2 \ ,$$

which simplifies to

$$\mathbf{d}_1^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = \mathbf{d}_{1/2}^T\mathbf{R}_2^1\mathbf{d}_2 \ . \tag{29}$$

Next, recalling that $\mathbf{d}_1 = \mathbf{d}_{1/2}$, then (3) can be substituted in (29) yielding

$$\mathbf{d}_1^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = {}^I\mathbf{d}_1^T\,{}^I\mathbf{d}_2 \ . \tag{30}$$

This concludes the first part of the proof. Indeed, (27) and (30) are verified for all θ, as long as $\mathbf{d}_1 = \mathbf{d}_{1/2}$. Taking, instead, the assumption that $\mathbf{d}_2 = \mathbf{d}_{2/1}$, assuming the positive sign without loss of generality, and substituting in (28) gives

$$\mathbf{d}_1^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = \mathbf{d}_1^T\mathbf{R}\left(\theta, -\mathbf{d}_{1/2}\right)\mathbf{R}_2^1\mathbf{d}_{2/1} \ . \tag{31}$$

Using (2) in (31) allows to write

$$\mathbf{d}_1^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = -\mathbf{d}_1^T\mathbf{R}\left(\theta, -\mathbf{d}_{1/2}\right)\mathbf{d}_{1/2} \ ,$$

which, recalling that a rotation of a vector around an axis parallel to that vector results in that same vector, gives

$$\mathbf{d}_1^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = \mathbf{d}_1^T\mathbf{R}_2^1\mathbf{d}_{2/1} \ .$$

Finally, using $\mathbf{d}_2 = \mathbf{d}_{2/1}$ and then applying (3) yields

$$\mathbf{d}_1^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = {}^I\mathbf{d}_1^T\,{}^I\mathbf{d}_2 \ .$$

Hence, there are infinite solutions for $\mathbf{A}\left(\theta\right)$ that satisfy both (2) and (3) when $\mathbf{d}_2 = \pm\mathbf{d}_{2/1}$. □

3.1.2. Partial solution for $\mathbf{R}_3^1$

The other branch of the problem includes vehicle 1 and vehicle 3. The solution for $\mathbf{R}_3^1$ is analogous to the one of $\mathbf{R}_2^1$, since both branches have equivalent structures, and therefore it is omitted. Moreover, the degenerate cases and the relation between the two candidates for $\mathbf{R}_3^1$ are also analogous, being omitted as well. Thus, we find $\left(\mathbf{R}_3^1\right)_C$ and $\left(\mathbf{R}_3^1\right)_D$ using the previous lemmas with the appropriate measurements of the branch with vehicles 1 and 3.

3.2. Inertial attitude solution

The problem of finding $\mathbf{R}_1^I$ is addressed using the relations between the inertial measurements and the respective reference, which are given by

$${}^I\mathbf{d}_1 = \mathbf{R}_1^I\ \mathbf{d}_1 \ , \tag{32}$$

$${}^I\mathbf{d}_2 = \mathbf{R}_1^I\ \mathbf{R}_2^1\ \mathbf{d}_2 \ , \tag{33}$$

and

$${}^I\mathbf{d}_3 = \mathbf{R}_1^I\ \mathbf{R}_3^1\ \mathbf{d}_3 \ . \tag{34}$$

3.2.1. Solution for $\mathbf{R}_1^I$

This new problem has only one solution for each candidate, assuming that there are no degeneracies. Moreover, it corresponds to the TRIAD framework, because there are two pairs of non-collinear vectors represented in two different coordinate frames [4]. Therefore, $\left(\mathbf{R}_1^I\right)_A$, $\left(\mathbf{R}_1^I\right)_B$, $\left(\mathbf{R}_1^I\right)_C$, and $\left(\mathbf{R}_1^I\right)_D$ are obtained using this algorithm.

Assuming that both ${}^I\mathbf{d}_1 \neq {}^I\mathbf{d}_2$ and ${}^I\mathbf{d}_1 \neq {}^I\mathbf{d}_3$, then the direct application of this algorithm gives the candidates for $\mathbf{R}_1^I$ using either $\mathbf{R}_2^1$ or $\mathbf{R}_3^1$. Indeed, from the first branch

$$\begin{aligned}\mathbf{R}_1^I = {}^I\mathbf{d}_1\mathbf{d}_1^T &+ \frac{{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_2}{\|{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_2\|}\left(\frac{\mathbf{d}_1 \times \mathbf{R}_2^1\mathbf{d}_2}{\|\mathbf{d}_1 \times \mathbf{R}_2^1\mathbf{d}_2\|}\right)^T \\ &+ \left({}^I\mathbf{d}_1 \times \frac{{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_2}{\|{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_2\|}\right)\left(\mathbf{d}_1 \times \frac{\mathbf{d}_1 \times \mathbf{R}_2^1\mathbf{d}_2}{\|\mathbf{d}_1 \times \mathbf{R}_2^1\mathbf{d}_2\|}\right)^T \ ,\end{aligned}$$

where $\left(\mathbf{R}_1^I\right)_A$ and $\left(\mathbf{R}_1^I\right)_B$ are obtained by replacing $\mathbf{R}_2^1$ with $\left(\mathbf{R}_2^1\right)_A$ and $\left(\mathbf{R}_2^1\right)_B$, respectively. For the other branch the TRIAD gives

$$\begin{aligned}\mathbf{R}_1^I = {}^I\mathbf{d}_1\mathbf{d}_1^T &+ \frac{{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_3}{\|{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_3\|}\left(\frac{\mathbf{d}_1 \times \mathbf{R}_3^1\mathbf{d}_3}{\|\mathbf{d}_1 \times \mathbf{R}_3^1\mathbf{d}_3\|}\right)^T \\ &+ \left({}^I\mathbf{d}_1 \times \frac{{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_3}{\|{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_3\|}\right)\left(\mathbf{d}_1 \times \frac{\mathbf{d}_1 \times \mathbf{R}_3^1\mathbf{d}_3}{\|\mathbf{d}_1 \times \mathbf{R}_3^1\mathbf{d}_3\|}\right)^T \ ,\end{aligned}$$

where $\left(\mathbf{R}_1^I\right)_C$ and $\left(\mathbf{R}_1^I\right)_D$ are obtained by replacing $\mathbf{R}_3^1$ with $\left(\mathbf{R}_3^1\right)_C$ and $\left(\mathbf{R}_3^1\right)_D$, respectively.

The TRIAD assumes that the two vectors used are not collinear, because that corresponds to a degeneracy, which has infinite solutions. For this situation the degenerate cases are shown in the next two lemmas. Although this is well known, it is presented here for the sake of completeness, applied to the problem at hand.

Lemma 4. *Let $\mathbf{R}_1^I$ be the rotation matrix that satisfies both* (32) *and* (33)*. If*

$$ {}^I\mathbf{d}_1 = \pm{}^I\mathbf{d}_2 \ , \tag{35}$$

then $\mathbf{R}_1^I$ has an infinite number of solutions.

Proof. Fix $\mathbf{R}_1^I$ such that both (32) and (33) are verified and consider a new candidate defined by

$$\mathbf{A}(\theta) := \mathbf{R}\left(\theta, {}^I\mathbf{d}_1\right)\mathbf{R}_1^I \ ,$$

where $\theta \in \mathbb{R}$ is an arbitrary angle. It will be shown by direct computation that this arbitrary rotation satisfies both (32) and (33), assuming (35). To that end, compute

$$\mathbf{A}(\theta)\,\mathbf{d}_1 = \mathbf{R}\left(\theta, {}^I\mathbf{d}_1\right)\mathbf{R}_1^I\mathbf{d}_1 \ ,$$

which, using (32), becomes

$$\mathbf{A}(\theta)\,\mathbf{d}_1 = \mathbf{R}\left(\theta, {}^I\mathbf{d}_1\right){}^I\mathbf{d}_1 \ ,$$

or, equivalently,

$$\mathbf{A}(\theta)\,\mathbf{d}_1 = {}^I\mathbf{d}_1 \ .$$

Next, compute

$$\mathbf{A}(\theta)\,\mathbf{R}_2^1\mathbf{d}_2 = \mathbf{R}\left(\theta, {}^I\mathbf{d}_1\right)\mathbf{R}_1^I\mathbf{R}_2^1\mathbf{d}_2 \ . \tag{36}$$

Applying (33) in (36) results in

$$\mathbf{A}(\theta)\,\mathbf{R}_2^1\mathbf{d}_2 = \mathbf{R}\left(\theta, {}^I\mathbf{d}_1\right){}^I\mathbf{d}_2 \ .$$

Without loss of generality, the assumption with positive sign, *i.e.*, ${}^I\mathbf{d}_1 = {}^I\mathbf{d}_2$, is now used, which gives

$$\mathbf{A}(\theta)\,\mathbf{R}_2^1\mathbf{d}_2 = \mathbf{R}\left(\theta, {}^I\mathbf{d}_1\right){}^I\mathbf{d}_1 \ ,$$

or, equivalently,

$$\mathbf{A}(\theta)\,\mathbf{R}_2^1\mathbf{d}_2 = {}^I\mathbf{d}_1 \ .$$

Finally, using ${}^I\mathbf{d}_1 = {}^I\mathbf{d}_2$ again, gives

$$\mathbf{A}(\theta)\,\mathbf{R}_2^1\mathbf{d}_2 = {}^I\mathbf{d}_2 \ .$$

This concludes the proof, since there are infinite possibilities for $\mathbf{R}_1^I$ that satisfy both (32) and (33) when ${}^I\mathbf{d}_1 = \pm{}^I\mathbf{d}_2$. ☐

Lemma 5. *Let $\mathbf{R}_1^I$ be the rotation matrix that satisfies both* (32) *and* (34)*. If*

$$ {}^I\mathbf{d}_1 = \pm{}^I\mathbf{d}_3$$

then $\mathbf{R}_1^I$ has an infinite number of solutions.

Proof. This proof is analogous to Lemma 4 and thus it is omitted. ☐

3.3. Comparison

Considering that the problem is not degenerate, then, from the four candidates for $\mathbf{R}_1^I$, there is a pair of identical matrices emerging from the two different branches. That means that one of the following equations must be verified: $\left(\mathbf{R}_1^I\right)_A = \left(\mathbf{R}_1^I\right)_C$, $\left(\mathbf{R}_1^I\right)_A = \left(\mathbf{R}_1^I\right)_D$, $\left(\mathbf{R}_1^I\right)_B = \left(\mathbf{R}_1^I\right)_C$, or $\left(\mathbf{R}_1^I\right)_B = \left(\mathbf{R}_1^I\right)_D$. This comparison is made resorting to the rotation defined by

$$\tilde{\mathbf{R}_1^I}\left(\theta, \mathbf{x}\right) := \left(\mathbf{R}_1^I\right)_X\left(\mathbf{R}_1^I\right)_Y^T \ , \tag{37}$$

where $\left(\mathbf{R}_1^I\right)_X$ and $\left(\mathbf{R}_1^I\right)_Y$ represent different candidates. This rotation is an identity matrix when $\left(\mathbf{R}_1^I\right)_X$ and $\left(\mathbf{R}_1^I\right)_Y$ are identical, which means that $\theta = 0$. Therefore, the absolute value of θ is used as the comparison parameter that gives the proximity between each pair of $\mathbf{R}_1^I$ candidates.

The trace of (37) is used to find θ, since the trace of a square matrix is the sum of its eigenvalues [18], which in the case of a rotation matrix is given as

$$\begin{aligned}\operatorname{trace}\left(\tilde{\mathbf{R}_1^I}\left(\theta, \mathbf{x}\right)\right) &= 1 + \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta} \\ &= 1 + 2\cos\left(\theta\right) \ .\end{aligned}$$

Then, defining $\phi = |\theta|$ and rearranging it, the comparison parameter is expressed as

$$\phi = \left|\arccos\left(\frac{\operatorname{trace}\left(\tilde{\mathbf{R}_1^I}\left(\theta, \mathbf{x}\right)\right) - 1}{2}\right)\right| \ . \tag{38}$$

3.4. Solution completion

The remaining attitude matrices are computed using the rotation matrices previously determined, as given by

$$\mathbf{R}_3^2 = \left(\mathbf{R}_2^1\right)^T \mathbf{R}_3^1 \ ,$$

$$\mathbf{R}_2^I = \mathbf{R}_1^I \mathbf{R}_2^1 \ ,$$

and

$$\mathbf{R}_3^I = \mathbf{R}_1^I \mathbf{R}_3^1 \ .$$

3.5. Solution in the presence of noise

In practice, the measurements will be corrupted by noise and hence the attitude rotation matrices that are determined will deviate from the actual attitude rotation matrices. This deviation creates a new problem for the solution described before.

In the comparison of the $\mathbf{R}_1^I$ candidates, it was assumed that the correct candidates would be exactly identical. Therefore, the comparison criteria must be changed in order to work in the presence of noise. The comparison parameter, ϕ, determined in (38), gets closer to zero as the two rotations get closer. So, with noise, instead of looking for $\phi = 0$, the correct solution is defined by the smaller ϕ, noticing that $\phi \geq 0$.

Furthermore, to improve the accuracy of $\mathbf{R}_1^I$, the average between the candidates with the smallest ϕ of each branch is used as the solution, whereas the correspondent candidates for $\mathbf{R}_2^1$ and $\mathbf{R}_3^1$ are used as firstly calculated.

The average used is matrix-based and is rooted on the Singular Value Decomposition (SVD) of the sum of both candidates [19]. Hence, considering the SVD given by

$$\frac{\left(\mathbf{R}_1^I\right)_X + \left(\mathbf{R}_1^I\right)_Y}{2} = \mathbf{XDY},$$

the average $\mathbf{R}_1^I$ is given by $\left(\mathbf{R}_1^I\right)_{avg} = \mathbf{XY}$.

4. Simulations

The solution for the problem proposed is now tested in a simulated environment, where noise is added to reproduce an approximation of a realistic scenario.

4.1. Sensor model

The sensor measurement model used in the simulations is based on a focal plane detector [15], thus it is assumed that the sensors used are vision-based, even for the inertial measurements. Denoting the image-space observation by the vector $\mathbf{m} \equiv [\chi \ \psi]^T$, then the measurement model is given by[1]

$$\tilde{\mathbf{m}} = \mathbf{m} + \mathbf{n} \ , \tag{39}$$

where $\tilde{\mathbf{m}}$ is the measurement and $\mathbf{n}$ is the random noise. The noise model describing the uncertainty of the image-space observations is supposed to follow a zero mean Gaussian distribution,

$$\mathbf{n} \sim \mathcal{N}\left(\mathbf{0}, R_{\mathrm{FOCAL}}\right) \ ,$$

with the covariance of the focal plane given by [20]

$$R_{\mathrm{FOCAL}} = \frac{\sigma^2}{1 + d\left(\chi^2 + \psi^2\right)} \begin{bmatrix} \left(1 + d\chi^2\right)^2 & \left(d\chi\psi\right)^2 \\ \left(d\chi\psi\right)^2 & \left(1 + d\psi^2\right)^2 \end{bmatrix} , \tag{40}$$

where σ^2 is the variance of the measurement errors associated with χ and ψ, and d is a parameter on the order of 1.

The focal length is assumed to be unitary and the sensor boresight is assumed to be the z-axis. Hence the measurement vector in the object space and sensor frame is given as

$${}^s\mathbf{d} = \frac{1}{\sqrt{1 + \chi^2 + \psi^2}} \begin{bmatrix} \chi \\ \psi \\ 1 \end{bmatrix} . \tag{41}$$

4.2. Simulation setup

The setup that is here proposed considers a fixed formation chosen a priori, which is used in 1000 runs of a Monte Carlo simulation. The estimates for the rotation matrices of the problem are found in each trial.

The true LOS vectors are given by

$${}^I\mathbf{d}_{1/2} = \begin{bmatrix} 0 \\ -\sin(30°) \\ -\cos(30°) \end{bmatrix} , {}^I\mathbf{d}_{1/3} = \begin{bmatrix} 0 \\ \sin(30°) \\ -\cos(30°) \end{bmatrix} , \tag{42a}$$

whereas the inertial vectors are given by

$${}^I\mathbf{d}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , {}^I\mathbf{d}_2 = \begin{bmatrix} 0 \\ \cos(60°) \\ -\sin(60°) \end{bmatrix} ,$$

$${}^I\mathbf{d}_3 = \begin{bmatrix} \cos(60°)\sin(30°) \\ \cos(60°)\cos(30°) \\ -\sin(60°) \end{bmatrix} . \tag{42b}$$

The true rotation matrices were chosen as

$$\mathbf{R}_I^1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \mathbf{R}_I^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} , \mathbf{R}_I^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} . \tag{42c}$$

The remaining matrices are a combination of those shown in (42c) and the remaining vectors of the problems are generated from (42a), (42b), and (42c).

The measurement vectors are computed in the focal plane, which contains noise as described by the sensor model. Therefore, the true vectors have to be transformed into the focal plane frame and then noise is added to each of them using (40), with $\sigma = 17 \times 10^{-6}$ rad. The chief is assumed to have three sensors, whereas deputies have only

[1] The image-space frame is the 2D coordinate system of the sensor, whereas the object-space frame is the vehicle body coordinate system.

two. The orthogonal transformations, from the respective body frame to the sensor frame, were chosen as

$$\mathbf{R}_1^{s_{\mathbf{d}_{1/2}}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{R}_1^{s_{\mathbf{d}_{1/3}}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \tag{43a}$$

$$\mathbf{R}_2^{s_{\mathbf{d}_{2/1}}} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{R}_3^{s_{\mathbf{d}_{3/1}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{43b}$$

$$\mathbf{R}_1^{s_{\mathbf{d}_1}} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{R}_2^{s_{\mathbf{d}_2}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{43c}$$

$$\mathbf{R}_3^{s_{\mathbf{d}_3}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \tag{43d}$$

with the superscript representing each sensor, and the subscript identifying the vehicle body frame. Note that the inertial vectors in the inertial frame are assumed to be known exactly, since these are the references for the inertial sensors, hence no noise is added to these vectors.

The process of adding noise to the measurements follows these steps. First, the true measurement vectors are transformed using (43) into the form of (41). Then, the respective image-space variables χ and ψ are computed. Next, $\mathbf{n}$ is sampled, recalling (40). Finally, the object-space measurements are obtained after applying (39), to add the noise component, and reusing (41) with the components of $\tilde{\mathbf{m}}$.

The standard deviations in the focal plane are given for both coordinates of each measurement in Table 1.

	$\mathbf{d}_{1/2}$	$\mathbf{d}_{2/1}$	$\mathbf{d}_{1/3}$	$\mathbf{d}_{3/1}$	$\mathbf{d}_1$	$\mathbf{d}_2$	$\mathbf{d}_3$
σ_χ [μrad]	8.50	8.50	8.50	8.50	17.0	14.7	15.9
σ_ψ [μrad]	34.0	34.0	34.0	34.0	17.0	19.6	18.4

Table 1: Focal plane standard deviations.

4.3. Results

The simulation results consider the error between estimates and true attitudes. The errors of each trial, for $\mathbf{R}_2^1$, $\mathbf{R}_3^1$ and $\mathbf{R}_1^I$, are given as Euler angles and are shown in Fig. 3. The respective root mean squared errors (RMSE) are given in Table 2.

	Yaw [μrad]	Pitch [μrad]	Roll [μrad]
$\mathbf{R}_2^1$	21.8319	12.4453	12.5491
$\mathbf{R}_3^1$	23.0114	11.5056	13.2582
$\mathbf{R}_1^I$	16.5815	16.7665	13.8375

Table 2: Simulation RMSE.

5. Conclusion

In this paper a new attitude determination problem is defined, considering three vehicles capable of measuring LOS and inertial vectors. The LOS vectors are constrained and, as a consequence, two of the vehicles cannot measure LOS relative to one another. The solution for this problem is devised based in the geometric relations between the measurements and resorting to a comparison between candidates for the same rotation matrix. The problem has, in general, a unique solution. However, degenerate configurations are possible, which were shown to have infinite solutions. Finally, simulations were conducted, which show the behavior of our solution in the presence of noise.

Future work will consist in the analysis of the solution in terms of the existence of a unique solution, both for the relative and the inertial attitudes. Indeed, in some specific cases, the relative attitude for each branch will be unique. On the other hand, there exist specific degenerate cases in which the inertial attitude cannot be determined uniquely, with two possible solutions. The covariance analysis of the solution is also interesting for the assessment of the performance of the proposed system in the presence of noise.

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Appendix A. Trigonometric Function Solution

In this appendix, the solution is shown for the trigonometric equation given by

$$a_p = a_c \cos(\theta) + a_s \sin(\theta) \ , \tag{A.1}$$

where a_p, a_s and a_c are known scalars. The method to solve (A.1) is found by simplifying the right side of (A.1) into a wave equation, as given in the next lemma.

Lemma 6. *Consider the trigonometric equation* (A.1) *and assume that* $a_s^2 + a_c^2 \neq 0$ *and that* $\left| \frac{a_p}{\sqrt{a_s^2+a_c^2}} \right| \leq 1$. *Then*

$$\theta = \operatorname{atan2}(a_s, a_c) \pm \arccos\left(\frac{a_p}{\sqrt{a_s^2 + a_c^2}} \right) .$$

Proof. Let $\lambda := \operatorname{atan2}(a_s,\ a_c)$, which means that

$$\sin(\lambda) = \frac{a_s}{\sqrt{a_s^2 + a_c^2}} \ , \tag{A.2a}$$

$$\cos(\lambda) = \frac{a_c}{\sqrt{a_s^2 + a_c^2}} \ . \tag{A.2b}$$

Next, rewrite (A.1) as given by

$$\frac{a_p}{\sqrt{a_s^2 + a_c^2}} = \frac{a_c}{\sqrt{a_s^2 + a_c^2}} \cos(\theta) + \frac{a_s}{\sqrt{a_s^2 + a_c^2}} \sin(\theta) \ . \tag{A.3}$$

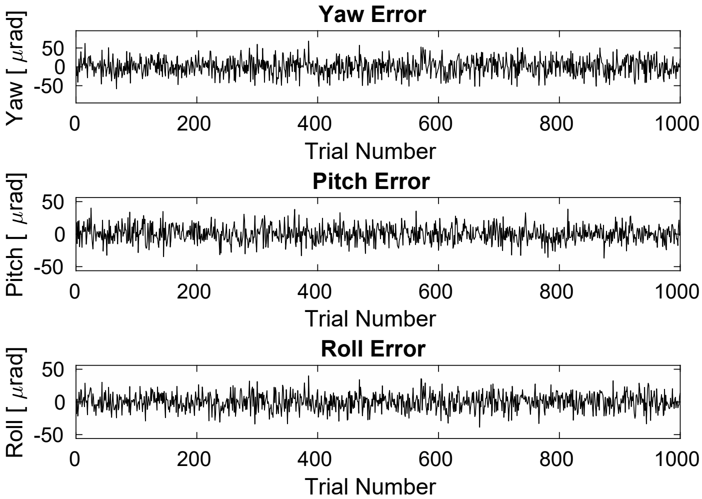
It becomes clear from (A.2) that (A.3) can be expressed as $\frac{a_p}{\sqrt{a_s^2+a_c^2}} = \cos(\lambda)\cos(\theta) + \sin(\lambda)\sin(\theta)$, which from the trigonometric identity of the cosine of the difference is given as

$$\cos(\theta - \lambda) = \frac{a_p}{\sqrt{a_s^2 + a_c^2}} \ . \tag{A.4}$$

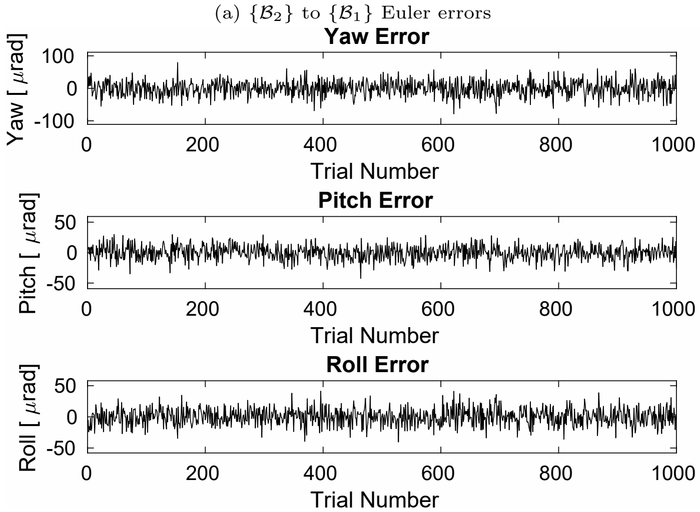
Finally, applying the inverse cosine to (A.4) leads to the solution for the angle, given by

$$\begin{aligned} \theta &= \lambda \pm \arccos\left(\frac{a_p}{\sqrt{a_s^2+a_c^2}} \right) \\ &= \operatorname{atan2}(a_s, a_c) \pm \arccos\left(\frac{a_p}{\sqrt{a_s^2+a_c^2}} \right) \ . \end{aligned}$$

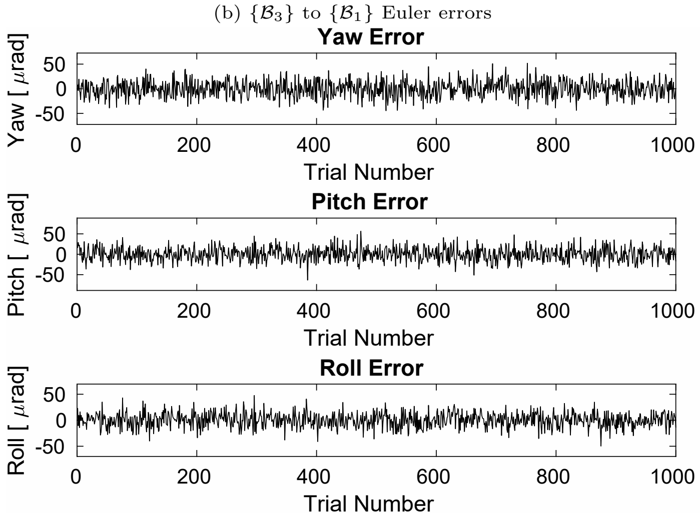
□



(a) $\{\mathcal{B}_2\}$ to $\{\mathcal{B}_1\}$ Euler errors



(b) $\{\mathcal{B}_3\}$ to $\{\mathcal{B}_1\}$ Euler errors



(c) $\{\mathcal{B}_1\}$ to $\{\mathcal{I}\}$ Euler errors

Figure 3: Relative and inertial attitude estimation error.