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Original Paper

Tightly Coupled Long Baseline / Ultra-short Baseline Integrated Navigation System

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This paper proposes a novel integrated navigation filter based on a combined Long Baseline / Ultra Short Baseline (LBL/USBL) acoustic positioning system with application to underwater vehicles. With a tightly-coupled structure, the position, linear velocity, attitude, and rate gyro bias are estimated, considering the full nonlinear system dynamics without resorting to any algebraic inversion or linearization techniques. The resulting solution ensures convergence of the estimation error to zero for all initial conditions, exponentially fast. Finally, it is shown, under simulation environment, that the filter achieves very good performance in the presence of sensor noise.

Keywords: Navigation; marine robotics; long baseline; ultra-short baseline; observability analysis; sensor fusion.

1. Introduction

Navigation systems are vital for the successful operation of autonomous vehicles. For aerial and ground vehicles the much celebrated Global Positioning System (GPS) is the usual choice, warranting aided navigation solutions such as the ones presented in Sukkarieh et al. (1999), Vik and Fossen (2001), and Batista et al. (2009), see also the references therein. In underwater scenarios other solutions must be devised due to the high attenuation that the electromagnetic field suffers. In particular, Long Baseline (LBL) and Short Baseline (SBL) acoustic positioning systems have been employed, see e.g. Whitcomb et al. (1999), Jouffroy and Opderbecke (2004), Kinsey and Whitcomb (2003), Larsen (2000), Vaganay et al. (1998), Ricordel et al. (2001), and references therein. Another commercially available solution is the GPS Intelligent Buoy (GIB) system, see Thomas (1998). Further work on the GIB underwater positioning system can be found in Alcocer et al. (2007). Position and linear velocity globally asymptotically stable (GAS) filters based on an Ultra-Short Baseline (USBL) positioning system were presented by the authors in Batista et al. (2010), while the Extended Kalman Filter (EKF) is the workhorse of the solution presented in Morgado et al. (2006). For interesting surveys on underwater navigation, please see Leonard et al. (1998) and Kinsey et al. (2006).

The GPS, LBL, SBL, USBL, or GIB positioning systems are essentially employed in the estimation of linear motion quantities (position, linear velocity, acceleration) and other sensors are usually considered for the problem of attitude estimation, which is still very active, as evidenced by numerous recent publications, see e.g. Metni et al. (2006), Tayebi et al. (2007), Campolo et al. (2006), Choukroun (2009). The Extended Kalman Filter (EKF) has been instrumental to many stochastic solutions, see e.g. Sabatini (2006), while nonlinear alternatives, aiming for

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stability and convergence properties, in deterministic settings, have been proposed in Sanyal et al. (2008), Vasconcelos et al. (2007), Rehbinder and Ghosh (2003), Mahony et al. (2008), Thienel and Sanner (2003), Grip et al. (2012), and Martin and Salaun (2010), to mention just a few, see Crassidis et al. (2007) for a thorough survey on attitude estimation. Recently, the authors have proposed two alternative solutions in Batista et al. (2012a) and Batista et al. (2012b). In the first, the Kalman filter is the workhorse, where no linearizations are carried out whatsoever, resulting in a design which guarantees globally asymptotically stable (GAS) error dynamics. In the later, a cascade observer is proposed that achieves globally exponentially stable (GES) error dynamics and that requires less computational power than the Kalman filter, at the expense of the filtering performance. Common to both solutions is the fact that the topological restrictions of the Special Orthogonal Group $SO(3)$ are not explicitly imposed, though they are verified asymptotically in the absence of noise. In the presence of sensor noise, the distance of the estimates provided by the cascade observer or the Kalman filter to $SO(3)$ remains close to zero and methods are proposed that give estimates of the attitude arbitrarily close to $SO(3)$. In Batista et al. (2012c) an alternative additional result gives attitude estimates explicitly on $SO(3)$, at the possible expense of the continuity of the solution during the initial transients, hence not violating the topological limitations that are thoroughly discussed in Bhat and Bernstein (2000).

For underwater vehicles, the usual sensing devices employed for attitude determination are two triads of orthogonally mounted accelerometers and magnetometers, coupled with a triad of orthogonally mounted rate gyros, used for filtering purposes. Essentially, the magnetometers and the accelerometers provide direct measurements, in body-fixed coordinates, of known vectors in inertial coordinates. Hence, an attitude estimate can be readily obtained from the solution of the Wahba's problem. With additional angular velocity measurements, it is then possible to design attitude filters, possibly including the estimation of rate gyro bias. The disadvantage of the use of magnetometers is that they are subject to magnetic field anomalies, such as the ones that can be encountered nearby objects with strong magnetic signatures, rendering the magnetic field measurements position dependent and therefore useless. This can be particularly dangerous in underwater intervention scenarios and as such alternatives need to be devised.

In previous work by the authors, see Batista et al. (2011), a novel complete navigation system was proposed based on a combined Long Baseline / Ultra-short Baseline (LBL/USBL) acoustic positing system. With a Long Baseline acoustic positioning system, an underwater vehicle has access to the distances to a set of known transponders, which are usually fixed in the mission scenario. Under some mild assumptions on the LBL configuration, it is possible to determine the inertial position of the vehicle. With an Ultra-Short Baseline acoustic position system installed on-board the vehicle, in the so-called inverted configuration, see Morgado et al. (2011), the vehicle has access to the distance to a fixed transponder in the mission scenario and the time (or range) differences of arrival between each pair of receivers of the USBL array. From those measurements, and under some mild assumptions on the USBL array configuration, the position of the external landmark relative to the vehicle, and expressed in body-fixed coordinates, is readily available. Using spread spectrum techniques, see Morgado et al. (2010), it is possible to combine LBL and USBL acoustic positioning devices, which gives, in essence, both the distance between the vehicle and each of the external landmarks and the time (or range) differences of arrival between pairs of receivers, for each landmark. In this way, with a LBL/USBL it is not only possible to determine the inertial position of the vehicle but also the positions of the external LBL landmarks with respect to the vehicle, expressed in body-fixed coordinates. In Batista et al. (2011), and for attitude determination purposes, the later were employed to obtain body-fixed vector measurements of known constant inertial vectors, hence allowing for attitude estimation, while the inertial position was used to the estimation of the linear motion quantities.

The actual measurements of a LBL/USBL acoustic positioning system are acoustic signals, which when processed yield ranges and range differences of arrival between the acoustic receivers of the USBL. In Batista et al. (2011) these were used, resorting to inversion or algebraic optimization techniques, to obtain the inertial position of the vehicle and the body-fixed positions of the landmarks. However, it would be beneficial if the actual range and range differences of arrival could be directly employed in the estimation solution, avoiding intermediate nonlinear computations that can distort noise and allowing for better tuning of the estimator parameters. Additional benefits would be the possibility of inclusion of outlier detection algorithms at the range or range-difference of arrival levels and better coping with loss of some of these measurements.

The main contribution of this paper is the design of a tightly coupled integrated navigation solution based on a LBL/USBL acoustic positioning system. In contrast with the solution proposed in Batista et al. (2011), the range and range differences of arrival are used directly in the observers feedback loop, hence avoiding intermediate computations, and no linearizations are carried out whatsoever. First, an attitude observer, that includes the estimation of rate gyro bias, is proposed, which is independent of the linear motion quantities. The proposed observer achieves globally exponentially stable error dynamics and it is computationally efficient. Topological limitations are avoided by relaxation of the constraints of the Special Orthogonal Group, which are nevertheless verified asymptotically. Additional references are provided that yield estimates on $SO(3)$ based directly on the output of the proposed observer with meaningless additional computational burden. Afterwards, a position and linear velocity observer is proposed assuming exact angular data information, which also yields globally exponentially stable error dynamics. Finally, the cascade structure is analyzed and it is shown that the error converges exponentially fast to zero for all initial conditions. This is, to the best of the authors' knowledge, the first contribution on the design of tightly coupled integrated LBL/USBL navigation system. Previous work by the authors can be found in Batista et al. (2013a) and Batista et al. (2013b), where the solutions for the estimation of the linear and angular motion quantities were presented independently. This paper improves those results by providing detailed proofs and by considering the complete interconnected estimation structure, including its stability analysis.

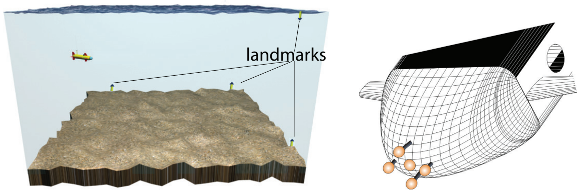
The paper is organized as follows. The problem statement and the nominal system dynamics are introduced in Section 2. The problem of attitude estimation is considered, independently, in Section 3, while that of estimating the linear motion quantities is addressed in Section 4. The complete integrated navigation system is proposed and analyzed in Section 5 and simulation results are presented in Section 6. Finally, Section 7 summarizes the main conclusions and results of the paper.

1.1 *Notation*

Throughout the paper the symbol $\mathbf{0}_{n\times m}$ denotes an $n \times m$ matrix of zeros, $\mathbf{I}_n$ an identity matrix with dimension $n \times n$, and $\text{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_\text{n})$ a block diagonal matrix. When the dimensions are omitted the matrices are assumed of appropriate dimensions. For $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \times \mathbf{y}$ and $\mathbf{x} \cdot \mathbf{y}$ represent the cross and inner products, respectively. Finally, the Dirac delta function is denoted by $\delta(t)$.

2. Problem statement

Consider an underwater vehicle moving in a scenario where there is a set of fixed landmarks installed in a Long Baseline configuration and suppose that the vehicle is equipped with an Ultra Short Baseline acoustic positioning system, which measures not only the distance between the vehicle and each landmark but also the range differences of arrival between the acoustic receivers of the USBL, from each landmark, as depicted in Fig. 1. For further details on the USBL, please refer to Morgado et al. (2011), Morgado et al. (2010), and references therein. Further assume that the vehicle is equipped with a Doppler Velocity Log, which measures the velocity of the vehicle relative to the water, and a triad of orthogonally mounted rate gyros, which measures the angular velocity up to some offset. Finally, it is considered that the vehicle



(a) AUV and LBL array (b) AUV with USBL array

Figure 1. Mission Scenario

moves in the presence of a constant unknown ocean current. The problem considered in the paper is the design of a highly integrated tightly coupled estimation solution for the inertial position of the vehicle and its attitude, the ocean current velocity, and the rate gyro bias, with convergence guarantees.

2.1 *System dynamics*

In order to set the problem framework, let $\{I\}$ denote a local inertial reference coordinate frame and $\{B\}$ a coordinate frame attached to the vehicle, usually called the body-fixed reference frame. The kinematics of the vehicle are described by

$$\begin{cases} \dot{\mathbf{p}}(t) = \mathbf{R}(t)\mathbf{v}(t) \\ \dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}\left(\boldsymbol{\omega}(t)\right) \end{cases}, \tag{1}$$

where $\mathbf{p}(t) \in \mathbb{R}^3$ denotes the inertial position of the vehicle, $\mathbf{v}(t) \in \mathbb{R}^3$ is the velocity of the vehicle relative to $\{I\}$ and expressed in body-fixed coordinates, $\mathbf{R}(t) \in SO(3)$ is the rotation matrix from $\{B\}$ to $\{I\}$, $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ is the angular velocity of $\{B\}$, expressed in body-fixed coordinates, and $\mathbf{S}\left(\boldsymbol{\omega}\right)$ is the skew-symmetric matrix such that $\mathbf{S}\left(\boldsymbol{\omega}\right)\boldsymbol{x}$ is the cross product $\boldsymbol{\omega} \times \boldsymbol{x}$.

The DVL provides the velocity of the vehicle relative to the water, expressed in body-fixed coordinates, denoted by $\mathbf{v}_r(t) \in \mathbb{R}^3$, such that

$$\mathbf{v}(t) = \mathbf{v}_r(t) + \mathbf{v}_c(t), \tag{2}$$

where $\mathbf{v}_c(t) \in \mathbb{R}^3$ is the ocean current velocity expressed in body-fixed coordinates, while the triad of rate gyros gives

$$\boldsymbol{\omega}_m(t) = \boldsymbol{\omega}(t) + \mathbf{b}_{\boldsymbol{\omega}}(t), \tag{3}$$

where $\mathbf{b}_{\boldsymbol{\omega}}(t) \in \mathbb{R}^3$ denotes the rate gyro bias, which is assumed constant, i.e.,

$$\dot{\mathbf{b}}_{\boldsymbol{\omega}}(t) = \mathbf{0}. \tag{4}$$

Let $\mathbf{s}_i \in \mathbb{R}^3$, $i = 1, \ldots, N$, denote the inertial positions of the landmarks, and $\mathbf{a}_i \in \mathbb{R}^3$, $i = 1, \ldots, M$, the positions of the array of receivers of the USBL relative to the origin of $\{B\}$, expressed in body-fixed coordinates. Then, the range measurement between the i-th landmark and the j-th acoustic receiver of the USBL is given by

$$r_{i,j}(t) = \|\mathbf{s}_i - \mathbf{p}(t) - \mathbf{R}(t)\mathbf{a}_j\| \in \mathbb{R}. \tag{5}$$

Define $\mathbf{u}(t) := \mathbf{R}(t)\mathbf{v}_r(t)$ and let ${}^{I}\mathbf{v}_c(t) := \mathbf{R}(t)\mathbf{v}_c(t)$ denote the ocean current velocity expressed in inertial coordinates. Assuming it is constant, and combining (1)-(5), yields the nonlinear system

$$\begin{cases} \dot{\mathbf{p}}(t) = {}^I\mathbf{v}_c(t) + \mathbf{u}(t) \\ \dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}\left(\boldsymbol{\omega}_m(t) - \mathbf{b}_{\boldsymbol{\omega}}(t)\right) \\ {}^I\dot{\mathbf{v}}_c(t) = \mathbf{0}(t) \\ \dot{\mathbf{b}}_{\boldsymbol{\omega}}(t) = \mathbf{0} \\ r_{1,1}(t) = \|\mathbf{s}_1 - \mathbf{p}(t) - \mathbf{R}(t)\mathbf{a}_1\| \\ \vdots \\ r_{N,M}(t) = \|\mathbf{s}_N - \mathbf{p}(t) - \mathbf{R}(t)\mathbf{a}_M\| \end{cases}. \tag{6}$$

The problem considered in the paper is the design of an estimator for (6) with global convergence guarantees.

2.2 *Long Baseline / Ultra Short Baseline configuration*

Long Baseline acoustic configurations are one of the earliest methods employed for underwater navigation. These are characterized by the property that the distance between the transponders is long or similar to the distance between the vehicle and the transponders. This is in contrast with Ultra Short Baseline systems, where the that between the transponder and the vehicle is much larger than the distance between receivers of the USBL system. In common is the fact that, under standard assumptions, both the inertial position of the vehicle (for the LBL) and the position of the landmarks with respect to the vehicle, expressed in body-fixed coordinates, (for the USBL, in the so-called inverted configuration) are uniquely determined. This happens with the following standard assumptions, which are considered in the remainder of the paper.

Assumption 1. The LBL acoustic positioning system includes at least 4 noncoplanar landmarks and the distance between the landmarks of the LBL is much larger than the distance between the receivers of the USBL acoustic positioning system.

Assumption 2. The USBL acoustic positioning system includes at least 4 noncoplanar receivers and the distance between the landmarks of the LBL is much larger than the distance between the receivers of the USBL acoustic positioning system.

Remark 1. When there exist at least 4 noncoplanar landmarks (receivers), it is always possible to determine the inertial position of the vehicle (the position of the landmark with respect to the vehicle, expressed in body-fixed coordinates) from the range measurements from each landmark to the vehicle (from the range and range differences of arrival between the landmark and the receivers of the array of the USBL). When there are fewer measurements that is not always possible and certain observability conditions must be met, see e.g. Batista et al. (2011) for the case of single range measurements. The scope of this paper is on the combination of the USBL and the LBL measurements, taking full advantage of the large data set to improve performance and robustness to temporary sensor failure, while still guaranteeing convergence of the error to zero. As such, particular cases that do not satisfy Assumptions 1 and 2 are not treated, though it is rather straightforward to extrapolate the results presented herein to other cases considering the analysis that is detailed in Batista et al. (2011).

3. Attitude and rate gyro bias estimation

This section details the design of an attitude observer that uses directly the ranges and range differences of arrival and that achieves globally exponentially stable error dynamics. The proposed approach builds vaguely on two different methodologies previously proposed by the authors. First, a sensor-based observer for the rate gyro bias is developed by appropriate state definition, which bears some resemblance with the design proposed in Batista et al. (2011), where the problems of source localization and navigation based on single range measurements were addressed. Secondly, a cascade attitude observer is proposed assuming that the rate gyro bias is known. Finally, the overall cascade observer is proposed and its stability is analyzed. The cascade design is similar, at large, to that proposed in Batista et al. (2012b). However, the structures of each individual observer are very different as they now rely on range and range differences of arrival measurements instead of vector measurements.

3.1 *Rate gyro bias observer*

The dependence of the attitude observer (and, consequently, the bias observer) on the inertial position of the vehicle is highly undesirable and in fact it should not be required. Indeed, in a LBL/USBL framework, the positions of the LBL landmarks with respect to the vehicle, expressed in body-fixed coordinates, are indirectly available (after some computations). If one takes the difference between pairs of these vectors, one obtains a set of body-fixed vectors that correspond to constant known inertial vectors, obtained from the differences of the inertial positions of the LBL landmarks. As such, this information suffices to determine the attitude of the vehicle without the need of the inertial position of the vehicle. In fact, this is the idea of the approach proposed in Batista et al. (2011). This section aims at achieving the same result but using directly the ranges and range differences of arrival, hence achieving a tightly-coupled structure.

Let $\mathcal{C}_\mathbf{s}$ denote a set of 2-combinations of elements of the set $\{1, \ldots, N\}$, e.g.

$$\mathcal{C}_\mathbf{s} = \{(1,2), (1,3), \ldots (1,N), (2,3), \ldots, (2,N), \ldots, (N-1,N)\},$$

and let $\mathcal{C}_\mathbf{a}$ denote a set of 2-combinations of elements of the set $\{1, \ldots, M\}$, e.g.

$$\mathcal{C}_\mathbf{a} = \{(1,2), (1,3), \ldots (1,M), (2,3), \ldots, (2,M), \ldots, (M-1,M)\}.$$

Define

$$q\left(m,n,i,j,t\right) := -\frac{1}{2}\left[r_{m,i}^2(t) - r_{n,i}^2(t)\right] + \frac{1}{2}\left[r_{m,j}^2(t) - r_{n,j}^2(t)\right] \tag{7}$$

for all $(m,n,i,j) \in \mathcal{C}_\mathbf{s} \times \mathcal{C}_\mathbf{a}$. First, notice that $q\left(m,n,i,j,t\right)$ is a direct function of the ranges and range differences of arrival, as it is possible to rewrite it as

$$q\left(m,n,i,j,t\right) = \frac{1}{2}\left[r_{n,i}(t) + r_{n,j}(t)\right]\left[r_{n,i}(t) - r_{n,j}(t)\right] - \frac{1}{2}\left[r_{m,i}(t) + r_{m,j}(t)\right]\left[r_{m,i}(t) - r_{m,j}(t)\right].$$

Next, substituting (5) in (7) gives

$$q\left(m,n,i,j,t\right) = \left(\mathbf{s}_m - \mathbf{s}_n\right)^T \mathbf{R}(t)\left(\mathbf{a}_i - \mathbf{a}_j\right). \tag{8}$$

As it can be seen, the inertial position of the vehicle does not influence $q\left(m,n,i,j,t\right)$. Yet, it depends on the attitude of the vehicle and, considering all 2-combinations of LBL landmarks and all 2-combinations of USBL receivers, it is related to the entire geometric structure of the LBL/USBL positioning system. The idea of the bias observer is to use $q\left(m,n,i,j,t\right)$, for all $(m,n,i,j) \in \mathcal{C}_\mathbf{s} \times \mathcal{C}_\mathbf{a}$, as system states, which are measured, in order to estimate the rate gyro bias $\mathbf{b}_{\boldsymbol{\omega}}(t)$, which is unknown.

Before proceeding some additional definitions are required. In particular, define, for all $(i,j) \in$

$\mathcal{C}_{\mathbf{a}}$, additional unit vectors $\mathbf{a}_{i,j}^{\perp_1} \in \mathbb{R}^3$ and $\mathbf{a}_{i,j}^{\perp_2} \in \mathbb{R}^3$ such that

$$\begin{cases} \frac{\mathbf{a}_i-\mathbf{a}_j}{\|\mathbf{a}_i-\mathbf{a}_j\|} \times \mathbf{a}_{i,j}^{\perp_1} = \mathbf{a}_{i,j}^{\perp_2} \\ \mathbf{a}_{i,j}^{\perp_1} \times \mathbf{a}_{i,j}^{\perp_2} = \frac{\mathbf{a}_i-\mathbf{a}_j}{\|\mathbf{a}_i-\mathbf{a}_j\|} \\ \mathbf{a}_{i,j}^{\perp_2} \times \frac{\mathbf{a}_i-\mathbf{a}_j}{\|\mathbf{a}_i-\mathbf{a}_j\|} = \mathbf{a}_{i,j}^{\perp_1} \end{cases}. \tag{9}$$

In short, the sets of vectors $\left\{\frac{\mathbf{a}_i-\mathbf{a}_j}{\|\mathbf{a}_i-\mathbf{a}_j\|}, \mathbf{a}_{i,j}^{\perp_1}, \mathbf{a}_{i,j}^{\perp_2}\right\}$, for all $(i,j) \in \mathcal{C}_{\mathbf{a}}$, form orthonormal bases of $\mathbf{R}^3$. Next, notice that under Assumption 2, it is always possible to express all additional vectors $\mathbf{a}_{i,j}^{\perp_1}$ and $\mathbf{a}_{i,j}^{\perp_2}$ as a linear combination of vectors $\mathbf{a}_k - \mathbf{a}_l$. Let these be defined as

$$\begin{cases} \mathbf{a}_{i,j}^{\perp_1} = \sum\limits_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_1(i,j,k,l)(\mathbf{a}_k - \mathbf{a}_l) \\ \mathbf{a}_{i,j}^{\perp_2} = \sum\limits_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_2(i,j,k,l)(\mathbf{a}_k - \mathbf{a}_l) \end{cases} \tag{10}$$

for all $(i,j) \in \mathcal{C}_{\mathbf{a}}$, where $\phi_1(i,j,k,l), \phi_2(i,j,k,l) \in \mathbb{R}$ are the linear combination coefficients.

The nominal system dynamics of the rate gyro bias observer are now derived. Taking the derivative of (8), and using (6), gives

$$\dot{q}(m,n,i,j,t) = (\mathbf{s}_m - \mathbf{s}_n)^T \mathbf{R}(t)\mathbf{S}(\boldsymbol{\omega}_m(t))(\mathbf{a}_i - \mathbf{a}_j) - (\mathbf{s}_m - \mathbf{s}_n)^T \mathbf{R}(t)\mathbf{S}(\mathbf{b}_{\boldsymbol{\omega}}(t))(\mathbf{a}_i - \mathbf{a}_j). \tag{11}$$

Express $\boldsymbol{\omega}_m(t)$ as the linear combination

$$\boldsymbol{\omega}_m(t) = \boldsymbol{\omega}_m(t) \cdot \frac{(\mathbf{a}_i - \mathbf{a}_j)}{\|(\mathbf{a}_i - \mathbf{a}_j)\|} \frac{(\mathbf{a}_i - \mathbf{a}_j)}{\|(\mathbf{a}_i - \mathbf{a}_j)\|} + \boldsymbol{\omega}_m(t) \cdot \mathbf{a}_{i,j}^{\perp_1}\mathbf{a}_{i,j}^{\perp_1} + \boldsymbol{\omega}_m(t) \cdot \mathbf{a}_{i,j}^{\perp_2}\mathbf{a}_{i,j}^{\perp_2}. \tag{12}$$

Using (12) first and then (9) it is possible to write

$$\begin{aligned} \boldsymbol{\omega}_m(t) \times (\mathbf{a}_i - \mathbf{a}_j) &= \boldsymbol{\omega}_m(t) \cdot \mathbf{a}_{i,j}^{\perp_1}\left[\mathbf{a}_{i,j}^{\perp_1} \times (\mathbf{a}_i - \mathbf{a}_j)\right] + \boldsymbol{\omega}_m(t) \cdot \mathbf{a}_{i,j}^{\perp_2}\left[\mathbf{a}_{i,j}^{\perp_2} \times (\mathbf{a}_i - \mathbf{a}_j)\right] \\ &= \boldsymbol{\omega}_m(t) \cdot \mathbf{a}_{i,j}^{\perp_2}\|\mathbf{a}_i - \mathbf{a}_j\|\mathbf{a}_{i,j}^{\perp_1} - \boldsymbol{\omega}_m(t) \cdot \mathbf{a}_{i,j}^{\perp_1}\|\mathbf{a}_i - \mathbf{a}_j\|\mathbf{a}_{i,j}^{\perp_2}. \end{aligned} \tag{13}$$

Substituting (10) in (13) gives

$$\begin{aligned} \boldsymbol{\omega}_m(t) \times (\mathbf{a}_i - \mathbf{a}_j) = \boldsymbol{\omega}_m(t) \cdot \mathbf{a}_{i,j}^{\perp_2}\|\mathbf{a}_i - \mathbf{a}_j\| &\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_1(i,j,k,l)(\mathbf{a}_k - \mathbf{a}_l) \\ -\boldsymbol{\omega}_m(t) \cdot \mathbf{a}_{i,j}^{\perp_1}\|\mathbf{a}_i - \mathbf{a}_j\| &\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_2(i,j,k,l)(\mathbf{a}_k - \mathbf{a}_l). \end{aligned} \tag{14}$$

Substituting (14) in the first term of the right side of (11), and using (8), gives

$$\begin{aligned} (\mathbf{s}_m - \mathbf{s}_n)^T \mathbf{R}(t)\mathbf{S}(\boldsymbol{\omega}_m(t))(\mathbf{a}_i - \mathbf{a}_j) = \boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\|\mathbf{a}_i - \mathbf{a}_j\| &\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_1(i,j,k,l)\, q(m,n,k,l,t) \\ -\boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\|\mathbf{a}_i - \mathbf{a}_j\| &\sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_2(i,j,k,l)\, q(m,n,k,l,t). \end{aligned} \tag{15}$$

Following the same circle of ideas it is possible to rewrite the second term of the right side of (11) as

$$
\begin{aligned}
(\mathbf{s}_m - \mathbf{s}_n)^T \mathbf{R}(t)\mathbf{S}\left(\mathbf{b}_{\boldsymbol{\omega}}(t)\right)(\mathbf{a}_i - \mathbf{a}_j) = \mathbf{b}_{\boldsymbol{\omega}}(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\left\|\mathbf{a}_i - \mathbf{a}_j\right\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_1(i,j,k,l)\, q(m,n,k,l,t) \\
-\mathbf{b}_{\boldsymbol{\omega}}(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\left\|\mathbf{a}_i - \mathbf{a}_j\right\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_2(i,j,k,l)\, q(m,n,k,l,t)\,.
\end{aligned}
\tag{16}
$$

Substituting (15) and (16) in (11) gives the nonlinear dynamics

$$
\begin{aligned}
\dot{q}(m,n,i,j,t) = {} & \boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\left\|\mathbf{a}_i - \mathbf{a}_j\right\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_1(i,j,k,l)\, q(m,n,k,l,t) \\
& -\boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\left\|\mathbf{a}_i - \mathbf{a}_j\right\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_2(i,j,k,l)\, q(m,n,k,l,t) \\
& +\mathbf{b}_{\boldsymbol{\omega}}(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\left\|\mathbf{a}_i - \mathbf{a}_j\right\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_2(i,j,k,l)\, q(m,n,k,l,t) \\
& -\mathbf{b}_{\boldsymbol{\omega}}(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\left\|\mathbf{a}_i - \mathbf{a}_j\right\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_1(i,j,k,l)\, q(m,n,k,l,t)\,.
\end{aligned}
\tag{17}
$$

for all $(m,n,i,j) \in \mathcal{C}_{\mathbf{s}} \times \mathcal{C}_{\mathbf{a}}$. Notice that (17) depends only on the USBL array geometry, the rate gyro measurements $\boldsymbol{\omega}_m(t)$, the additional quantities $q(m,n,i,j,t)$, the linear coefficients $\phi_1(i,j,k,l)$ and $\phi_2(i,j,k,l)$, all available, and the unknown rate gyro bias $\mathbf{b}_{\boldsymbol{\omega}}(t)$.

Consider the rate gyro bias observer dynamics given by

$$
\begin{aligned}
\dot{\hat{q}}(m,n,i,j,t) = {} & \boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\left\|\mathbf{a}_i - \mathbf{a}_j\right\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_1(i,j,k,l)\, \hat{q}(m,n,k,l,t) \\
& -\boldsymbol{\omega}_m(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\left\|\mathbf{a}_i - \mathbf{a}_j\right\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_2(i,j,k,l)\, \hat{q}(m,n,k,l,t) \\
& +\hat{\mathbf{b}}_{\boldsymbol{\omega}}(t)\cdot\mathbf{a}_{i,j}^{\perp_1}\left\|\mathbf{a}_i - \mathbf{a}_j\right\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_2(i,j,k,l)\, q(m,n,k,l,t) \\
& -\hat{\mathbf{b}}_{\boldsymbol{\omega}}(t)\cdot\mathbf{a}_{i,j}^{\perp_2}\left\|\mathbf{a}_i - \mathbf{a}_j\right\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_1(i,j,k,l)\, q(m,n,k,l,t) \\
& +\alpha(m,n,i,j)\left[q(m,n,i,j,t) - \hat{q}(m,n,i,j,t)\right]
\end{aligned}
\tag{18}
$$

for all $(m,n,i,j) \in \mathcal{C}_{\mathbf{s}} \times \mathcal{C}_{\mathbf{a}}$, and

$$
\begin{aligned}
\dot{\hat{\mathbf{b}}}_{\boldsymbol{\omega}}(t) = & \sum_{(m,n,i,j)\in\mathcal{C}_{\mathbf{s}}\times\mathcal{C}_{\mathbf{a}}} \beta(m,n,i,j)\left\|\mathbf{a}_i - \mathbf{a}_j\right\|\left[q(m,n,i,j,t) - \hat{q}(m,n,i,j,t)\right] \\
& \left[\mathbf{a}_{i,j}^{\perp_1} \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_2(i,j,k,l)\, q(m,n,k,l,t) - \mathbf{a}_{i,j}^{\perp_2} \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_1(i,j,k,l)\, q(m,n,k,l,t)\right],
\end{aligned}
\tag{19}
$$

where $\alpha(m,n,i,j) > 0$ and $\beta(m,n,i,j) > 0$, for all $(m,n,i,j) \in \mathcal{C}_{\mathbf{s}} \times \mathcal{C}_{\mathbf{a}}$, are observer tuning parameters.

Let $\tilde{q}(m,n,i,j,t) := q(m,n,i,j,t) - \hat{q}(m,n,i,j,t)$, for all $(m,n,i,j) \in \mathcal{C}_{\mathbf{s}} \times \mathcal{C}_{\mathbf{a}}$ and $\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t) := \mathbf{b}_{\boldsymbol{\omega}}(t) - \hat{\mathbf{b}}_{\boldsymbol{\omega}}(t)$ denote the observer error. Then, the observer error dynamics are given by

$$\begin{aligned}\dot{\tilde{q}}(m,n,i,j,t) = {} & \boldsymbol{\omega}_m(t) \cdot \mathbf{a}_{i,j}^{\perp_2} \|\mathbf{a}_i - \mathbf{a}_j\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_1(i,j,k,l)\, \tilde{q}(m,n,k,l,t) \\ & - \boldsymbol{\omega}_m(t) \cdot \mathbf{a}_{i,j}^{\perp_1} \|\mathbf{a}_i - \mathbf{a}_j\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_2(i,j,k,l)\, \tilde{q}(m,n,k,l,t) \\ & + \tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t) \cdot \mathbf{a}_{i,j}^{\perp_1} \|\mathbf{a}_i - \mathbf{a}_j\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_2(i,j,k,l)\, q(m,n,k,l,t) \\ & - \tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t) \cdot \mathbf{a}_{i,j}^{\perp_2} \|\mathbf{a}_i - \mathbf{a}_j\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_1(i,j,k,l)\, q(m,n,k,l,t) \\ & - \alpha(m,n,i,j)\, \tilde{q}(m,n,i,j,t)\end{aligned}$$

for all $(m,n,i,j) \in \mathcal{C}_{\mathbf{s}} \times \mathcal{C}_{\mathbf{a}}$, and

$$\dot{\tilde{\mathbf{b}}}_{\boldsymbol{\omega}}(t) = -\sum_{(m,n,i,j)\in\mathcal{C}_{\mathbf{s}}\times\mathcal{C}_{\mathbf{a}}} \beta(m,n,i,j)\, \tilde{q}(m,n,i,j,t) \|\mathbf{a}_i - \mathbf{a}_j\|$$
$$\left[\mathbf{a}_{i,j}^{\perp_1} \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_2(i,j,k,l)\, q(m,n,k,l,t) - \mathbf{a}_{i,j}^{\perp_2} \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_1(i,j,k,l)\, q(m,n,k,l,t)\right].$$

The following theorem establishes that the resulting rate gyro bias observer has globally exponentially stable error dynamics.

Theorem 3.1. *Suppose that Assumptions 1 and 2 are satisfied and consider the rate gyro bias observer given by* (18) *and* (19), *where* $\alpha(m,n,i,j) > 0$ *and* $\beta(m,n,i,j) > 0$ *for all* $(m,n,i,j) \in \mathcal{C}_{\mathbf{s}} \times \mathcal{C}_{\mathbf{a}}$. *Then, the origin of the error dynamics is a globally exponentially stable equilibrium point.*

Proof. Let $\tilde{\boldsymbol{x}}_1(t) := \left[\ldots\ \tilde{q}(m,n,i,j,t)\ \ldots\ \tilde{\mathbf{b}}_{\boldsymbol{\omega}}^T(t)\right]^T \in \mathbb{R}^{{}_2^NC\,{}_2^MC+3}$, $(m,n,i,j) \in \mathcal{C}_{\mathbf{s}} \times \mathcal{C}_{\mathbf{a}}$, denote the estimator error, in compact form, where ${}_2^NC = N(N-1)/2$ and ${}_2^MC = M(M-1)/2$ denote the number of 2-combinations of N and M elements, respectively. Define

$$V_1(\tilde{\boldsymbol{x}}_1(t)) := \frac{1}{2} \sum_{(i,j,k,l)\in\mathcal{C}_{\mathbf{s}}\times\mathcal{C}_{\mathbf{a}}} \beta(m,n,i,j) \left[\tilde{q}(m,n,i,j,t)\right]^2 + \frac{1}{2} \left\|\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right\|^2$$

as a Lyapunov function candidate. Clearly,

$$\gamma_1 \|\tilde{\boldsymbol{x}}_1(t)\|^2 \leq V_1(\tilde{\boldsymbol{x}}_1(t)) \leq \gamma_2 \|\tilde{\boldsymbol{x}}_1(t)\|^2, \tag{20}$$

where

$$\gamma_1 := \frac{1}{2} \min(1, \beta(m,n,i,j)),\ (m,n,i,j) \in \mathcal{C}_{\mathbf{s}} \times \mathcal{C}_{\mathbf{a}}$$

and

$$\gamma_2 := \frac{1}{2} \max(1, \beta(m,n,i,j)),\ (m,n,i,j) \in \mathcal{C}_{\mathbf{s}} \times \mathcal{C}_{\mathbf{a}}.$$

The time derivative of $V_1(\tilde{\boldsymbol{x}}_1(t))$ can be written, after some straightforward computations, as

$$\dot{V}_1(\tilde{\boldsymbol{x}}_1(t)) = -\tilde{\boldsymbol{x}}_1^T(t)\mathbf{C}_1^T\mathbf{C}_1\tilde{\boldsymbol{x}}_1(t) = -\sum_{(i,j,k,l)\in\mathcal{C}_{\mathbf{s}}\times\mathcal{C}_{\mathbf{a}}} \alpha(m,n,i,j)\,\beta(m,n,i,j)\,[\tilde{q}(m,n,i,j,t)]^2,$$

where $\mathbf{C}_1 = \left[\operatorname{diag}\left(\sqrt{\alpha(\mathrm{m,n,i,j})\,\beta(\mathrm{m,n,i,j})}\right)\ \mathbf{0}\right]$. Hence,

$$\dot{V}_1(\tilde{\boldsymbol{x}}_1(t)) \leq 0. \tag{21}$$

Now, notice that the error dynamics can be written as the linear time-varying (LTV) system

$$\dot{\tilde{\boldsymbol{x}}}_1(t) = \boldsymbol{\mathcal{A}}_1(t)\tilde{\boldsymbol{x}}_1(t), \tag{22}$$

where

$$\boldsymbol{\mathcal{A}}_1(t) = \begin{bmatrix} \boldsymbol{\mathcal{A}}_{11}(t) & \boldsymbol{\mathcal{A}}_{12}(t) \\ \boldsymbol{\mathcal{A}}_{21}(t) & \mathbf{0} \end{bmatrix}$$

and each row of the matrix $\boldsymbol{\mathcal{A}}_{12}(t)$, corresponding to the state error $\tilde{q}(m,n,i,j,t)$, is given by

$$\begin{aligned} &\|\mathbf{a}_i - \mathbf{a}_j\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_2(i,j,k,l)\, q(m,n,k,l,t) \left(\mathbf{a}_{i,j}^{\perp_1}\right)^T \\ &- \|\mathbf{a}_i - \mathbf{a}_j\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_1(i,j,k,l)\, q(m,n,k,l,t) \left(\mathbf{a}_{i,j}^{\perp_2}\right)^T. \end{aligned}$$

The definitions of $\boldsymbol{\mathcal{A}}_{11}(t)$ and $\boldsymbol{\mathcal{A}}_{21}(t)$ are omitted as they are not required in the sequel. If in addition to (20) and (21), the pair $(\boldsymbol{\mathcal{A}}_1(t), \mathbf{C}_1)$ is uniformly completely observable, then the origin of the linear time-varying system (22) is a globally exponentially stable equilibrium point, see (Khalil 2001, Example 8.11). The remainder of the proof amounts to show that the pair $(\boldsymbol{\mathcal{A}}_1(t), \mathbf{C}_1)$ is uniformly completely observable. For any piecewise continuous, bounded matrix $\mathbf{K}_1(t)$, of compatible dimensions, uniform complete observability of the pair $(\boldsymbol{\mathcal{A}}_1(t), \mathbf{C}_1)$ is equivalent to uniform complete observability of the pair $(\mathbf{A}_1(t), \mathbf{C}_1)$, with $\mathbf{A}_1(t) := \boldsymbol{\mathcal{A}}_1(t) - \mathbf{K}_1(t)\mathbf{C}_1$, see (Ioannou and Sun 1995, Lemma 4.8.1). Now, notice that, attending to the particular forms of $\mathbf{C}_1$ and $\boldsymbol{\mathcal{A}}_1(t)$, there exists a continuous bounded matrix $\mathbf{K}_1(t)$, which depends explicitly on the observer parameters, the rate gyro readings, $\boldsymbol{\omega}_m(t)$, the USBL structure, the linear coefficients $\phi_1(i,j,k,l)$ and $\phi_2(i,j,k,l)$, and $q(m,n,i,j,t)$, $(m,n,i,j)\in\mathcal{C}_{\mathbf{s}}\times\mathcal{C}_{\mathbf{a}}$, such that

$$\mathbf{A}_1(t) = \begin{bmatrix} \mathbf{0} & \boldsymbol{\mathcal{A}}_{12}(t) \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

The expression of $\mathbf{K}_1(t)$ is not presented here as it is evident from the context and it is not required in the sequel. It remains to show that the pair $(\mathbf{A}_1(t), \mathbf{C}_1)$ is uniformly completely observable, i.e., that there exist positive constants ϵ_1, ϵ_2, and δ such that

$$\epsilon_1\mathbf{I} \preceq \boldsymbol{\mathcal{W}}(t, t+\delta) \preceq \epsilon_2\mathbf{I} \tag{23}$$

for all $t \geq t_0$, where $\boldsymbol{\mathcal{W}}_1(t_0, t_f)$ is the observability Gramian associated with the pair $(\mathbf{A}_1(t), \mathbf{C}_1)$ on $[t_0, t_f]$. Since the entries of both $\mathbf{A}_1(t)$ and $\mathbf{C}_1$ are continuous and bounded, the right side of (23) is evidently verified. Therefore, only the left side of (23) requires verification. Let

$$\mathbf{d} = \left[\ldots\ d_{m,n,i,j}\ \ldots\ \mathbf{d}_2^T\right]^T \in \mathbb{R}^{{}^N C_2^M C + 3},\ d_{m,n,i,j}\in\mathbb{R},\ (m,n,i,j)\in\mathcal{C}_{\mathbf{s}}\times\mathcal{C}_{\mathbf{a}},\ \mathbf{d}_2\in\mathbb{R}^3$$

be a unit vector and define

$$\mathbf{f}\left(\tau,t\right) := \left[\ldots\; f_{m,n,i,j}\left(\tau,t\right)\;\ldots\right]^T \in \mathbb{R}^{{}^N_2C^M_2C},\; (m,n,i,j) \in \mathcal{C}_{\mathbf{s}} \times \mathcal{C}_{\mathbf{a}},$$

where

$$\begin{aligned} f_{m,n,i,j}\left(\tau,t\right) &:= \sqrt{\alpha\left(m,n,i,j\right)\beta\left(m,n,i,j\right)}\,\Big(d_{m,n,i,j} \\ &+ \int_t^\tau \|\mathbf{a}_i - \mathbf{a}_j\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_2\left(i,j,k,l\right) q\left(m,n,k,l,\sigma\right) \mathbf{a}_{i,j}^{\perp_1} \cdot \mathbf{d}_2 d\sigma \\ &- \int_t^\tau \|\mathbf{a}_i - \mathbf{a}_j\| \sum_{(k,l)\in\mathcal{C}_{\mathbf{a}}} \phi_1\left(i,j,k,l\right) q\left(m,n,k,l,\sigma\right) \mathbf{a}_{i,j}^{\perp_2} \cdot \mathbf{d}_2 d\sigma \Big), \end{aligned} \tag{24}$$

$\tau \in [t, t+\delta]$, $t \geq t_0$. It is easy to show that

$$\mathbf{d}^T \boldsymbol{\mathcal{W}}_1\left(t,t+\delta\right)\mathbf{d} = \int_t^{t+\delta} \left\|\mathbf{f}\left(\tau,t\right)\right\|^2 d\tau.$$

Reversing the train of thought used to obtain (15) but considering $\mathbf{d}_2$ instead of $\boldsymbol{\omega}_m(t)$, i.e., substituting (8) in (24), and then using (10) and (9), it is trivial to rewrite $f_{m,n,i,j}\left(\tau,t\right)$ as

$$f_{m,n,i,j}\left(\tau,t\right) := \sqrt{\alpha\left(m,n,i,j\right)\beta\left(m,n,i,j\right)} \left(d_{m,n,i,j} - \int_t^\tau \left(\mathbf{s}_m - \mathbf{s}_n\right)^T \mathbf{R}(\sigma)\mathbf{S}\left(\mathbf{d}_2\right)\left(\mathbf{a}_i - \mathbf{a}_j\right) d\sigma\right).$$

The derivative of $f_{m,n,i,j}\left(\tau,t\right)$ with respect to τ is given by

$$\frac{\partial}{\partial\tau} f_{m,n,i,j}\left(\tau,t\right) := -\sqrt{\alpha\left(m,n,i,j\right)\beta\left(m,n,i,j\right)} \left(\mathbf{s}_m - \mathbf{s}_n\right)^T \mathbf{R}(t)\mathbf{S}\left(\mathbf{d}_2\right)\left(\mathbf{a}_i - \mathbf{a}_j\right).$$

Under Assumptions 1 and 2 it is trivial to conclude that there exists a positive constant μ such that $\left\|\frac{\partial}{\partial\tau}\mathbf{f}\left(t,t\right)\right\| > \mu\left\|\mathbf{d}_2\right\|$ for all non-null vectors $\mathbf{d}_2$ and $t \geq t_0$. Fix $\delta > 0$. Resorting to (Batista et al. 2011, Proposition 2), it follows that there exists $\nu_1 > 0$ such that

$$\left\|\int_t^{t+\delta} \frac{\partial}{\partial\tau}\mathbf{f}\left(\sigma,t\right) d\sigma\right\| > \nu_1\left\|\mathbf{d}_2\right\|$$

for all non-null vectors $\mathbf{d}_2$ and $t \geq t_0$. Fix $\epsilon > 0$ sufficiently small such that

$$\left|d_{m,n,i,j}\right| < \epsilon \tag{25}$$

for all $(m,n,i,j) \in \mathcal{C}_{\mathbf{s}} \times \mathcal{C}_{\mathbf{a}}$ and

$$\epsilon < \frac{1}{2}\nu_1\left\|\mathbf{d}_2\right\|.$$

Notice that this is always possible as the smallest ϵ is the largest $\|\mathbf{d}_2\|$ is, as $\mathbf{d}$ is a unit vector. Then, it is clear that there exists ν_2 such that $\left\|\mathbf{f}\left(t+\delta,t\right)\right\| \geq \nu_2$ for all $t \geq t_0$ and all unit vectors $\mathbf{d}$ that satisfy (25). Resorting to (Batista et al. 2011, Proposition 2) again, it follows that there exists $\nu_3 > 0$ such that, for all unit vectors $\mathbf{d}$ that satisfy (25), (23) holds for all $t \geq t_0$, with $\epsilon_1 = \nu_3$. Suppose now that there exists $d_{m,n,i,j}$ such that

$$|d_{m,n,i,j}| \geq \epsilon. \tag{26}$$

In that case, it is trivial to see that $\|\mathbf{f}(t,t)\| \geq \epsilon$ for all $t \geq t_0$. Hence, resorting to (Batista et al. 2011, Proposition 2) again, it follows that there exists $\nu_4 > 0$ such that, for all unit vectors $\mathbf{d}$ that satisfy (26) for some $(m,n,i,j) \in \mathcal{C}_{\mathbf{s}} \times \mathcal{C}_{\mathbf{a}}$, (23) holds for all $t \geq t_0$, with $\epsilon_1 = \nu_4$. But then it follows that (23) holds for all $t \geq t_0$ and unit vectors $\mathbf{d}$, with $\epsilon_1 := \min(\nu_3, \nu_4)$, which means that the pair $(\mathbf{A}_1(t), \mathbf{C}_1)$ is uniformly completely observable, hence concluding the proof. $\square$

3.2 *Attitude observer*

Let

$$\boldsymbol{x}_2(t) := \left[\mathbf{z}_1^T(t)\ \mathbf{z}_2^T(t)\ \mathbf{z}_3^T(t)\right]^T \in \mathbb{R}^9$$

be a column representation of $\mathbf{R}(t)$, where

$$\mathbf{R}(t) = \begin{bmatrix} \mathbf{z}_1^T(t) \\ \mathbf{z}_2^T(t) \\ \mathbf{z}_3^T(t) \end{bmatrix},$$

with $\mathbf{z}_i(t) \in \mathbb{R}^3$, $i = 1,2,3$. Then, it is easy to show that $\dot{\boldsymbol{x}}_2(t) = -\mathbf{S}_3\left(\boldsymbol{\omega}_m(t) - \mathbf{b}_{\boldsymbol{\omega}}(t)\right)\boldsymbol{x}_2(t)$, where $\mathbf{S}_3(\mathbf{x}) := \operatorname{diag}(\mathbf{S}(\mathbf{x}), \mathbf{S}(\mathbf{x}), \mathbf{S}(\mathbf{x})) \in \mathbb{R}^{9\times 9}$.

From (8) it is possible to write $q(m,n,i,j,t)$ as a linear combination of elements of $\boldsymbol{x}_2(t)$, i.e., $q(m,n,i,j,t) = \mathbf{c}_{m,n,i,j} \cdot \boldsymbol{x}_2(t)$, where

$$\mathbf{c}_{m,n,i,j} := \begin{bmatrix} (\mathbf{a}_i - \mathbf{a}_j) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{a}_i - \mathbf{a}_j) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\mathbf{a}_i - \mathbf{a}_j) \end{bmatrix} (\mathbf{s}_m - \mathbf{s}_n) \in \mathbb{R}^9.$$

Let $\mathbf{q}(t) := \left[\ldots q(m,n,i,j,t) \ldots\right]^T \in \mathbb{R}^{{}^{M}_{2}C{}^{M}_{2}C}$, $(m,n,i,j) \in \mathcal{C}_{\mathbf{s}} \times \mathcal{C}_{\mathbf{a}}$. Then, it is possible to write $\mathbf{q}(t) = \mathbf{C}_2\boldsymbol{x}_2(t)$, where $\mathbf{C}_2 \in \mathbb{R}^{{}^{N}_{2}C{}^{M}_{2}C\times 9}$ is omitted as it is evident from the context. Under Assumptions 1 and 2 is is trivial to show that $\mathbf{C}_2$ has full rank.

Consider the attitude observer given by

$$\dot{\hat{\boldsymbol{x}}}_2(t) = -\mathbf{S}_3\left(\boldsymbol{\omega}_m(t) - \mathbf{b}_{\boldsymbol{\omega}}(t)\right)\hat{\boldsymbol{x}}_2(t) + \mathbf{C}_2^T\mathbf{Q}^{-1}\left[\mathbf{q}(t) - \mathbf{C}_2\hat{\boldsymbol{x}}_2(t)\right], \tag{27}$$

where $\mathbf{Q} = \mathbf{Q}^T \in \mathbb{R}^{{}^{N}_{2}C{}^{M}_{2}C\times{}^{N}_{2}C{}^{M}_{2}C}$ is a positive definite matrix, and define the error variable $\tilde{\boldsymbol{x}}_2(t) = \boldsymbol{x}_2(t) - \hat{\boldsymbol{x}}_2(t)$. Then, the observer error dynamics are given by

$$\dot{\tilde{\boldsymbol{x}}}_2(t) = \boldsymbol{\mathcal{A}}_2(t)\tilde{\boldsymbol{x}}_2(t), \tag{28}$$

where $\boldsymbol{\mathcal{A}}_2(t) := -\left[\mathbf{S}_3\left(\boldsymbol{\omega}_m(t) - \mathbf{b}_{\boldsymbol{\omega}}(t)\right) + \mathbf{C}_2^T\mathbf{Q}^{-1}\mathbf{C}_2\right]$.

The following theorem is the main result of this section.

Theorem 3.2. *Suppose that the rate gyro bias is known and consider the attitude observer* (27)*, where* $\mathbf{Q} \succ \mathbf{0}$ *is a design parameter. Then, under Assumptions 1 and 2, the origin of the observer error dynamics* (28) *is a globally exponentially stable equilibrium point.*

Proof. The proof follows by considering the Lyapunov candidate function

$$V_2\left(\tilde{\boldsymbol{x}}_2(t)\right) := \frac{1}{2}\left\|\tilde{\boldsymbol{x}}_2(t)\right\|^2.$$

It is similar to that of (Batista et al. 2012b, Theorem 2) and therefore it is omitted. The only difference is, in fact, in the definition of $\mathbf{C}_2$, which is nevertheless full rank, the only requirement for the proof. □

3.3 *Cascade observer*

This section presents the overall cascade observer and its stability analysis. In Section 3.1 an observer was derived, based directly on the ranges and range differences of arrival, that provides an estimate of the bias, with GES dynamics. The idea of the cascade observer is to feed the attitude observer proposed in Section 3.2 with the bias estimate provided by the bias observer proposed in Section 3.1. The bias observer remains the same, given by (18) and (19), whereas the attitude observer is now written as

$$\dot{\hat{\boldsymbol{x}}}_2(t) = -\mathbf{S}_3\left(\boldsymbol{\omega}_m(t) - \hat{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right)\hat{\boldsymbol{x}}_2(t) + \mathbf{C}_2^T\mathbf{Q}^{-1}\left[\mathbf{q}(t) - \mathbf{C}_2\hat{\boldsymbol{x}}_2(t)\right]. \tag{29}$$

The error dynamics corresponding to the bias observer are the same and therefore Theorem 3.1 applies. Evidently, the use of an estimate of the bias instead of the bias itself in the attitude observer introduces an error, and the stability of the system must be further examined. In this situation, the error dynamics of the cascade observer can be written as

$$\begin{cases} \dot{\tilde{\boldsymbol{x}}}_1(t) = \boldsymbol{\mathcal{A}}_1(t)\tilde{\boldsymbol{x}}_1(t) \\ \dot{\tilde{\boldsymbol{x}}}_2(t) = \left[\boldsymbol{\mathcal{A}}_2(t) - \mathbf{S}_3\left(\hat{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right)\right]\tilde{\boldsymbol{x}}_2(t) + \mathbf{u}_2(t), \end{cases} \tag{30}$$

where $\mathbf{u}_2(t) := \mathbf{S}_3\left(\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)\right)\boldsymbol{x}_2(t)$.

The following theorem is the main result of the paper.

Theorem 3.3. *Consider the cascade attitude observer given by* (18), (19), *and* (29). *Then, in the conditions of Theorem 3.1 and Theorem 3.2, the origin of the observer error dynamics* (30) *is a globally exponentially stable equilibrium point.*

Proof. The proof follows exactly the same steps of (Batista et al. 2012b, Theorem 3) and therefore it is omitted, even though the specific system dynamics are different. It is omitted due to space limitations. □

3.4 *Further discussion*

3.4.1 Estimates on $SO(3)$

The attitude solution previously proposed does not yield estimates on $SO(3)$ as the Special Orthogonal Group restrictions have been relaxed, in a similar fashion to the approaches proposed in Batista et al. (2012a) or Batista et al. (2012b). In the absence of noise, the estimates converge asymptotically to elements of $SO(3)$, while in the presence of noise their distance to $SO(3)$ remains close to zero. Additional refinements are possible such as those discussed in (Batista et al. 2012b, Section 3.4). This is not the focus of the paper and as such it is omitted. Furthermore, explicit estimates on $SO(3)$ could be obtained, based in the attitude observer here proposed, resorting to (Batista et al. 2012c, Theorem 7).

3.4.2 *Computational complexity*

The design herein proposed consists in a cascade observer where the number of states of the second observer is 9 and the number of states of the first observer is $N(N-1)M(M-1)/4+3$, with a total number of states of $N(N-1)M(M-1)/4+12$. For a typical LBL/USBL configuration with 4 landmarks and 4 acoustic receivers in the USBL array, that corresponds to 48 states. While this number may seem relatively high, it is very important to stress that the resulting observer is computationally efficient and of simple implementation. Indeed, all the observer coefficients are computed offline and no differential equations are required to compute the observer gains.

4. Position and linear velocity estimation

This section addresses the design of an estimation solution for the inertial position and inertial ocean current velocity based on the LBL/USBL positioning system assuming exact angular information, i.e., assuming that both the attitude and the angular velocity are available. First, state and output augmentation are performed, in Section 4.1, to attain a nominal system that, although nonlinear, can be regarded as linear for observability analysis and observer design purposes. Afterwards, the observability of that system is analyzed in Section 4.2. Finally, in Section 4.3, a Kalman filter for the resulting system, with globally exponentially stable error dynamics, is briefly discussed.

4.1 *State and output augmentation*

In the recent past, a novel observer analysis and design technique has been proposed by the authors for navigation systems based on nonlinear range measurements, which consists basically in: i) include the range measurements in the system state; ii) identify the nonlinear terms of the dynamics of the range measurements as additional state variables; iii) define augmented outputs, when appropriate, to capture the structure of arrays of landmarks or receivers; and iv) work with the resulting nonlinear system, which can actually be regarded as linear time-varying, for observability analysis and observer design purposes. This approach has been successfully employed considering single measurements, see Batista et al. (2011), LBL configurations, see Batista et al. (2010, 2013), and USBL configurations, see Morgado et al. (2011), where different auxiliary sensors were considered, for example DVLs or triads of accelerometers. The design presented herein consists in the integration of both LBL/USBL measurements with this approach.

The time derivative of the range measurements (5) is given by

$$\dot{r}_{i,j}(t) = \frac{\mathbf{u}(t)+\mathbf{R}(t)\mathbf{S}\left(\boldsymbol{\omega}(t)\right)\mathbf{a}_j}{r_{i,j}(t)}\cdot\mathbf{p}(t) + \frac{-\mathbf{s}_i+\mathbf{R}(t)\mathbf{a}_j}{r_{i,j}(t)}\cdot {}^{I}\mathbf{v}_c(t) + \frac{1}{r_{i,j}(t)}\mathbf{p}(t)\cdot {}^{I}\mathbf{v}_c(t) + u_{r_{i,j}}(t), \quad (31)$$

where

$$u_{r_{i,j}}(t) := \frac{\mathbf{u}^T(t)\mathbf{R}(t)\mathbf{a}_j - \mathbf{u}^T(t)\mathbf{s}_i - \mathbf{s}_i^T\mathbf{R}(t)\mathbf{S}\left(\boldsymbol{\omega}(t)\right)\mathbf{a}_j}{r_{i,j}(t)}.$$

Identifying the nonlinear term $\mathbf{p}(t)\cdot {}^{I}\mathbf{v}_c(t)$ in (31) with a new variable and taking its time derivative gives

$$\frac{d}{dt}\left[\mathbf{p}(t)\cdot {}^{I}\mathbf{v}_c(t)\right] = \mathbf{u}(t)\cdot {}^{I}\mathbf{v}_c(t) + \left\| {}^{I}\mathbf{v}_c(t)\right\|^2. \quad (32)$$

Finally, identifying the nonlinear term $\left\| {}^{I}\mathbf{v}_c(t)\right\|^2$ in (32) and taking its time derivative gives

$\frac{d}{dt}\left[\left\|{}^I\mathbf{v}_c(t)\right\|^2\right] = 0$.

For the sake of clarity of presentation, let $x_{1,1}(t) := r_{1,1}(t), \ldots, x_{N,M}(t) := r_{N,M}(t)$, $x_3(t) := \mathbf{p}(t) \cdot {}^I\mathbf{v}_c(t)$, and $x_4(t) := \left\|{}^I\mathbf{v}_c(t)\right\|^2$, and define the augmented state vector as

$$\boldsymbol{x}_3(t) := \left[\mathbf{p}^T(t)\ {}^I\mathbf{v}_c^T(t)\ x_{1,1}(t)\ x_{1,2}(t) \ldots x_{N,M}(t)\ x_3(t)\ x_4(t)\right]^T \in \mathbb{R}^{3+3+NM+1+1}.$$

Then, the system dynamics can be written as

$$\dot{\boldsymbol{x}}_3(t) := \mathbf{A}_3(t)\boldsymbol{x}_3(t) + \mathbf{B}_3\mathbf{u}_a(t),$$

where $\mathbf{A}_3(t) \in \mathbb{R}^{(6+NM+2)\times(6+NM+2)}$,

$$\mathbf{A}_3(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\mathbf{u}^T(t)-\mathbf{a}_1^T\mathbf{S}(\boldsymbol{\omega}(t))\mathbf{R}^T(t)}{r_{1,1}(t)} & \frac{-\mathbf{s}_1^T+\mathbf{a}_1^T\mathbf{R}^T(t)}{r_{1,1}(t)} & \mathbf{0} & \frac{1}{r_{1,1}(t)} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\mathbf{u}^T(t)-\mathbf{a}_M^T\mathbf{S}(\boldsymbol{\omega}(t))\mathbf{R}^T(t)}{r_{N,M}(t)} & \frac{-\mathbf{s}_N^T+\mathbf{a}_M^T\mathbf{R}^T(t)}{r_{N,M}(t)} & \mathbf{0} & \frac{1}{r_{N,M}(t)} & 0 \\ \mathbf{0} & \mathbf{u}^T(t) & \mathbf{0} & 0 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 \end{bmatrix},$$

$$\mathbf{B}_3 = \begin{bmatrix} \mathbf{I}\ \mathbf{0} \\ \mathbf{0}\ \mathbf{0} \\ \mathbf{0}\ \mathbf{I} \\ \mathbf{0}\ \mathbf{0} \\ \mathbf{0}\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(6+NM+2)\times(3+NM)},$$

and

$$\mathbf{u}_a(t) := \left[\mathbf{u}^T(t)\ u_{r_{1,1}}(t) \ldots u_{r_{N,M}}(t)\right]^T \in \mathbb{R}^{3+NM}.$$

In order to define the output, notice first that the states $x_{1,1}(t), \ldots, x_{N,M}(t)$ are measured. Note, however, that the range differences of arrival (RDOA) between pairs of receivers to the same landmark are measured more accurately with the USBL when compared to the distance between the landmark and any given receiver of the USBL. Selecting a reference sensor on the array, for instance receiver 1 for now, all the other ranges are easily reconstructed from the range measured at receiver 1 and the RDOA between receiver 1 and the other receivers. Hence, the first set of measurements that is considered is

$$\mathbf{y}_1(t) = \begin{bmatrix} r_{1,1}(t) \\ r_{1,1}(t) - r_{1,2}(t) \\ \vdots \\ r_{1,1}(t) - r_{1,M}(t) \\ \vdots \\ r_{N,1}(t) \\ \vdots \\ r_{N,1}(t) - r_{N,M}(t) \end{bmatrix} \in \mathbb{R}^{NM}. \tag{33}$$

However, if that was the only output to be considered, the LBL/USBL structure would not be encoded in the output. In order to capture the LBL/USBL structure, consider first the square of the range measurements, which is given by

$$r_{i,j}^2(t) = \|\mathbf{p}(t)\|^2 + \|\mathbf{s}_i\|^2 + \|\mathbf{a}_j\|^2 - 2\left[\mathbf{s}_i - \mathbf{R}(t)\mathbf{a}_j\right] \cdot \mathbf{p}(t) - 2\mathbf{s}_i^T\mathbf{R}(t)\mathbf{a}_j$$

for all $i = 1, \ldots, N$, $j = 1, \ldots, M$. Then,

$$r_{m,j}^2(t) - r_{n,j}^2(t) = \|\mathbf{s}_m\|^2 - \|\mathbf{s}_n\|^2 - 2\left(\mathbf{s}_m - \mathbf{s}_n\right) \cdot \left[\mathbf{p}(t) + \mathbf{R}(t)\mathbf{a}_j\right] \tag{34}$$

and

$$r_{i,m}^2(t) - r_{i,n}^2(t) = \|\mathbf{a}_m\|^2 - \|\mathbf{a}_n\|^2 - 2\left[\mathbf{R}(t)\left(\mathbf{a}_m - \mathbf{a}_n\right)\right] \cdot \left[\mathbf{s}_i - \mathbf{p}(t)\right]. \tag{35}$$

Breaking the differences of the squares, using $a^2 - b^2 = (a+b)(a-b)$, it follows from (34) and (35) that

$$2\frac{\left(\mathbf{s}_m - \mathbf{s}_n\right)^T}{r_{m,j}(t) + r_{n,j}(t)}\mathbf{p}(t) + x_{m,j}(t) - x_{n,j}(t) = \frac{\|\mathbf{s}_m\|^2 - \|\mathbf{s}_n\|^2 - 2\left(\mathbf{s}_m - \mathbf{s}_n\right)^T\mathbf{R}(t)\mathbf{a}_j}{r_{m,j}(t) + r_{n,j}(t)} \tag{36}$$

and

$$-2\frac{\left(\mathbf{a}_m - \mathbf{a}_n\right)^T\mathbf{R}^T(t)}{r_{i,m}(t) + r_{i,n}(t)}\mathbf{p}(t) + x_{i,m}(t) - x_{i,n}(t) = \frac{\|\mathbf{a}_m\|^2 - \|\mathbf{a}_n\|^2 - 2\left(\mathbf{a}_m - \mathbf{a}_n\right)^T\mathbf{R}^T(t)\mathbf{s}_i}{r_{i,m}(t) + r_{i,n}(t)}, \tag{37}$$

which capture the LBL/USBL structure. The augmented output can then be written as

$$\mathbf{y}_3(t) = \mathbf{C}_3(t)\boldsymbol{x}_3(t),$$

with $\mathbf{C}_3(t) \in \mathbb{R}^{(NM + M\,{}^N_2C + N\,{}^M_2C)\times(3+3+NM+1+1)}$,

$$\mathbf{C}_3(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{C}_{13} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_{21}(t) & \mathbf{0} & \mathbf{C}_{23} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_{31}(t) & \mathbf{0} & \mathbf{C}_{33} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{C}_{13} := \operatorname{diag}\left(\mathbf{C}_{13}^0, \ldots, \mathbf{C}_{13}^0\right) \in \mathbb{R}^{\mathrm{NM}\times\mathrm{NM}}$, with

$$\mathbf{C}_{13}^0 := \begin{bmatrix} 1 & 0 & \ldots & \ldots & 0 \\ 1 & -1 & \ddots & & \vdots \\ 1 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \ldots & 0 & -1 \end{bmatrix} \in \mathbb{R}^{M\times M},$$

$$\mathbf{C}_{21}(t) := \begin{bmatrix} \mathbf{C}_{21}^1(t) \\ \vdots \\ \mathbf{C}_{21}^M(t) \end{bmatrix} \in \mathbb{R}^{(M\,{}^N_2C)\times 3},$$

$$\mathbf{C}_{21}^i(t) := 2\begin{bmatrix} \frac{(\mathbf{s}_1-\mathbf{s}_2)^T}{r_{1,i}(t)+r_{2,i}(t)} \\ \frac{(\mathbf{s}_1-\mathbf{s}_3)^T}{r_{1,i}(t)+r_{3,i}(t)} \\ \vdots \\ \frac{(\mathbf{s}_{N-1}-\mathbf{s}_N)^T}{r_{N-1,i}(t)+r_{N,i}(t)} \end{bmatrix} \in \mathbb{R}^{{}^N_2C\times 3},$$

$$\mathbf{C}_{31}(t) := \begin{bmatrix} \mathbf{C}_{31}^1(t) \\ \vdots \\ \mathbf{C}_{31}^N(t) \end{bmatrix} \in \mathbb{R}^{(N\,{}^M_2C)\times 3},$$

$$\mathbf{C}_{31}^i(t) := -2\begin{bmatrix} \frac{(\mathbf{a}_1-\mathbf{a}_2)^T\mathbf{R}^T(t)}{r_{i,1}(t)+r_{i,2}(t)} \\ \frac{(\mathbf{a}_1-\mathbf{a}_3)^T\mathbf{R}^T(t)}{r_{i,1}(t)+r_{i,3}(t)} \\ \vdots \\ \frac{(\mathbf{a}_{M-1}-\mathbf{a}_M)^T\mathbf{R}^T(t)}{r_{i,M-1}(t)+r_{i,M}(t)} \end{bmatrix} \in \mathbb{R}^{{}^M_2C\times 3},$$

where ${}^N_2C = N(N-1)/2$ and ${}^M_2C = M(M-1)/2$ correspond to the numbers of 2-combinations of N and M elements, respectively, and $\mathbf{C}_{23}$ and $\mathbf{C}_{33}$ encode the differences of range measurements in (36) and (37), respectively, which are omitted as they are evident from the context. In short, $\mathbf{C}_{31}$ encodes (33), matrices $\mathbf{C}_{21}(t)$ and $\mathbf{C}_{23}$ encode (36) for all $j \in \{1, \ldots, M\}$ and $m, n \in \{1, \ldots, N\}$, with $n \neq m$, and matrices $\mathbf{C}_{31}(t)$ and $\mathbf{C}_{33}$ encode (37) for all $i \in \{1, \ldots, N\}$ and $m, n \in \{1, \ldots, M\}$, with $n \neq m$.

Considering the augmented system state and outputs, the final augmented system dynamics can be written as

$$\begin{cases} \dot{\boldsymbol{x}}_3(t) = \mathbf{A}_3(t)\boldsymbol{x}_3(t) + \mathbf{B}_3\mathbf{u}_a(t) \\ \mathbf{y}_3(t) = \mathbf{C}_3(t)\boldsymbol{x}_3(t) \end{cases}. \tag{38}$$

4.2 *Observability analysis*

The observability of the nonlinear system (38) and its relation with the original nonlinear system

$$\begin{cases} \dot{\mathbf{p}}(t) = {}^I\mathbf{v}_c(t) + \mathbf{u}(t) \\ {}^I\dot{\mathbf{v}}_c(t) = \mathbf{0}(t) \\ r_{1,1}(t) = \|\mathbf{s}_1 - \mathbf{p}(t) - \mathbf{R}(t)\mathbf{a}_1\| \\ \vdots \\ r_{N,M}(t) = \|\mathbf{s}_N - \mathbf{p}(t) - \mathbf{R}(t)\mathbf{a}_M\| \end{cases}. \tag{39}$$

is analyzed in this section.

Even though the system dynamics (38) resemble a linear time-varying system, it is, in fact, nonlinear, as the system matrices depend both on the output and the input. However, this is not a problem for observability and observer design purposes and the results for linear time-varying systems still apply, see (Batista et al. 2011, Lemma 1). Before presenting the main results, it is therefore convenient to compute the transition matrix associated with $\mathbf{A}_3(t)$ and the observability Gramian associated with the pair $(\mathbf{A}_3(t), \mathbf{C}_3(t))$. Long, tedious but straightforward computations allow to show that the transition matrix associated with $\mathbf{A}_3(t)$ is given by

$$\boldsymbol{\phi}(t,t_0) = \begin{bmatrix} \boldsymbol{\phi}_A(t,t_0) & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\phi}_{BA}(t,t_0) & \mathbf{I} & \boldsymbol{\phi}_{BC}(t,t_0) \\ \boldsymbol{\phi}_{CA}(t,t_0) & \mathbf{0} & \boldsymbol{\phi}_{CC}(t,t_0) \end{bmatrix},$$

where

$$\boldsymbol{\phi}_A(t,t_0) = \begin{bmatrix} \mathbf{I} & (t-t_0)\,\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \in \mathbb{R}^{6\times 6},$$

$$\boldsymbol{\phi}_{BA}(t,t_0) = \begin{bmatrix} \boldsymbol{\phi}_{BA1}(t,t_0) & \boldsymbol{\phi}_{BA2}(t,t_0) \end{bmatrix} \in \mathbb{R}^{NM\times 6},$$

$$\boldsymbol{\phi}_{BA1}(t,t_0) = \begin{bmatrix} \boldsymbol{\phi}_{BA1(1,1)}(t,t_0) \\ \vdots \\ \boldsymbol{\phi}_{BA1(N,M)}(t,t_0) \end{bmatrix} \in \mathbb{R}^{NM\times 3},$$

$$\boldsymbol{\phi}_{BA1(i,j)}(t,t_0) = \int_{t_0}^{t} \frac{\mathbf{u}^T(\sigma) - \mathbf{a}_j^T \mathbf{S}(\boldsymbol{\omega}(\sigma))\,\mathbf{R}^T(\sigma)}{r_{i,j}(\sigma)} d\sigma,$$

$$\boldsymbol{\phi}_{BA2}(t,t_0) = \begin{bmatrix} \boldsymbol{\phi}_{BA2(1,1)}(t,t_0) \\ \vdots \\ \boldsymbol{\phi}_{BA2(N,M)}(t,t_0) \end{bmatrix} \in \mathbb{R}^{NM\times 3},$$

$$\begin{aligned} \boldsymbol{\phi}_{BA2(i,j)}(t,t_0) = & \int_{t_0}^{t} \frac{-\mathbf{s}_i^T + \mathbf{a}_j^T \mathbf{R}^T(\sigma_1)}{r_{i,j}(\sigma)} d\sigma_1 \\ & + \int_{t_0}^{t} \frac{(\sigma - t_0)\left[\mathbf{u}(\sigma_1) + \mathbf{R}(\sigma_1)\,\mathbf{S}(\boldsymbol{\omega}(\sigma_1))\,\mathbf{a}_j\right]^T + \int_{t_0}^{\sigma_1} \mathbf{u}^T(\sigma_2)\,d\sigma_2}{r_{i,j}(\sigma)} d\sigma_1, \end{aligned}$$

$$\boldsymbol{\phi}_{BC}(t,t_0) = \begin{bmatrix} \int_{t_0}^{t} \frac{1}{r_{1,1}(\sigma)} d\sigma & \int_{t_0}^{t} \frac{\sigma - t_0}{r_{1,1}(\sigma)} d\sigma \\ \vdots & \vdots \\ \int_{t_0}^{t} \frac{1}{r_{N,M}(\sigma)} d\sigma & \int_{t_0}^{t} \frac{\sigma - t_0}{r_{N,M}(\sigma)} d\sigma \end{bmatrix} \in \mathbb{R}^{NM\times 2},$$

$$\boldsymbol{\phi}_{CA}(t,t_0) = \begin{bmatrix} \mathbf{0} & \int_{t_0}^{t} \mathbf{u}^T(\sigma)\,d\sigma \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{2\times 6},$$

and

$$\boldsymbol{\phi}_{CC}(t,t_0) = \begin{bmatrix} 1 & t - t_0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2},$$

It is, however, straightforward to verify that $\boldsymbol{\phi}(t_0, t_0) = \mathbf{I}$ and $\frac{d}{dt}\boldsymbol{\phi}(t, t_0) = \mathbf{A}_3(t)\boldsymbol{\phi}(t, t_0)$. The observability Gramian associated with the pair $(\mathbf{A}_3(t), \mathbf{C}_3)$ is simply given by

$$\boldsymbol{\mathcal{W}}_3(t_0, t_f) = \int_{t_0}^{t_f} \boldsymbol{\phi}^T(t, t_0)\,\mathbf{C}_3^T(t)\mathbf{C}_3(t)\boldsymbol{\phi}(t, t_0)\,dt. \tag{40}$$

The following theorem addresses the observability of (38).

Theorem 4.1. *Under Assumptions 1 or 2 (or both), the nonlinear system* (38) *is observable on* $\mathcal{I} := [t_0, t_f]$, $t_0 < t_f$, *in the sense that, given the system input* $\{\mathbf{u}_a(t), t \in \mathcal{I}\}$ *and the system output* $\{\mathbf{y}_3(t), t \in \mathcal{I}\}$, *the initial condition* $\boldsymbol{x}_3(t_0)$ *is uniquely determined.*

Proof. The proof follows by contradiction. Suppose that the nonlinear system (38) is not observable in $\mathcal{I}$. Then, the observability Gramian $\boldsymbol{\mathcal{W}}(t_0, t_f)$ is not invertible, see (Batista et al. 2011, Lemma 1), which means that there exists a unit vector

$$\mathbf{d} = \begin{bmatrix}\mathbf{d}_1^T \; \mathbf{d}_2^T \; \mathbf{d}_3^T \; d_4^T d_5\end{bmatrix}^T \in \mathbb{R}^{3+3+NM+1+1},$$

with $\mathbf{d}_1 \in \mathbb{R}^3$, $\mathbf{d}_2 \in \mathbb{R}^3$, $\mathbf{d}_3 \in \mathbb{R}^{NM}$, and $d_4, d_5 \in \mathbb{R}$, such that

$$\mathbf{d}^T \boldsymbol{\mathcal{W}}(t_0, t)\,\mathbf{d} = 0 \tag{41}$$

for all $t \in \mathcal{I}$. Substituting (40) in (41) yields

$$\int_{t_0}^{t} \left\|\mathbf{C}_3(\tau)\,\boldsymbol{\phi}(\tau, t_0)\,\mathbf{d}\right\|^2 = 0 \tag{42}$$

for all $t \in \mathcal{I}$. Taking the time derivative of (42) gives $\left\|\mathbf{C}_3(t)\,\boldsymbol{\phi}(t, t_0)\,\mathbf{d}\right\|^2 = 0$ for all $t \in \mathcal{I}$, which in turn implies that

$$\mathbf{C}_3(t)\,\boldsymbol{\phi}(t, t_0)\,\mathbf{d} = \mathbf{0} \tag{43}$$

for all $t \in \mathcal{I}$. With $t = t_0$ in (43) gives

$$\begin{cases}\mathbf{C}_{13}\mathbf{d}_3 = \mathbf{0}\\ \mathbf{C}_{21}(t_0)\,\mathbf{d}_1 + \mathbf{C}_{23}\mathbf{d}_3 = \mathbf{0}\\ \mathbf{C}_{31}(t_0)\,\mathbf{d}_1 + \mathbf{C}_{33}\mathbf{d}_3 = \mathbf{0}\end{cases}. \tag{44}$$

Notice first that $\mathbf{C}_{13}$ has full rank, which means that $\mathbf{d}_3 = \mathbf{0}$. On the other hand, under Assumption 1 matrix $\mathbf{C}_{21}(t_0)$ has full rank, while under Assumption 2 matrix $\mathbf{C}_{31}(t_0)$ has full rank. Hence, under the conditions of the theorem, is has been shown so far that the only solution of (44) is $\mathbf{d}_1 = \mathbf{0}$ and $\mathbf{d}_3 = \mathbf{0}$. Taking in the time derivative of (43) gives $\frac{d}{dt}\mathbf{C}_3(t)\,\boldsymbol{\phi}(t, t_0)\,\mathbf{d} = \mathbf{0}$ for all $t \in \mathcal{I}$. In particular, for $t = t_0$, and considering $\mathbf{d}_1 = \mathbf{0}$ and $\mathbf{d}_3 = \mathbf{0}$, it is possible to write that

$$\left[-\mathbf{s}_i + \mathbf{R}(t_0)\,\mathbf{a}_j\right]^T \mathbf{d}_2 + d_4 = 0 \tag{45}$$

for all $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, M\}$. Now, under Assumption 1 or 2 (or both), it is straightforward to show that the only solution of (45) is $\mathbf{d}_2 = \mathbf{0}$ and $d_4 = 0$. Finally, taking the second time derivative of (43), for $t = t_0$, and considering $\mathbf{d}_1 = \mathbf{d}_2 = \mathbf{0}$, $\mathbf{d}_3 = \mathbf{0}$, and $d_4 = 0$, it is straightforward to show that it must also be $d_5 = 0$. But this contradicts the hypothesis of existence of a unit vector $\mathbf{d}$ such that (41) holds. Hence, by contradicton, the observability

Gramian $\boldsymbol{\mathcal{W}}(t_0, t_f)$ is invertible and hence the nonlinear system (38) is observable in the sense established in the theorem, see (Batista et al. 2011, Lemma 1). $\square$

The fact that (38) is observable does not immediately entail that the nonlinear system (39) is observable nor that an observer for (38) is also an observer for (39), as there is nothing in the system dynamics (38) imposing the nonlinear algebraic relations that were at its own origin. Moreover, the range measurements as a nonlinear function of the state were also discarded. However, all that turns out to be true, as shown in the following theorem.

Theorem 4.2. *Under Assumptions 1 or 2 (or both), the nonlinear system* (39) *is observable on* $\mathcal{I} := [t_0, t_f]$, $t_0 < t_f$, *in the sense that, given the system input* $\mathbf{u}(t)$ *and the system output* $r_{1,1}(t), \ldots, r_{N,M}(t)$ *for* $t \in \mathcal{I}$, *the initial condition* $\mathbf{p}(t_0)$ *and* ${}^{I}\mathbf{v}_c(t_0)$ *is uniquely determined. Moreover, the initial conditions of the augmented nonlinear system* (38) *match those of* (39) *and hence an observer with globally asymptotically stable error dynamics for* (38) *is also an observer for* (39) *with globally asymptotically stable error dynamics.*

Proof. Let

$$\boldsymbol{x}_3(t_0) := \begin{bmatrix} \mathbf{p}'(t_0) \\ {}^{I}\mathbf{v}_c'(t_0) \\ x_{1,1}(t_0) \\ x_{1,2}(t_0) \\ \vdots \\ x_{N,M}(t_0) \\ x_3(t_0) \\ x_4(t_0) \end{bmatrix} \in \mathbb{R}^{3+3+NM+1+1}$$

be the initial condition of (38), which, from Theorem 4.1, is uniquely determined, and let $\mathbf{p}(t_0)$ and ${}^{I}\mathbf{v}_c(t_0)$ be the initial condition of (39). First, notice that it must be $x_{1,1}(t_0) = r_{1,1}(t_0), \ldots, x_{N,M}(t_0) = r_{N,M}(t_0)$ as these states are actually measured. Moreover, evaluating the outputs of the nonlinear system (38) that capture the LBL and USBL structure, given by (36) and (37), at $t = t_0$, gives

$$2\frac{(\mathbf{s}_m - \mathbf{s}_n)^T}{r_{m,j}(t_0) + r_{n,j}(t_0)}\mathbf{p}'(t_0) + x_{m,j}(t_0) - x_{n,j}(t_0) = \frac{\|\mathbf{s}_m\|^2 - \|\mathbf{s}_n\|^2 - 2(\mathbf{s}_m - \mathbf{s}_n)^T \mathbf{R}(t_0)\mathbf{a}_j}{r_{m,j}(t_0) + r_{n,j}(t_0)}$$

and

$$-2\frac{(\mathbf{a}_m - \mathbf{a}_n)^T \mathbf{R}^T(t_0)}{r_{i,m}(t_0) + r_{i,n}(t_0)}\mathbf{p}'(t_0) + x_{i,m}(t_0) - x_{i,n}(t_0) = \frac{\|\mathbf{a}_m\|^2 - \|\mathbf{a}_n\|^2 - 2(\mathbf{a}_m - \mathbf{a}_n)^T \mathbf{R}^T(t_0)\mathbf{s}_i}{r_{i,m}(t_0) + r_{i,n}(t_0)}$$

or, equivalently,

$$2(\mathbf{s}_m - \mathbf{s}_n)^T \mathbf{p}'(t_0) + r_{m,j}^2(t_0) - r_{n,j}^2(t_0) = \|\mathbf{s}_m\|^2 - \|\mathbf{s}_n\|^2 - 2(\mathbf{s}_m - \mathbf{s}_n)^T \mathbf{R}(t_0)\mathbf{a}_j \tag{46}$$

and

$$-2(\mathbf{a}_m - \mathbf{a}_n)^T \mathbf{R}^T(t_0)\mathbf{p}'(t_0) + r_{i,m}^2(t_0) - r_{i,n}^2(t_0) = \|\mathbf{a}_m\|^2 - \|\mathbf{a}_n\|^2 - 2(\mathbf{a}_m - \mathbf{a}_n)^T \mathbf{R}^T(t_0)\mathbf{s}_i. \tag{47}$$

Substituting (34) and (35) in (46) and (47), respectively, gives

$$2(\mathbf{s}_m - \mathbf{s}_n)^T \left[\mathbf{p}'(t_0) - \mathbf{p}(t_0)\right] = 0 \tag{48}$$

for all $m, n \in \{1, \ldots, N\}$, $n \neq n$, and

$$2\left(\mathbf{a}_m - \mathbf{a}_n\right)^T \mathbf{R}^T\left(t_0\right)\left[\mathbf{p}'\left(t_0\right) - \mathbf{p}\left(t_0\right)\right] = 0 \tag{49}$$

for all $m, n \in \{1, \ldots, M\}$, $n \neq n$. Now, it is straightforward to show that, under Assumption 1 the only solution of (48) is $\mathbf{p}'\left(t_0\right) = \mathbf{p}\left(t_0\right)$, while under Assumption 2 the only solution of (49) is also $\mathbf{p}'\left(t_0\right) = \mathbf{p}\left(t_0\right)$. Thus, so far it has been shown that

$$\begin{cases} \mathbf{p}'\left(t_0\right) = \mathbf{p}\left(t_0\right) \\ x_{1,1}\left(t_0\right) = r_{1,1}\left(t_0\right) \\ \vdots \\ x_{N,M}\left(t_0\right) = r_{N,M}\left(t_0\right) \end{cases}. \tag{50}$$

As a function of the initial state of (39), the square of the range readings can actually be written as

$$\begin{aligned} r_{i,j}^2(t) = &\left\|\int_{t_0}^{t} \mathbf{u}\left(\tau\right) d\tau\right\|^2 + 2\left[\mathbf{u}\left(t\right) + \mathbf{R}(t)\mathbf{a}_j\right] \cdot \mathbf{p}\left(t_0\right) \\ &+2\left(t - t_0\right)\left[-\mathbf{s}_i + \mathbf{R}(t)\mathbf{a}_j + \int_{t_0}^{t} \mathbf{u}\left(\sigma\right) d\sigma\right] \cdot {}^I\mathbf{v}_c\left(t_0\right) \\ &+2\left(t - t_0\right)\mathbf{p}\left(t_0\right) \cdot {}^I\mathbf{v}_c\left(t_0\right) + \left(t - t_0\right)^2 \left\|{}^I\mathbf{v}_c\left(t_0\right)\right\|^2 \\ &+r_{i,j}^2\left(t_0\right) - 2\mathbf{p}^T\left(t_0\right)\mathbf{R}\left(t_0\right)\mathbf{a}_j + 2\mathbf{s}_i^T\mathbf{R}\left(t_0\right)\mathbf{a}_j \\ &-2\mathbf{s}_i^T\mathbf{R}(t)\mathbf{a}_j - 2\left[\mathbf{s}_i - \mathbf{R}(t)\mathbf{a}_j\right] \cdot \int_{t_0}^{t} \mathbf{u}\left(\tau\right) d\tau, \end{aligned} \tag{51}$$

while as a function of the initial states of (38) it is possible to write

$$\begin{aligned} r_{i,j}^2(t) = &\left\|\int_{t_0}^{t} \mathbf{u}\left(\tau\right) d\tau\right\|^2 + 2\left[\mathbf{u}\left(t\right) + \mathbf{R}(t)\mathbf{a}_j\right] \cdot \mathbf{p}'\left(t_0\right) \\ &+2\left(t - t_0\right)\left[-\mathbf{s}_i + \mathbf{R}(t)\mathbf{a}_j + \int_{t_0}^{t} \mathbf{u}\left(\sigma\right) d\sigma\right] \cdot {}^I\mathbf{v}_c'\left(t_0\right) \\ &+2\left(t - t_0\right)x_3\left(t_0\right) + \left(t - t_0\right)^2 x_4\left(t_0\right) \\ &+x_{i,j}^2\left(t_0\right) - 2\mathbf{x}_1^T\left(t_0\right)\mathbf{R}\left(t_0\right)\mathbf{a}_j + 2\mathbf{s}_i^T\mathbf{R}\left(t_0\right)\mathbf{a}_j \\ &-2\mathbf{s}_i^T\mathbf{R}(t)\mathbf{a}_j - 2\left[\mathbf{s}_i - \mathbf{R}(t)\mathbf{a}_j\right] \cdot \int_{t_0}^{t} \mathbf{u}\left(\tau\right) d\tau. \end{aligned} \tag{52}$$

Now, comparing the differences of the squares of the ranges $r_{m,j}^2(t) - r_{n,j}^2(t)$ and $r_{i,m}^2(t) - r_{i,n}^2(t)$, using (50), (51), and (52), it is possible to write

$$\begin{cases} \left[\mathbf{s}_i - \mathbf{s}_j\right]^T\left[{}^I\mathbf{v}_c'\left(t_0\right) - {}^I\mathbf{v}_c\left(t_0\right)\right] = 0 \\ \left[\mathbf{a}_m - \mathbf{a}_n\right]^T \mathbf{R}^T(t)\left[{}^I\mathbf{v}_c'\left(t_0\right) - {}^I\mathbf{v}_c\left(t_0\right)\right] = 0 \end{cases} \tag{53}$$

for all $i, j \in \{1, \ldots, N\}$, $i \neq j$, and all $m, n \in \{1, \ldots, M\}$, $m \neq n$. Under Assumption 1, Assumption 2, or both, the only solution of (53) is

$${}^I\mathbf{v}_c'\left(t_0\right) = {}^I\mathbf{v}_c\left(t_0\right). \tag{54}$$

Now, comparing (51) with (52) and using (50) and (54) it follows that

$$2\left(t - t_0\right)\left[x_3\left(t_0\right) - \mathbf{p}\left(t_0\right) \cdot {}^I\mathbf{v}_c\left(t_0\right)\right] + \left(t - t_0\right)^2\left[x_4\left(t_0\right) - \left\|{}^I\mathbf{v}_c\left(t_0\right)\right\|^2\right] = 0. \tag{55}$$

As the functions $2(t-t_0)$ and $(t-t_0)^2$ are linearly independent, it follows from (55) that

$$\begin{cases} x_3(t_0) = \mathbf{p}(t_0) \cdot {}^I\mathbf{v}_c(t_0) \\ x_4(t_0) = \left\| {}^I\mathbf{v}_c(t_0) \right\|^2 \end{cases}.$$

But this concludes the proof: i) it has been shown that the initial conditions of (39) match those of (38), which are uniquely determined as shown in Theorem 4.1, hence concluding the proof of the first part of the theorem; and ii) the second part of the theorem follows from the first: the estimation error of an observer for (38) with globally asymptotically stable error dynamics converges to zero, which means that its estimates asymptotically approach the true state. But as the true state of (38) matches that of the nonlinear system (39), that means that an observer for (38) is also an observer for the original nonlinear system, with globally asymptotically stable error dynamics. □

4.3 *Kalman filter*

As a result of Theorem 4.2, a filtering solution for the nonlinear system (39) is simply obtained with the design of a Kalman filter for the augmented system (38), which can be regarded as LTV for this purpose as the output and input are available. The design is trivial and therefore it is omitted. Notice that the proposed solution is not an EKF, which would not offer global convergence guarantees, and no approximate linearizations are carried out.

In order to guarantee that the Kalman filter has globally exponentially stable error dynamics, stronger forms of observability are required, in particular uniform complete observability, see Sastry and Desoer (1982) and Jazwinski (1970). The pair $(\mathbf{A}_3(t), \mathbf{C}_3(t))$ can be easily shown to be uniformly completely observable following the same reasoning as in Theorem 4.1 but considering uniform bounds. The proof is omitted due to the lack of space.

5. Integrated LBL/USBL navigation system

In Section 3 a cascade observer was proposed for the attitude based on the measurements provided by the rate gyros and the LBL/USBL system, which gives in addition estimates of the rate gyro bias. The error dynamics were shown to be globally exponentially stable and the estimation system does not depend on any other quantities. In Section 4 the problem of estimating the linear motion quantities (inertial position and ocean current velocity) was addressed assuming perfect angular information, i.e., assuming that the attitude and the angular velocity are known. In practice, these quantities are provided by the estimator developed in Section 3 and as such the overall LBL/USBL navigation system consists in a cascade system, where the attitude observer feeds the position and velocity filter, as depicted in Fig. 1. In short, the rate gyro bias estimate is employed to obtain an estimate of the angular velocity, which is fed, together with the estimate of the attitude, to the estimator for linear motion quantities.

The fact that the exact values of $\mathbf{R}(t)$ and $\boldsymbol{\omega}(t)$ are not available for the Kalman filter proposed in Section 4.3 induces errors in the system matrices $\mathbf{A}_3(t)$ and $\mathbf{C}_3(t)$, as well as in the system input $\mathbf{u}_a(t)$, and only estimates of these quantities are available, i.e., the Kalman filter for the

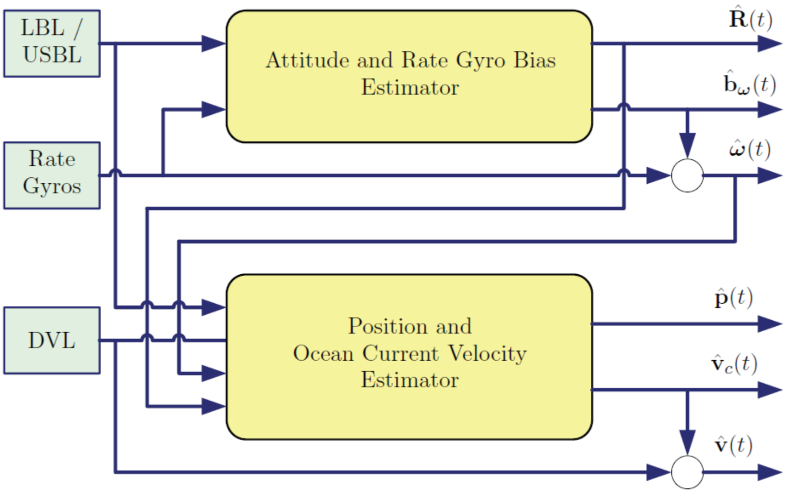


Figure 1. Integrated LBL/USBL navigation system

estimation of linear motion quantities has available

$$\hat{\mathbf{A}}_3(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{\hat{\mathbf{u}}^T(t) - \mathbf{a}_1^T \mathbf{S}(\hat{\boldsymbol{\omega}}(t))\hat{\mathbf{R}}^T(t)}{r_{1,1}(t)} & \frac{-\mathbf{s}_1^T + \mathbf{a}_1^T \hat{\mathbf{R}}^T(t)}{r_{1,1}(t)} & \mathbf{0} & \frac{1}{r_{1,1}(t)} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\hat{\mathbf{u}}^T(t) - \mathbf{a}_M^T \mathbf{S}(\hat{\boldsymbol{\omega}}(t))\hat{\mathbf{R}}^T(t)}{r_{N,M}(t)} & \frac{-\mathbf{s}_N^T + \mathbf{a}_M^T \hat{\mathbf{R}}^T(t)}{r_{N,M}(t)} & \mathbf{0} & \frac{1}{r_{N,M}(t)} & 0 \\ \mathbf{0} & \hat{\mathbf{u}}^T(t) & \mathbf{0} & 0 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 \end{bmatrix},$$

instead of $\mathbf{A}_3(t)$,

$$\hat{\mathbf{C}}_3(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{C}_{13} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_{21}(t) & \mathbf{0} & \mathbf{C}_{23} & \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{C}}_{31}(t) & \mathbf{0} & \mathbf{C}_{33} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

instead of $\mathbf{C}_3(t)$, with

$$\hat{\mathbf{C}}_{31}(t) := \begin{bmatrix} \hat{\mathbf{C}}_{31}^1(t) \\ \vdots \\ \hat{\mathbf{C}}_{31}^N(t) \end{bmatrix} \in \mathbb{R}^{(N\binom{M}{2}C)\times 3},$$

$$\hat{\mathbf{C}}_{31}^i(t) := -2 \begin{bmatrix} \frac{(\mathbf{a}_1 - \mathbf{a}_2)^T \hat{\mathbf{R}}^T(t)}{r_{i,1}(t) + r_{i,2}(t)} \\ \frac{(\mathbf{a}_1 - \mathbf{a}_3)^T \hat{\mathbf{R}}^T(t)}{r_{i,1}(t) + r_{i,3}(t)} \\ \vdots \\ \frac{(\mathbf{a}_{M-1} - \mathbf{a}_M)^T \hat{\mathbf{R}}^T(t)}{r_{i,M-1}(t) + r_{i,M}(t)} \end{bmatrix} \in \mathbb{R}^{\binom{M}{2}C\times 3},$$

and

$$\hat{\mathbf{u}}_a(t) := \begin{bmatrix} \hat{\mathbf{u}}(t) \\ \hat{u}_{r_{1,1}}(t) \\ \vdots \\ \hat{u}_{r_{N,M}}(t) \end{bmatrix}$$

instead of $\mathbf{u}_a(t)$, with $\hat{\mathbf{u}}(t) := \hat{\mathbf{R}}(t)\mathbf{v}_r(t)$ and

$$\hat{u}_{r_{i,j}}(t) := \frac{\hat{\mathbf{u}}^T(t)\hat{\mathbf{R}}(t)\mathbf{a}_j - \hat{\mathbf{u}}^T(t)\mathbf{s}_i - \mathbf{s}_i^T\hat{\mathbf{R}}(t)\mathbf{S}\left(\hat{\boldsymbol{\omega}}(t)\right)\mathbf{a}_j}{r_{i,j}(t)}.$$

Moreover, the augmented measurements are also estimated, and

$$\hat{\mathbf{y}}_3(t) = \begin{bmatrix} \mathbf{y}_1(t) \\ \frac{\|\mathbf{s}_1\|^2-\|\mathbf{s}_2\|^2-2(\mathbf{s}_1-\mathbf{s}_2)^T\hat{\mathbf{R}}(t)\mathbf{a}_1}{r_{1,1}(t)+r_{2,1}(t)} \\ \frac{\|\mathbf{s}_1\|^2-\|\mathbf{s}_3\|^2-2(\mathbf{s}_1-\mathbf{s}_3)^T\hat{\mathbf{R}}(t)\mathbf{a}_1}{r_{1,1}(t)+r_{3,1}(t)} \\ \vdots \\ \frac{\|\mathbf{s}_{N-2}\|^2-\|\mathbf{s}_N\|^2-2(\mathbf{s}_{N-2}-\mathbf{s}_N)^T\hat{\mathbf{R}}(t)\mathbf{a}_M}{r_{N-2,M}(t)+r_{N,M}(t)} \\ \frac{\|\mathbf{s}_{N-1}\|^2-\|\mathbf{s}_N\|^2-2(\mathbf{s}_{N-1}-\mathbf{s}_N)^T\hat{\mathbf{R}}(t)\mathbf{a}_M}{r_{N-1,M}(t)+r_{N,M}(t)} \\ \frac{\|\mathbf{a}_1\|^2-\|\mathbf{a}_2\|^2-2(\mathbf{a}_1-\mathbf{a}_2)^T\hat{\mathbf{R}}^T(t)\mathbf{s}_1}{r_{1,1}(t)+r_{1,2}(t)} \\ \frac{\|\mathbf{a}_1\|^2-\|\mathbf{a}_3\|^2-2(\mathbf{a}_1-\mathbf{a}_3)^T\hat{\mathbf{R}}^T(t)\mathbf{s}_1}{r_{1,1}(t)+r_{1,3}(t)} \\ \vdots \\ \frac{\|\mathbf{a}_{M-2}\|^2-\|\mathbf{a}_M\|^2-2(\mathbf{a}_{M-2}-\mathbf{a}_M)^T\hat{\mathbf{R}}^T(t)\mathbf{s}_N}{r_{N,M-2}(t)+r_{N,M}(t)} \\ \frac{\|\mathbf{a}_{M-1}\|^2-\|\mathbf{a}_M\|^2-2(\mathbf{a}_{M-1}-\mathbf{a}_M)^T\hat{\mathbf{R}}^T(t)\mathbf{s}_N}{r_{N,M-1}(t)+r_{N,M}(t)} \end{bmatrix}$$

is employed instead of $\mathbf{y}_3(t)$.

Let $\mathbf{w}(t)$ denote the system disturbances, assumed as zero-mean white Gaussian noise, with $\mathrm{E}\left[\mathbf{w}(\mathrm{t})\,\mathbf{w}^{\mathrm{T}}(\mathrm{t}-\tau)\right] = \boldsymbol{\Xi}\delta(\tau)$, and $\mathbf{n}(t)$ be the output noise, assumed as zero-mean white Gaussian noise, with $\mathrm{E}\left[\mathbf{n}(\mathrm{t})\,\mathbf{n}^{\mathrm{T}}(\mathrm{t}-\tau)\right] = \boldsymbol{\Theta}\delta(\tau)$ and $\mathrm{E}\left[\mathbf{w}(\mathrm{t})\,\mathbf{n}^{\mathrm{T}}(\mathrm{t}-\tau)\right] = \mathbf{0}$. The resulting Kalman filter is given by

$$\dot{\hat{\boldsymbol{x}}}_3(t) = \hat{\mathbf{A}}_3(t)\hat{\boldsymbol{x}}_3(t) + \mathbf{B}_3\hat{\mathbf{u}}_a(t) + \hat{\mathbf{K}}(t)\left[\hat{\mathbf{y}}_3(t) - \hat{\mathbf{C}}_3(t)\hat{\boldsymbol{x}}_3(t)\right],$$

where $\hat{\mathbf{K}}(t)$ is the Kalman gain,

$$\hat{\mathbf{K}}(t) = \hat{\mathbf{P}}(t)\,\hat{\mathbf{C}}_3^T(t)\boldsymbol{\Theta}^{-1},$$

where $\hat{\mathbf{P}}(t)$ is the covariance matrix, which satisfies

$$\dot{\hat{\mathbf{P}}}(t) = \hat{\mathbf{A}}_3(t)\,\hat{\mathbf{P}}(t) + \hat{\mathbf{P}}(t)\,\hat{\mathbf{A}}_3^T(t) + \boldsymbol{\Xi} - \hat{\mathbf{P}}(t)\,\hat{\mathbf{C}}_3^T(t)\boldsymbol{\Theta}^{-1}\hat{\mathbf{C}}_3(t)\hat{\mathbf{P}}(t).$$

Naturally, it is necessary to show that the error of the perturbed Kalman filter converges to zero for all initial conditions. This is a theoretical problem, that of the study of the convergence of the error of the Kalman filter when the system matrices $\mathbf{A}(t)$ and $\mathbf{C}(t)$, as well as the system output $\mathbf{y}(t)$, are perturbed by exponential decaying errors. Assuming: i) bounds on the system matrices;

ii) that the nominal system is uniformly completely observable; and iii) that the system state is bounded, it can actually be shown that the error of the Kalman filter converges exponentially fast for all initial conditions. This falls out of the scope of this paper and will be detailed in a future article. However, all required assumptions are verified, in practice, for the proposed LBL/USBL setup, as the mission scenario is bounded in space and the linear and angular velocities must also be bounded due to the actuation bounds of any real system.

6. Simulations

This section provides simulation results in order to demonstrate the achievable performance with the proposed solution. In the simulations, the 3-D kinematic model for an underwater vehicle was employed. It is not necessary to consider the dynamics as the estimators are purely kinematic, hence the results apply to all underwater vehicles, regardless of the dynamics. The trajectory described by the vehicle is shown in Fig. 1. The LBL configuration is composed of

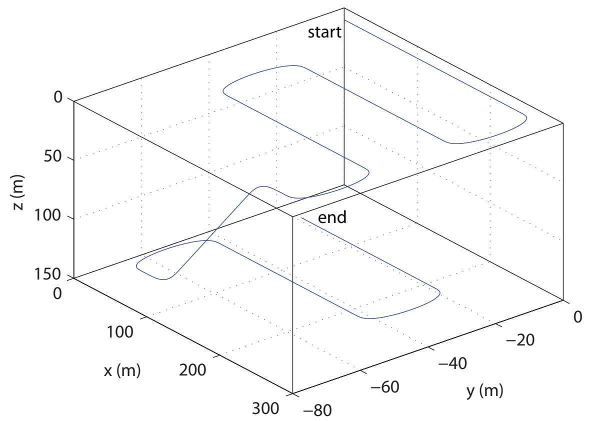


Figure 1. Trajectory described by the vehicle

4 acoustic transponders and their inertial positions are $\mathbf{s}_1 = \begin{bmatrix} 0\ 0\ 0 \end{bmatrix}$ (m), $\mathbf{s}_2 = \begin{bmatrix} 0\ 0\ 250 \end{bmatrix}$ (m), $\mathbf{s}_3 = \begin{bmatrix} 1000\ 0\ 250 \end{bmatrix}$ (m), $\mathbf{s}_4 = \begin{bmatrix} 0\ 1000\ 250 \end{bmatrix}$ (m), while the positions of the USBL array receivers, in body-fixed coordinates, are $\mathbf{a}_1 = \begin{bmatrix} 0\ 0\ 0 \end{bmatrix}$ (m), $\mathbf{a}_2 = \begin{bmatrix} 0\ 0.3\ 0 \end{bmatrix}$ (m), $\mathbf{a}_3 = \begin{bmatrix} 0.20\ 0.15\ 0.15 \end{bmatrix}$ (m), $\mathbf{a}_4 = \begin{bmatrix} 0.20\ 0.15\ {-0.15} \end{bmatrix}$ (m), hence both Assumptions 1 and 2 are satisfied.

Sensor noise was considered for all sensors. In particular, the LBL range measurements, the USBL range differences of arrival, and the DVL relative velocity readings are assumed to be corrupted by additive uncorrelated zero-mean white Gaussian noise, with standard deviations of 1 m, 6×10^{-3} m, and 0.01 m, respectively. The angular velocity measurements are also assumed to be perturbed by additive, zero mean, white Gaussian noise, with standard deviation of 0.05 $^\circ$/s.

To tune the Kalman filter for the estimation of the linear motion quantities, the state disturbance intensity matrix was chosen as

$$\operatorname{diag}\left(10^{-2}\mathbf{I},\, 10^{-4}\mathbf{I},\, 10^{-2},\, \ldots, 10^{-2},\, 10^{-2},\, 10^{-3}\right)$$

and the output noise intensity matrix as

$$\operatorname{diag}\left(\mathbf{Q}_0, \mathbf{Q}_0, \mathbf{Q}_0, \mathbf{Q}_0, 1, \ldots, 1\right),$$

where $\mathbf{Q}_0 := \operatorname{diag}(1,\, 0.6,\, 0.6,\, 0.6)$. The parameters of the attitude observer were chosen as $\alpha(m,n,i,j) = 0.1$, $\beta(m,n,i,j) = 5 \times 10^{-8}$ for all $(m,n,i,j) \in \mathcal{C}_\mathbf{s} \times \mathcal{C}_\mathbf{a}$, and $\mathbf{Q} = 10^4\mathbf{I}$. All initial conditions were set to zero but the initial attitude estimate, which was set with a large error, with a rotation of 180 degrees about the z-axis.

The convergence of the attitude observer error is very fast, as it is possible to observe from the evolution of the errors of the components of the rotation matrix and the rate gyro bias error, which are depicted in Fig. 2. The error of the additional states of the attitude observer, $\tilde{\mathbf{q}}(t)$ also

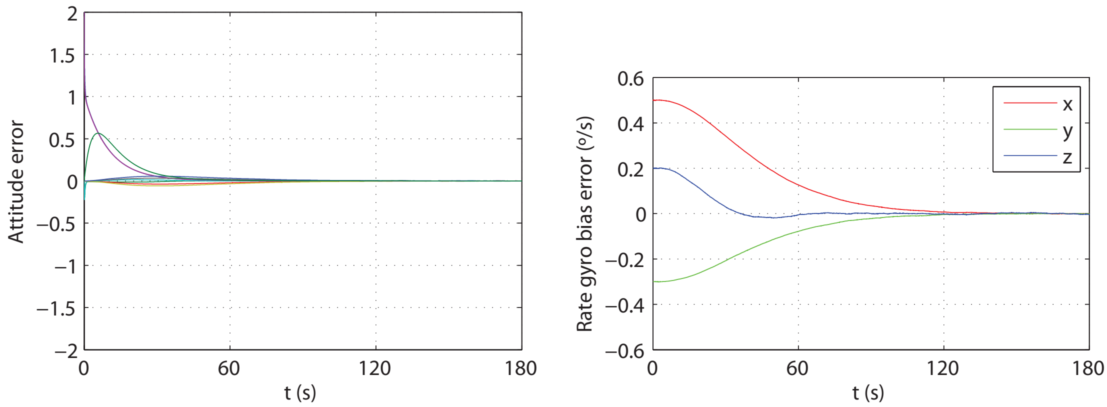


Figure 2. Initial convergence of the attitude observer errors $\tilde{\mathbf{R}}(t)$ and $\tilde{\mathbf{b}}_{\boldsymbol{\omega}}(t)$

converges and is not shown here only because it corresponds to intermediate states with no use in practice.

The initial evolution of the position and velocity errors are depicted in Fig. 3. As it can be seen from the various plots, the convergence rate of the filter for the estimation of the linear motion quantities is quite fast.

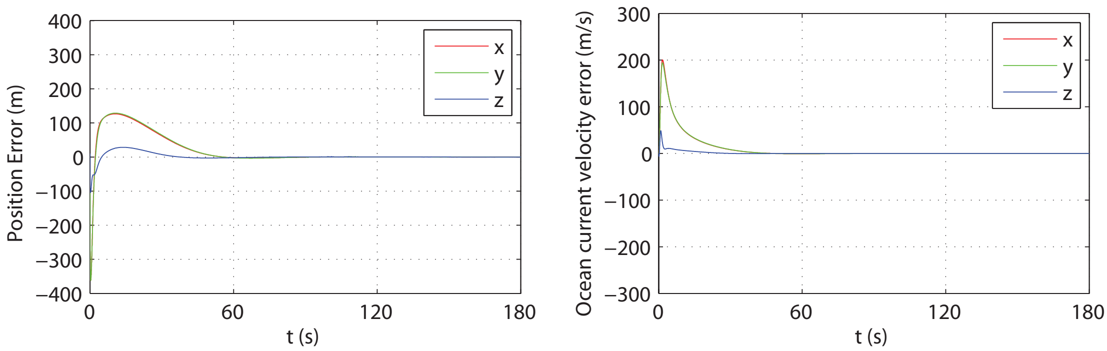


Figure 3. Initial convergence of the position error $\tilde{\mathbf{p}}(t)$ and the current velocity error $\tilde{\mathbf{v}}_c(t)$

In order to evaluate the performance of the attitude observer, and for the purpose of performance evaluation only, an additional error variable is defined as $\tilde{\mathbf{R}}_p(t) = \mathbf{R}^T(t)\hat{\mathbf{R}}(t)$, which corresponds to the rotation matrix error. Using the Euler angle-axis representation for this new error variable,

$$\tilde{\mathbf{R}}_p(t) = \mathbf{I}\cos\left(\tilde{\theta}(t)\right) + \left[1 - \cos\left(\tilde{\theta}(t)\right)\right]\tilde{\mathbf{d}}(t)\tilde{\mathbf{d}}^T(t) - \mathbf{S}\left(\tilde{\mathbf{d}}(t)\right)\sin\left(\tilde{\theta}(t)\right),$$

where $0 \leq \tilde{\theta}(t) \leq \pi$ and $\tilde{\mathbf{d}}(t) \in \mathbb{R}^3$, $\left\|\tilde{\mathbf{d}}(t)\right\| = 1$, are the angle and axis that represent the rotation error, the performance of the observer is identified with the evolution of $\tilde{\theta}$. After the initial transients fade out, the resulting angle mean error is around $0.06\ ^\circ$.

Finally, in order to better evaluate the performance of the proposed solution, the Monte Carlo method was applied, and 1000 simulations were carried out with different, randomly generated noise signals. The standard deviation of the errors were computed for each simulation and

Table 1. Standard deviation of the steady-state estimation error, averaged over 1000 runs of the simulation

Variable	Standard deviation
$\tilde{\mathbf{p}}_x$ (m)	3.6×10^{-2}
$\tilde{\mathbf{p}}_y$ (m)	4.0×10^{-2}
$\tilde{\mathbf{p}}_z$ (m)	4.4×10^{-2}
$\tilde{\mathbf{v}}_x$ (m/s)	2.3×10^{-3}
$\tilde{\mathbf{v}}_y$ (m/s)	2.4×10^{-3}
$\tilde{\mathbf{v}}_z$ (m/s)	3.0×10^{-3}
$\tilde{\mathbf{b}}_{\boldsymbol{\omega}x}$ (°/s)	1.2×10^{-3}
$\tilde{\mathbf{b}}_{\boldsymbol{\omega}y}$ (°/s)	0.9×10^{-3}
$\tilde{\mathbf{b}}_{\boldsymbol{\omega}z}$ (°/s)	2.0×10^{-3}

averaged over the set of simulations. The results are depicted in Table 1. The mean attitude angle error is 0.05 °. As it is possible to observe, the standard deviation of the errors is very low, adequate for the sensor suite that was considered.

7. Conclusions

This paper proposed a novel integrated tightly-coupled navigation filter for autonomous vehicles based on a combined Long Baseline / Ultra-short Baseline (LBL/USBL) positioning system. First, rate gyro bias is proposed, which feeds a second attitude observer that yields estimates of the rotation matrix from body-fixed to inertial coordinates. The error of the cascade rate gyro bias and attitude observer was shown to be globally exponentially stable (GES). Secondly, a framework for the estimation of the position of the vehicle and the ocean current velocity was proposed, which also features GES error dynamics assuming perfect knowledge of the attitude of the vehicle. This quantity is actually provided by the previous observer, which results in an overall cascade system. The structure is tightly-coupled in the sense that the actual measurements of the LBL/USBL are directly employed in the estimator dynamics. Simulation results were carried out, including Monte Carlo simulations, that evidence excellent performance of the proposed solution in the presence of realistic sensor noise. Future work includes: i) explicitly account for measurement delays; ii) comparison with the Extended Kalman filter, which does not offer global convergence guarantees; iii) design of an outlier rejection algorithm that takes advantage of the fact that each range or range difference of arrival is used directly in the filter, meaning that it is possible to exclude some measurements while still operating with the others; iv) study of the convergence of the error of the Kalman filter when the system matrices, as well as the system output, are perturbed by exponential decaying errors; and v) experimental validation of the proposed estimation solution.

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