# Markov chains and stochastic stability

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Reading group on Sequential Decision Making

# Outline of the presentation

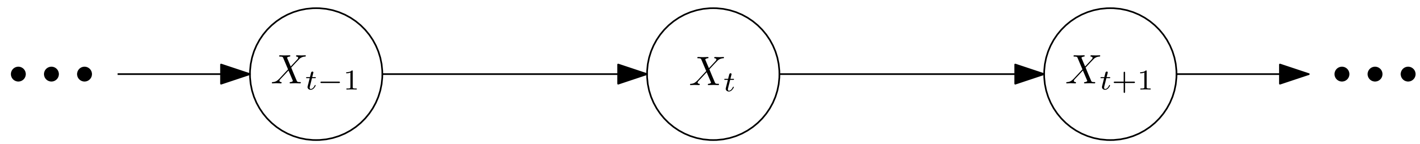
- **Markov chains as dynamic models**
- Transition probabilities
- Irreducibility and $\psi$-irreducibility
- Cyclic behavior
- Transience and recurrence
- Stationarity and invariance
- Ergodicity

## Markov chains as dynamic models

- Markov chains generalize the (discrete-time) dynamic model

$$X_{t+1} = f(X_t); \tag{1}$$

- In a Markov chain, the state $X_t$ at time $t$ depends only on the state at time $t-1$;
- Unlike dynamic models such as that in $(1)$, $X_t$ is not a deterministic function of $X_{t-1}$ (stochastic process).



# Markov chains as dynamic models (2)

Purposes of the presentation:

- Describe several “interesting behaviors” of a Markov chain (cyclic behavior, recurrence, etc.);
- Generalize the idea of *stability* to the Markov chain framework;
- Refer a Lyapunov-like criterion for stability (if there is time);
- Refer some important applications (LLN, CLT, LIL, Poisson equation, etc).

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## Markov chains and transition probabilities

**Definition 1.** *A* homogeneous Markov chain *is a stochastic process* $\{X_t\}$*, where each random variable* $X_t$ *is defined in a set* $\mathcal{X}$ *(the state-space) and verifying*

$$\mathbb{P}\left[X_{t+1} \in U \mid \mathcal{F}_t\right] = \mathbb{P}\left[X_{t+1} \in U \mid X_t = x\right] = \mathsf{P}(x, U).$$

We will consider 2 different cases:

- Countable state-space (P is a probability matrix);
- General state-space (P is a probability kernel).

# Markov chains and transition probabilities (2)

A Markov chain is defined from:

- The state-space $\mathcal{X}$;
- The transition probabilities in P;
- An *initial distribution* $\mu_0$.

Often $\mu_0$ is implicit. The Markov chain can then be referred as a pair $(\mathcal{X}, \mathsf{P})$.

## $n$-skeleton chain

The kernel $\mathsf{P}^n$, defined recursively as

$$\mathsf{P}^1(x,z) = \mathsf{P}(x,z); \qquad \mathsf{P}^{n+1}(x,z) = \sum_{y \in \mathcal{X}} \mathsf{P}(y,z)\mathsf{P}^n(x,y),$$

or

$$\mathsf{P}^1(x,U) = \mathsf{P}(x,U) \qquad \mathsf{P}^{n+1}(x,U) = \int_{\mathcal{X}} \mathsf{P}(y,U)\mathsf{P}^n(x,dy),$$

determines the $n$-step transition probabilities

$$\mathsf{P}^n(x,U) = \mathbb{P}\left[X_{t+n} \in U \mid X_t = x\right].$$

# $n$-skeleton chain (2)

The definition of $\mathsf{P}^n$ leads to the known *Chapman-Kolmogorov equations*. For any $m < n$,

$$P^n(x,z) = \sum_{y \in \mathcal{X}} \mathsf{P}^m(x,y) P^{n-m}(y,z)$$

or

$$P^n(x,U) = \int_{\mathcal{X}} \mathsf{P}^m(y,U) P^{n-m}(x,dy).$$

Also, the pair $(\mathcal{X}, \mathsf{P}^n)$ is a new Markov chain—the $n$-*skeleton* chain for the chain $(\mathcal{X}, \mathsf{P})$.

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## Irreducibility

**Case 1:** Countable state-space $\mathcal{X}$

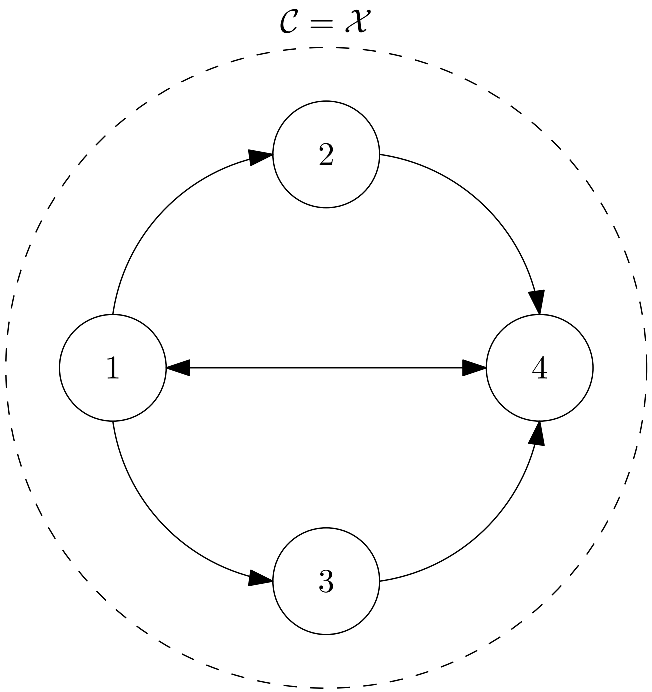
- A state $y$ is *accessible* from $x$ (denoted $x \to y$) if

$$P^n(x, y) > 0$$

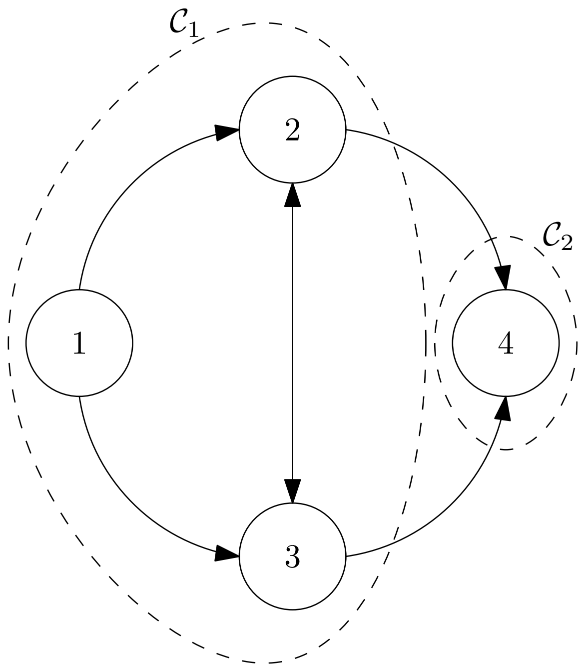
  for some $n$;

- If $x \to y$ and $y \to x$, $x$ and $y$ *communicate* (denoted $x \leftrightarrow y$);
- The relation "$\leftrightarrow$" partitions $\mathcal{X}$ into disjoint classes $\mathcal{C}_1, \ldots, \mathcal{C}_m$;
- The chain $(\mathcal{X}, \mathsf{P})$ is *irreducible* if $m = 1$, *i.e.*, , $x \leftrightarrow y$ for any $x, y \in \mathcal{X}$.

# Irreducibility (2)




Irreducible chain




Non-irreducible chain

# $\psi$-Irreducibility

**Case 2:** General state-space $\mathcal{X}$

- For a set $U \subset \mathcal{X}$, let $\tau_U$ be the *first return time to* $U$,

$$\tau_U = \min\{t > 0 \mid X_t \in U\};$$

- A Markov chain $(\mathcal{X}, \mathsf{P})$ is $\varphi$-irreducible if there is a measure $\varphi$ such that

$$\mathbb{P}[\tau_U < \infty \mid X_0 = x] > 0$$

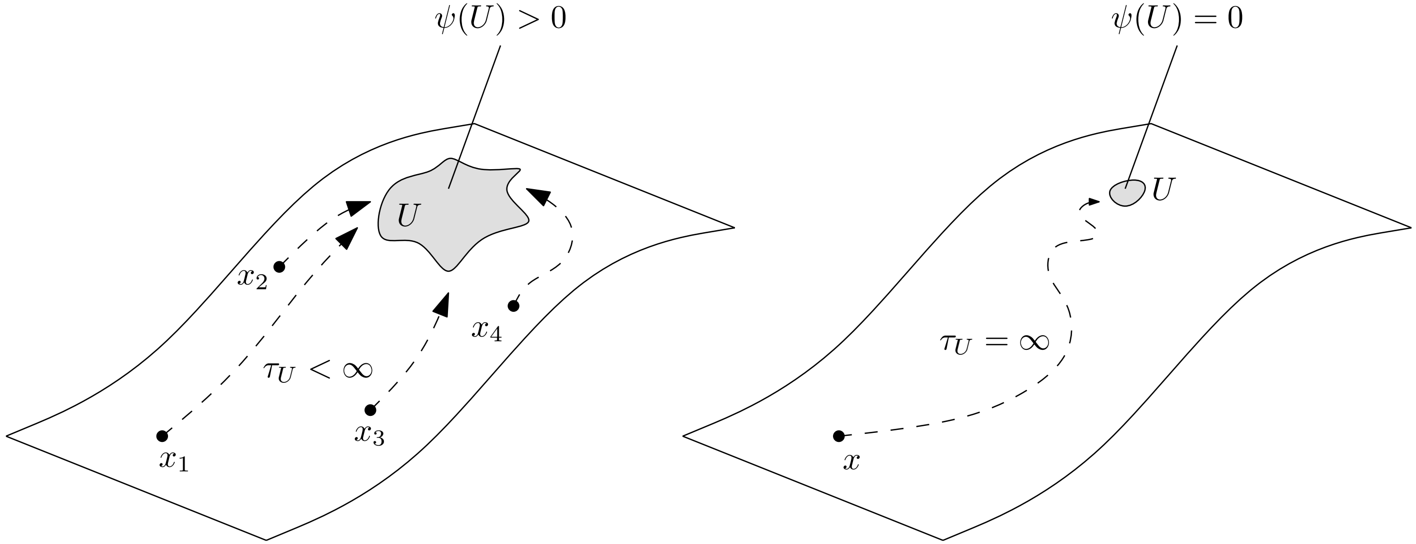
for any set $U$ with $\varphi(U) > 0$ and any $x \in \mathcal{X}$;

- If $(\mathcal{X}, \mathsf{P})$ is $\varphi$-irreducible, there is a maximal irreducibility measure $\psi$ such that

$$\psi(U) = 0 \quad \Rightarrow \quad \psi(\{x \in \mathcal{X} \mid \mathbb{P}[\tau_U < \infty \mid X_0 = x] > 0\}) = 0.$$

# $\psi$-Irreducibility (2)

- Any set $U$ with $\psi(U) > 0$ is reached *in finite time* from any point $x \in \mathcal{X}$;
- Any set $U$ with $\psi(U) = 0$ is not reached *in finite time* except for an "insignificant" set of points.



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# Cyclic behavior

**Case 1:** Countable state-space $\mathcal{X}$

- Given a state $y$, the *period* of $y$, $d(y)$, is

$$d(y) = g.c.d. \{n \mid \mathsf{P}^n(y,y) > 0\} \,.$$

- If $x, y \in \mathcal{C}_i$, $d(x) = d(y)$;
- If $(\mathcal{X}, \mathsf{P})$ is irreducible, $d = d(x)$ for any $x$ is the *period* of the chain;
- The chain is *aperiodic* if $d = 1$ and *periodic* otherwise.

## Cyclic behavior (2)

- If the chain $(\mathcal{X}, \mathsf{P})$ has period $d$, $\mathcal{X}$ can be partitioned into a family $\mathcal{D}$ of $d$ sets, $\mathcal{D} = \{D_1, \ldots, D_d\}$, such that

$$\mathsf{P}(x, D_{i+1}) = 1, \qquad i = 1, \ldots, d-1 \pmod{d},$$

for all $x \in D_i$.

- The family $\mathcal{D}$ is called a $d$-*cycle* for the chain $(\mathcal{X}, \mathsf{P})$.

## Cyclic behavior (3)

**Case 2:** General state-space $\mathcal{X}$

Though technically more elaborate, for a $\psi$-irreducible chain $(\mathcal{X}, \mathsf{P})$, we can define a maximal family $\mathcal{D}$ of sets, $\mathcal{D} = \{D_1, \ldots, D_d\}$, such that:

- $\mathsf{P}(x, D_{i+1}) = 1, \qquad i = 1, \ldots, d-1 \pmod d$, for all $x \in D_i$;
- $\psi(\mathcal{X} - \cup_i D_i) = 0$.

The family $\mathcal{D}$ is also a $d$-cycle for the chain $(\mathcal{X}, \mathsf{P})$; the chain is *aperiodic* if $d = 1$ and *periodic* otherwise.

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## Transience and recurrence

**Case 1:** Countable state-space $\mathcal{X}$

- A state $y \in \mathcal{X}$ is *transient* if

$$\mathbb{E}\left[\sum_{t=1}^{\infty} \mathbb{I}_y(X_t)\right] < \infty;$$

- On the other hand, it is *recurrent* if

$$\mathbb{E}\left[\sum_{t=1}^{\infty} \mathbb{I}_y(X_t)\right] = \infty;$$

- If the chain $(\mathcal{X}, \mathsf{P})$ is irreducible, then either *all states* in $\mathcal{X}$ are transient or all are recurrent;
- An irreducible chain is *transient* or *recurrent*, according to all its states being transient or recurrent.

# Transience and recurrence (2)

**Case 2:** General state-space $\mathcal{X}$

- For a set $U \subset \mathcal{X}$, let $\eta_U$ be the *occupation time of* $U$,

$$\eta_U = \sum_{t=1}^{\infty} \mathbb{I}_U(X_t);$$

- A set $U \subset \mathcal{X}$ is *transient* if

$$\mathbb{E}\left[\eta_U \mid X_0 = x\right] < \infty$$

for all $x \in \mathcal{X}$ and *recurrent* if

$$\mathbb{E}\left[\eta_U \mid X_0 = x\right] = \infty$$

for all $x \in \mathcal{X}$;

# Transience and recurrence (3)

- A chain $(\mathcal{X}, \mathsf{P})$ is *recurrent* if it is $\psi$-irreducible and

$$\mathbb{E}\left[\eta_U \mid X_0 = x\right] = \infty$$

for every $x \in \mathcal{X}$ and every set $U$ such that $\psi(U) > 0$;

- It is *transient* if $\mathcal{X}$ is transient.

## Transience and recurrence (4)

Summarizing,

- A recurrent chain (expectedly) visits each state infinitely often;
- A transient chain (expectedly) visits each state only a finite number of times.

In terms of long-term behavior of the chain, recurrent chains are more interesting.

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# Stationarity and invariance

**Case 1:** Countable state-space $\mathcal{X}$

- A probability distribution $\mu$ over $\mathcal{X}$ is *invariant* if

$$\mu(y) = \sum_{x \in \mathcal{X}} \mathsf{P}(x, y)\mu(x).$$

- Such distribution *remains unchanged* after a transition;
- Given any initial distribution $\mu_0$ over $\mathcal{X}$, if the limit distribution

$$\bar{\mu} = \lim_{t \to \infty} \mu^\top \mathsf{P}^t$$

exists, it must be invariant;

## Stationarity and invariance (2)

- Invariant distributions describe *stationary behavior* as do equilibrium points in dynamic systems;
- A Markov chain for which there is an invariant probability distribution $\mu$ is said to be *positive*.
- A recurrent chain $(\mathcal{X}, \mathsf{P})$ is positive if

$$\mathbb{E}\left[\tau_x \mid X_0 = x\right] < \infty,$$

  *i.e.*, every state is returned upon in finite time;
- In this case, the invariant distribution $\mu$ is uniquely defined.

# Stationarity and invariance (3)

**Case 2:** General state-space $\mathcal{X}$

- A probability measure $\mu$ over $\mathcal{X}$ is *invariant* if

$$\mu(U) = \int_{\mathcal{X}} \mathsf{P}(x, U) d\mu(x).$$

- A Markov chain for which there is an invariant probability measure $\mu$ is said to be *positive*.
- A recurrent chain $(\mathcal{X}, \mathsf{P})$ is positive if there is a set $C$ verifying some technical conditions and such that

$$\mathbb{E}\left[\tau_C \mid X_0 = x\right] < \infty,$$

for every $c \in C$.

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- **Ergodicity**

# Ergodicity

Ergodicity is related with the "convergence" of the chain to stationarity.

**Definition 2.** *A Markov chain* $(\mathcal{X}, \mathsf{P})$ *is* ergodic *if*

$$\left\|\mathsf{P}^t(x,\cdot) - \mu(U)\right\| \to 0$$

*for any* $x \in \mathcal{X}$ *and any* $U \subset \mathcal{X}$*.*

# Further definitions of ergodicity

**Definition 3.** *A Markov chain* $(\mathcal{X}, \mathsf{P})$ *is* geometrically ergodic *if, given any initial measure* $\mu_0$,

$$\sum_{t=0}^{\infty} r^t \left\|\mu_0 \mathsf{P}^t - \mu^*\right\| < \infty$$

*where* $r$ *is some constant such that* $r > 1$.

**Definition 4.** *An ergodic Markov chain* $(\mathcal{X}, \mathsf{P})$ *is* uniformly ergodic *if*

$$\sup_{x \in \mathcal{X}} \left\|\mathsf{P}^t(x, \cdot) - \mu\right\| \to 0$$

*as* $t \to \infty$.

## The use of ergodicity

Ergodicity of Markov chains is central in many aspects of RL:

- The implicit stationarity assumption of many on-line learning methods can be stated in terms of ergodicity of the underlying Markov chain [2];
- In advanced algorithms [6, 7, 8], geometric ergodicity is required to ensure that the "transient" of the chain quickly vanishes;
- Stochastic approximation algorithms (of which many RL methods are examples) strongly rely on geometric ergodicity to ensure convergence [1, 3, 4].

# The use of ergodicity

More fundamentally,

- Ergodicity allows the derivation of several limit theorems such as [5]:
  - Law of Large Numbers (LLN)
  - Law of the Iterated Logarithm (LIL)
  - Central Limit Theorem (CLT)

  applied to any measurable real-function $H$ defined on $\mathcal{X}$.

- It provides conditions for the existence of solutions for the Poisson equation

$$(\mathbf{I} - \mathsf{P})\nu(x) = H(x) - (\mu H),$$

  a fundamental tool to evaluate approximation errors along sample paths.

## If there is time...

Several results identify the conditions under which a Markov chain is ergodic/geometrically ergodic. Two simple theorems are

**Theorem 5.** *Every irreducible, aperiodic, positive Markov chain* $(\mathcal{X}, \mathsf{P})$ *defined on a countable space* $\mathcal{X}$ *is ergodic.*

**Theorem 6.** *Every ergodic Markov chain* $(\mathcal{X}, \mathsf{P})$ *defined in a countable space* $\mathcal{X}$ *is geometrically ergodic.*

## If there is time... (2)

- For general state-spaces, these results can be generalized, but require several more elaborate concepts that have not been covered here.
- However, one criterium for (geometric) ergodicity relies on the existence of a non-negative, real function $V$ defined on $\mathcal{X}$ such that

$$(\Delta V)(x) \leq -\beta V(x) + b\mathbb{I}_C(x).$$

- Roughly speaking, the function $V$ can be interpreted as a Lyapunov function for the chain!

*

## References


[1] A. Benveniste, M. Métivier, and P. Priouret. *Adaptive Algorithms and Stochastic Approximations*, volume 22 of *Applications of Mathematics*. Springer-Verlag, Berlin, 1990.

[2] D. P. Bertsekas and J. N. Tsitsiklis. *Neuro-Dynamic Programming*. Optimization and Neural Computation Series. Athena Scientific, Belmont, Massachusetts, 1996.

[3] B. Bharath and V. S. Borkar. Stochastic approximation algorithms: overview and recent trends. *Sādhanā*, 24(4, 5):425–452, 1999.

[4] H. J. Kushner and G. G. Yin. *Stochastic Approximation and Recursive Algorithms and Applications*, volume 35 of *Applications of Mathematics*. Springer-Verlag New York, Inc., second edition, 2003.

[5] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability*. Communications and Control Engineering Series. Springer-Verlag, New York, 1993.

[6] C. Szepesvári and W. D. Smart. Interpolation-based $Q$-learning. In *Proceedings of the 21st International Conference on Machine learning (ICML'04)*, pages 100–107, New York, USA, July 2004. ACM Press.

[7] J. N. Tsitsiklis and B. Van Roy. An analysis of temporal-difference learning with function approximation. *IEEE Transactions on Automatic Control*, 42(5): 674–690, May 1996.

[8] B. Van Roy. *Learning and value function approximation in complex decision processes*. PhD thesis, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, June 1998.