# SMOOTH PATH PLANNING

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April.2002

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# References

- Smooth Path Planning
  - Y.Kanayama, B. Hartman, “Smooth Local Path Planning for Autonomous Vehicles”, in *Autonomous Robot Vehicles*, edited by Cox, Wilfong, Springer, 1990, pp.62-67.

Reading assignment

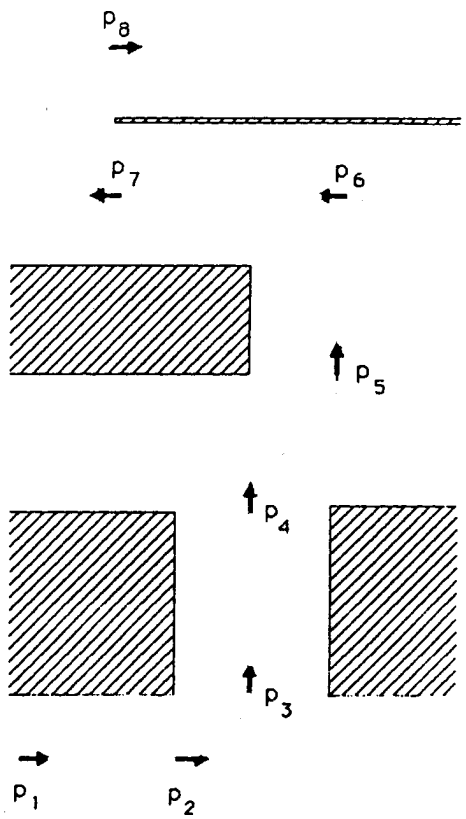
- Basis on Curves
  - Taylor Mann, “Advanced Calculus”, Xerox College Publishing, 1972.

# Problem Statement

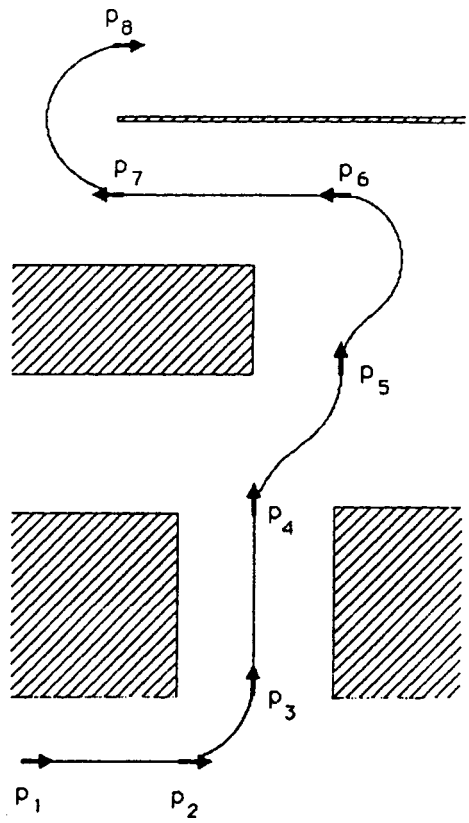
- Unsmooth motions cause slippage of wheels which degrades the robot dead-reckoning ability
- **Problem**: define a smooth path between start and goal posture
  - What is a smooth path ?
  - How do we compare smoothness?
  - Is there any other characteristic that we may use to have a smooth motion

## Example

Path define by $(p_1,p_2,p_3,p_4,p_5,p_6)$

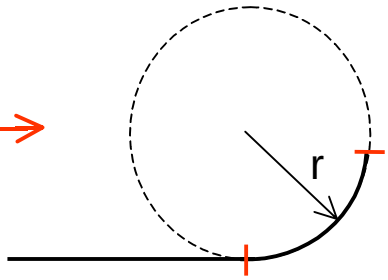


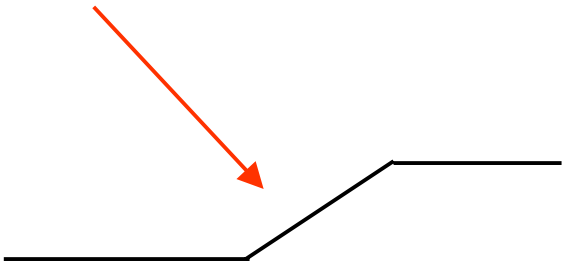

Solve a **local path** finding problem for pairs of postures $(p_1,p_{2)},(p_2\ p_3)\ (p_3,p_4),(p_4,p_5)(p_5,p_6)$



# Definitions

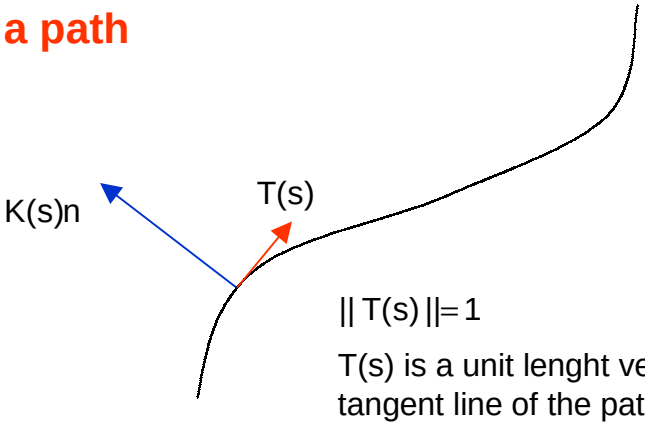
- **Smooth path**
  - A curve that does not intersect itself, and
  - It has a tangent at each point whose direction varies continuously as the point moves along the curve
- **Sectionally smooth path**
  - A curve composed by a finite number of smooth arcs joined end to end



straight line + arc of circle

- **Curvature of a path**




$$\frac{dT(s)}{ds} = K(s)n$$

$$\| n \| = 1$$

$$\| T(s) \| = 1$$

T(s) is a unit lenght vector along the tangent line of the path in the motion direction

**K(s)** is the **curvature** of the path

$$\theta(s) = \text{orientation of a posture corresponding to } s$$

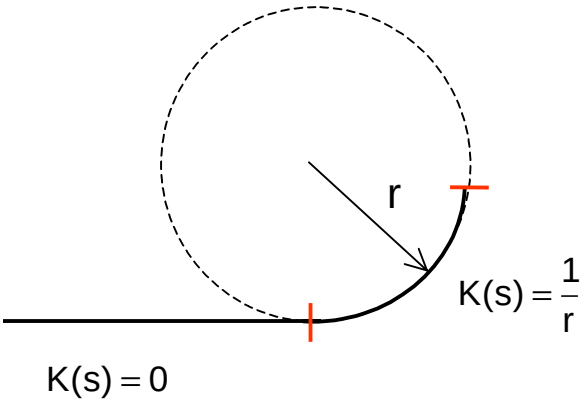
$$\dot{\theta}(s) = K(s)$$

# Definitions

**K(s)** – curvature of a path in a given point

$$K(s) = \frac{1}{r(s)}$$ radious of curvature of a path in a given point

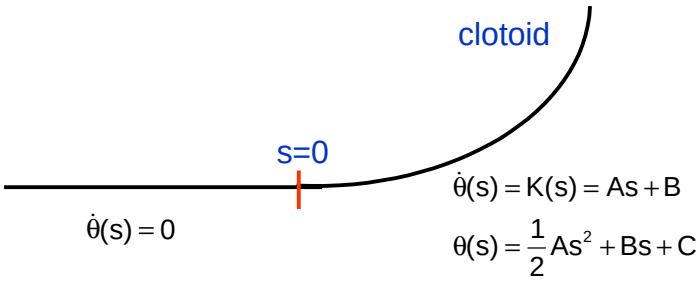
$$\dot{\theta}(s) = K(s)$$




straight line + arc of circle

r-radius of curvature

- smooth path
  - –The direction of a tangent vector in each point is continuous
- the curvature is not continuous




- clotoid
  - –The curvature varies linearly along the path

If B=0 the curvature is continuous
If C=0 the curve is smooth

# Path Representation

$\Pi$ path $\Rightarrow$ pair $(\ell, K)$

continuous curvature function

positive length

$$K : [-\ell/2, \ell/2] \rightarrow \Re$$

$$x(0) = y(0) = \theta(0) = 0$$

standard convention for path positioning

**Tangent direction function**

$$\theta(s) = \int_0^s K(t)dt, \qquad -\ell/2 \le s \le \ell/2$$

$$K(s) = \dot{\theta}(s)$$

**Coordinates of a point (x,y)=(x(s),y(s)) on $\Pi$**

$$x = x(s) = \int_0^s \cos\theta(t)dt$$

$$y = y(s) = \int_0^s \sin\theta(t)dt$$

**End postures of a path $\Pi$**

$$x(-\ell/2), y(-\ell/2), \theta(-\ell/2)$$

$$x(\ell/2), y(\ell/2), \theta(\ell/2)$$

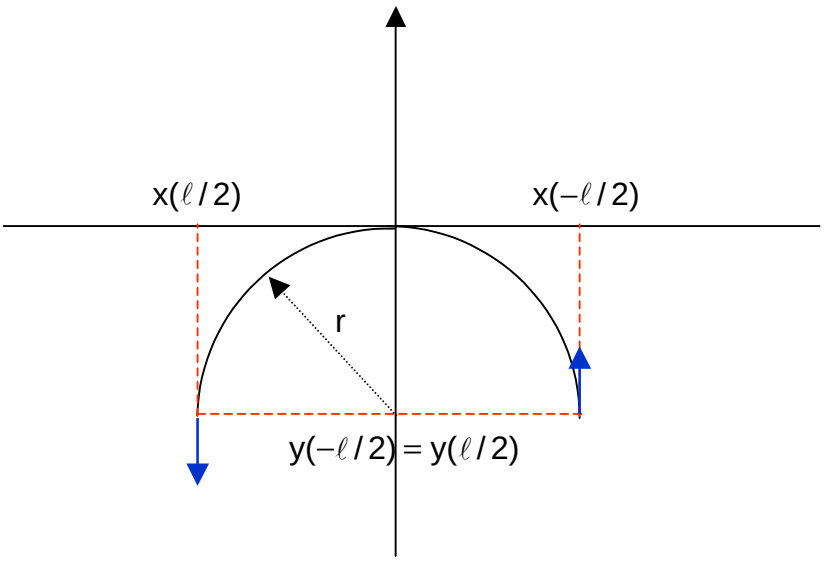
- **Path Property**
  - If a curvatura function K(s) is symmetric

    $$K(-s)=K(s)$$

    then

    $$\theta(s) = -\theta(-s) \quad \text{odd}$$
    $$x(s) = -x(-s) \quad \text{odd}$$
    $$y(s) = y(-s) \quad \text{even}$$

for $s = \ell / 2$ the endpostures are symmetric

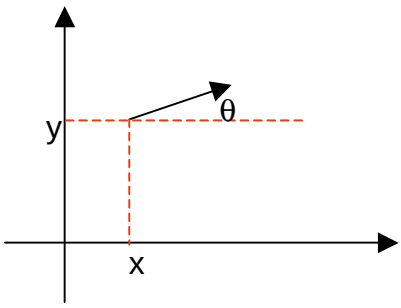


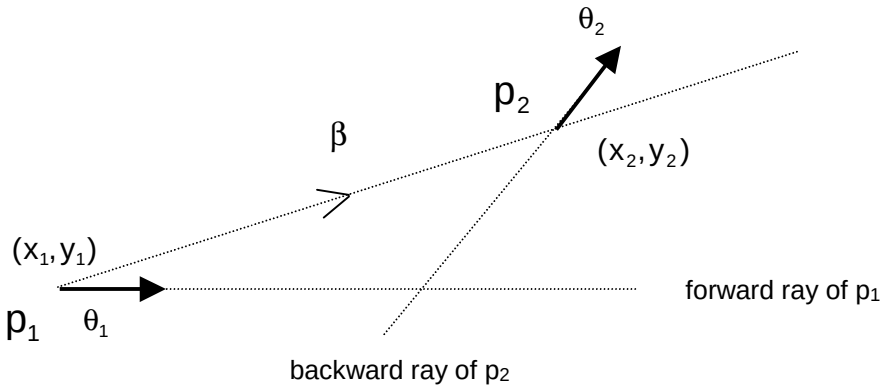

$$\ell = \pi r$$

# Overview of the Method

$p(x, y, \theta)$ posture

$(x, y)$ point



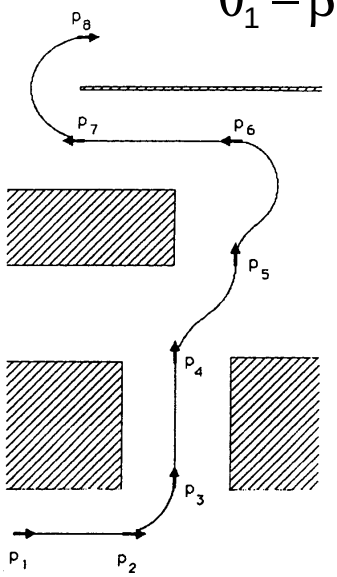




- **Orientation $\beta$ from point $p_1$ to point $p_2$**

$$\beta = \operatorname{arctg}\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

- **$p_1$ and $p_2$ are symmetric - sym($p_1$, $p_2$) - iff**

$$\theta_1 - \beta = -(\theta_2 - \beta)$$




$\mathrm{sym}(p_1, p_2)$ $\sim \mathrm{sym}(p_4, p_5)$

$\mathrm{sym}(p_2, p_3)$ $\sim \mathrm{sym}(p_5, p_6)$

$\mathrm{sym}(p_3, p_4)$

$\mathrm{sym}(p_6, p_7)$

$\mathrm{sym}(p_7, p_8)$

# Overview of the Method

- A **size d** of a **deflection α** of a pair ($p_1$,$p_2$) of a **symmetric posture pair**, sym ($p_1$,$p_2$)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

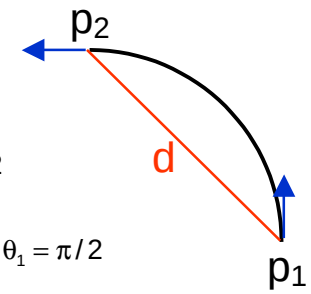
$$\alpha = \theta_2 - \theta_1$$

Variation in orientation evaluated counterclowise

$\theta_1 = \pi/2$

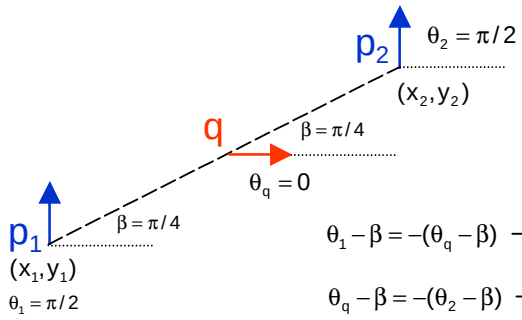
$\theta_2 = \pi$

$\alpha = \theta_2 - \theta_1 = \pi/2$




- A posture **q** is a **split posture** of a pair of postures ($p_1$,$p_2$) iff

sym($p_1$,q) e sym(q,$p_2$)




~sym($p_1$,$p_2$)

$\theta_1 - \beta = -(\theta_q - \beta) \longrightarrow \pi/2 - \beta = 0 + \beta \longrightarrow \beta = \pi/4$

$\theta_q - \beta = -(\theta_2 - \beta) \longrightarrow 0 - \beta = -(\pi/2 - \beta) \longrightarrow \beta = \pi/4$

sym($p_1$,q)

sym(q,$p_2$)

- A path (with a finite lenght) is said to be **simple** if its end postures are symmetric
  - Example: every circular arc

# Problem Solution

- **Problem:** define a path between postures $p_1$ and $p_2$ that minizes a given cost (related with smoothness)
- **Solution**:
  1. If sym($p_1$,p2) the solution is a single smooth path obtained by the minimization of a cost functional.
  2. If ~sym($p_1$,$p_2$) find a split postures q and then apply 1 for the local paths ($p_1$,q) and (q,$p_2$). Chose q in such a way that the total cost is minimzed.

# Cost Functional

- Cost Function 1

$$\cos t_1(\Pi) = \int_{-\ell/2}^{\ell/2} K^2(s)ds = \int_{-\ell/2}^{\ell/2} \dot{\theta}^2(s)ds$$

- Cost Function 2

$$\cos t_2(\Pi) = \int_{-\ell/2}^{\ell/2} \dot{K}^2(s)ds = \int_{-\ell/2}^{\ell/2} \ddot{\theta}^2(s)ds$$

Interpretation

<u>Hypothesis</u>:

constant-velocity navigation of the robot along its path

centripetal acceleration $a_c(s)$

$$a_c(s) = r(s)\,\dot{\theta}^2(s)$$

$$a_c(s) = r(s)\,\dot{\theta}^2(s) = \frac{1}{K(s)}K^2(s) = K(s)$$

The centripetal acceleration equals, at each path point, the path curvature

$$\text{Cost}_1 = \int \text{quadrado de } a_c$$

$$\text{Cost}_2 = \int \text{quadrado de } \dot{a}_c = \int \text{quadrado de jerk}$$

maximizing comfortable vehicle control

# Problem Solution (Cost 1)

- **Cost Function 1**

$$\cos t_1(\Pi) = \int_{-\ell/2}^{\ell/2} K^2(s)ds = \int_{-\ell/2}^{\ell/2} \dot{\theta}^2(s)ds$$

**Solution**

for a fixed path lenght $\ell$

$\min \ \cos t_1(\Pi)$ ←

$$\dot{\theta}(s) = A = K(s)$$
$$\theta(s) = As + B$$

$A, B$ integral constants

$A, \ B \neq 0$
$A \neq 0, \ B = 0$ } circular arcs

$A = 0$ — straight lines

- **Cost Function 2**

$$\cos t_2(\Pi) = \int_{-\ell/2}^{\ell/2} \dot{K}^2(s)ds = \int_{-\ell/2}^{\ell/2} \ddot{\theta}^2(s)ds$$

**Solution**

for a fixed path lenght $\ell$

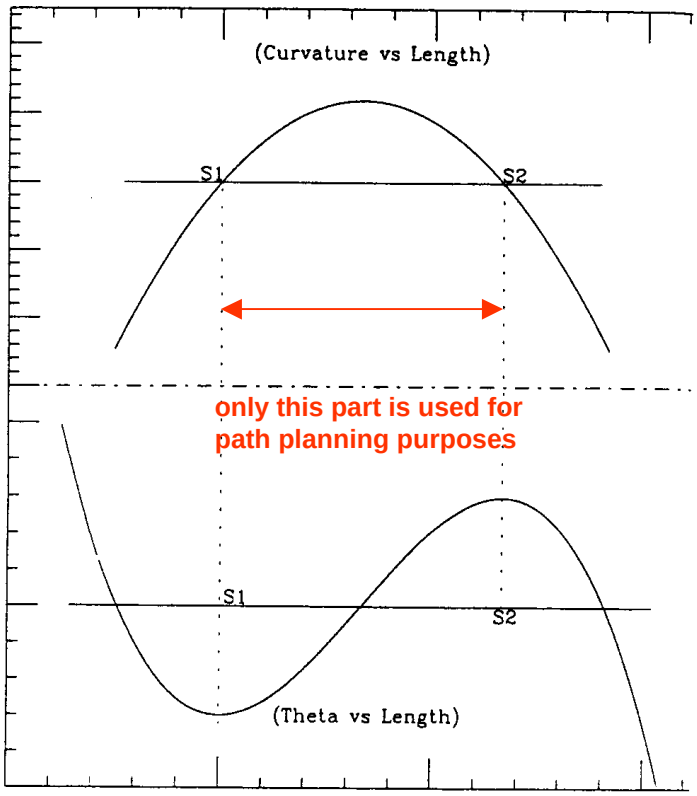
$\min \ \cos t_2(\Pi)$ ←

$$\dot{\theta}(s) = K(s) = \frac{1}{2}As^2 + Bs + C$$
$$\theta(s) = \frac{1}{6}As^3 + \frac{1}{2}Bs^2 + Cs + D$$

$A, B, C, D$ integral constants

⬍

cubic spirals




K(s) symmetric
The path is simple

Fig. 3 Curvature and Tangent Direction of Cubic Spiral

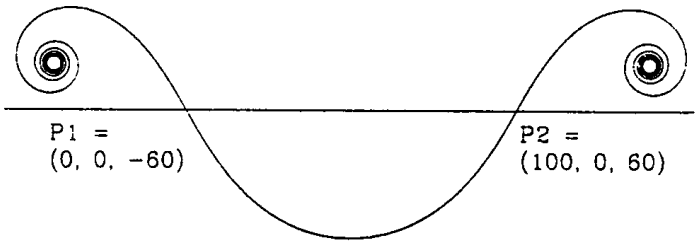



Fig. 4 A Whole Cubic Spiral

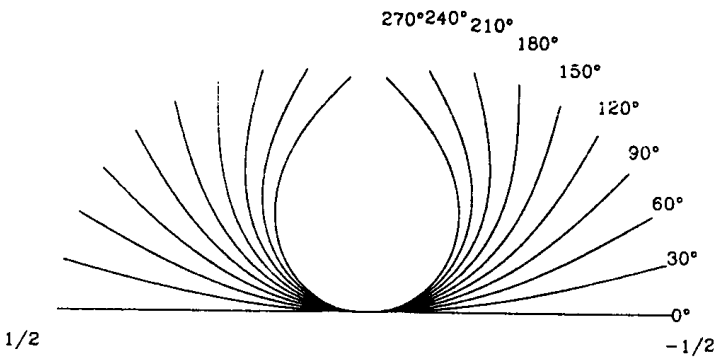



Fig. 5 Standard Cubic Spirals

# Problem solution: particular cases

## Case A - Start and end postures are symmetric pairs

$(p_1, p_2)$ —— known start and end postures

$d$ – size
$\alpha$ – deflection
known

Left turn

- Circular path – $cost_1$

$$\ell = \frac{\alpha/2}{\sin(\alpha/2)} d$$

$$K(s) = \frac{2\sin(\alpha/2)}{d}$$

$$cost_1 = \frac{2\alpha\sin(\alpha/2)}{d}$$

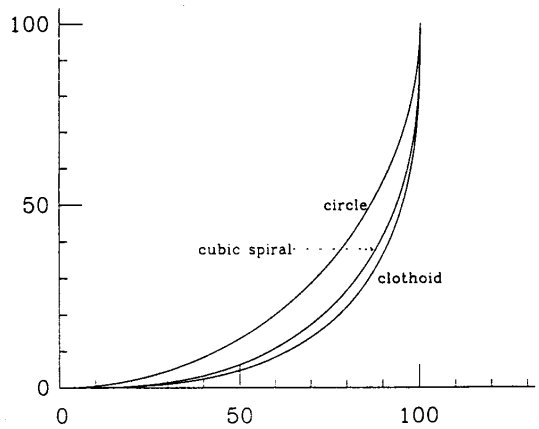



Fig. 6 Simple Curves with $\alpha = \pi/2$

- Cubic spiral path – $cost_2$

$$\ell = \frac{d}{D(\alpha)}$$

$$D(\alpha) = 2\int_0^{1/2} \cos[\alpha(3/2 - 2s^2)s]ds$$

$$K(s) = \frac{6\alpha D^3(\alpha)}{d^3}\left(\frac{d^2}{4D^2(\alpha)} - s^2\right)$$

$$cost_2 = \frac{12\alpha^2 D^3(\alpha)}{d^3}$$

U-turn

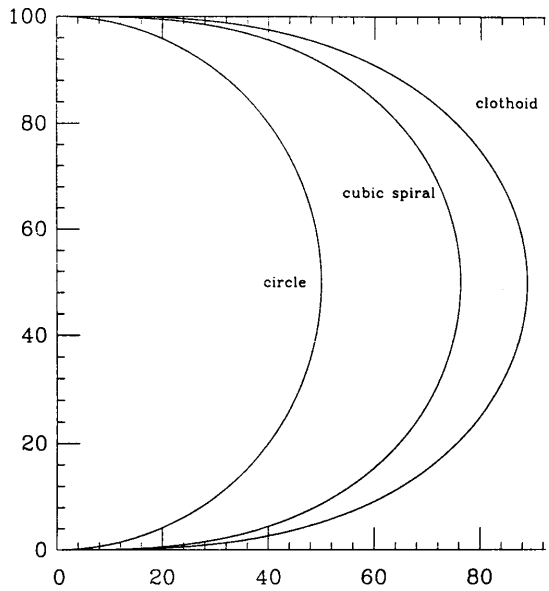



Fig. 7 Simple Curves with $\alpha = \pi$

Circle
Minimum lenght

Clotoid
Maximum lenght

The maximum curvature of the cubic spiral is less than that of the clotoid

## Case B - Start and end postures are NON symmetric pairs

$\sim \mathrm{sym}(p_1,p_2)$ $\quad p_1=(x_1,y_1,\theta_1)$

$p_2=(x_2,y_2,\theta_2)$

**General approach**

- Find the locus of split postures $q=(x,y,\theta)$ of the pair $(p_1,p_2)$
- Now, $\mathrm{sym}(p_1,q)$, $\mathrm{sym}(q,p_2)$
- Solve (locally) the smooth path planning problem for $(p_1,q)$ and $(q,p_2)$ chosing the split posture q that minimizes the sum

$$\mathrm{cost}(\Pi(p_1,q))+\mathrm{cost}(\Pi(q,p_2))$$

- Case B-1 $\quad \theta_1=\theta_2$
  - $p_1$ and $p_2$ are parallel

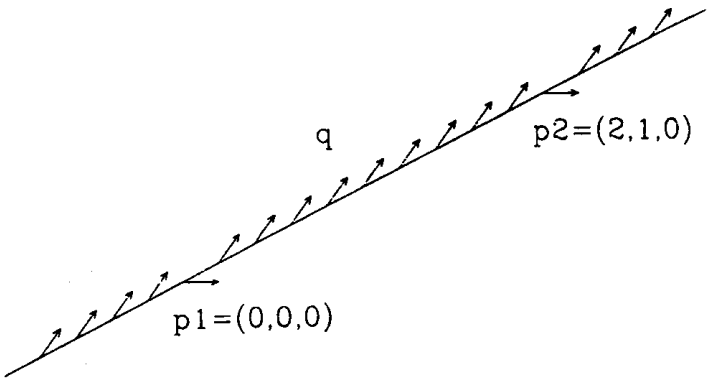



Fig. 8 Split Postures in Parallel Case

The locus of split postures is the line determined by $p_1$ and $p_2$

For either $\mathrm{cost}_1$ and $\mathrm{cost}_2$ the least cost split point of $p_1$ and $p_2$ is

$$q=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\beta-(\theta-\beta)\right)$$

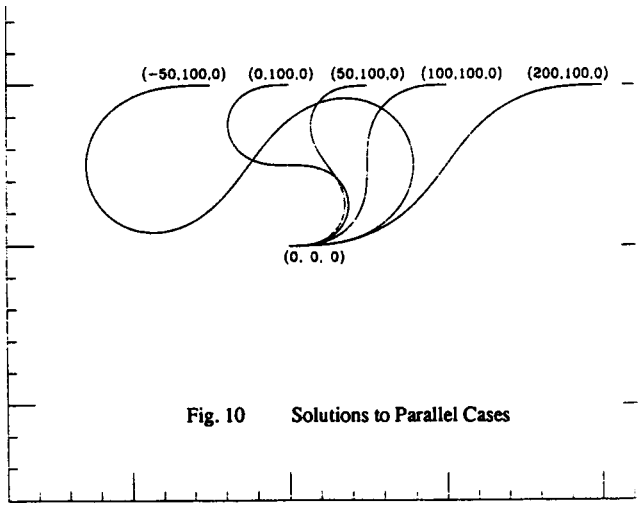



Fig. 10 Solutions to Parallel Cases

# Problem solution: particular cases

## Case B - Start and end postures are NON symmetric pairs

$\sim \mathrm{sym}(p_1, p_2)$ $\quad p_1 = (x_1, y_1, \theta_1)$

$p_2 = (x_2, y_2, \theta_2)$

- Case B-2 $\quad \theta_1 \neq \theta_2$
  - $p_1$ and $p_2$ are NON parallel

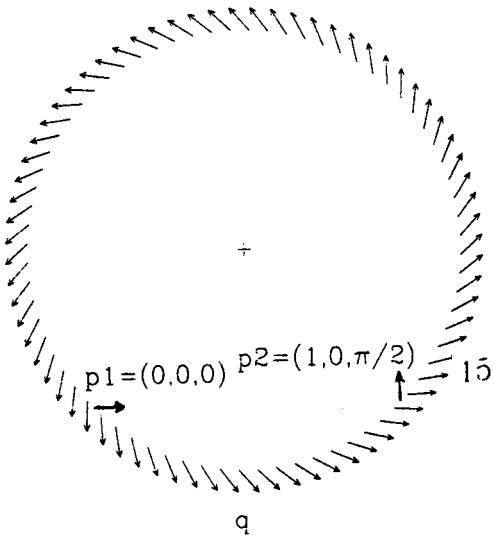



Fig. 9 Split Postures in Non-Parallel Ca

The locus of split postures is a circle

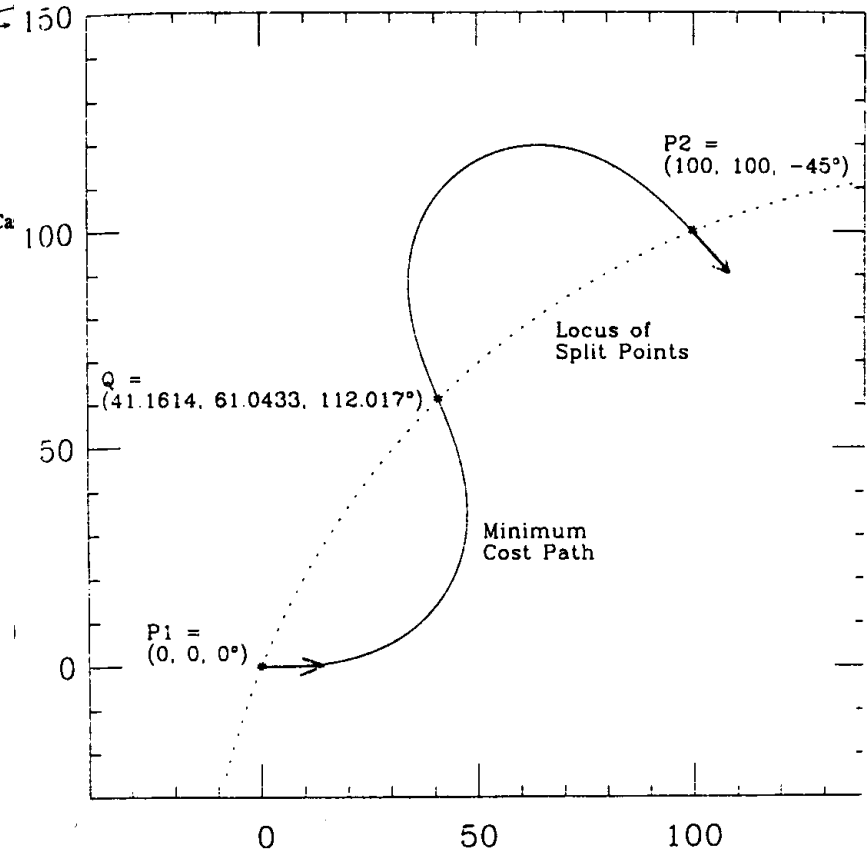



Fig. 11 Example of Non-Parallel Case