

# Optimal Motion Planning for Range-Based Marine Vehicle Positioning in the Presence of Unknown Currents

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**Abstract:** We address the problem of range-based marine vehicle positioning in the presence of unknown but constant ocean currents. The goal is to estimate the position of one or more vehicles from a sequence of range measurements to fixed or moving acoustic beacons with known locations. In contrast to most range-based positioning algorithms, we address the case where the currents are unknown and seek to estimate them explicitly as well. This increases the complexity of the problem at hand and raises interesting observability issues. In particular, the vehicles must undergo sufficiently exciting maneuvers so as to maximize the range-based information available for joint current /multiple vehicle position estimation. The main contribution of the paper is the computation of vehicle trajectories for range-based vehicle positioning system in the presence of constant, unknown currents by maximizing the determinant of a suitable Fisher information matrix (FIM), subject to collision avoidance and maneuvering constraints. A numerical solution is proposed for the general set-up of multiple vehicles and beacons. Analytical solutions are obtained for the case of one vehicle and one static beacon. The efficacy of the strategies proposed for vehicle trajectory optimization is shown by numerical simulations.

*Keywords:* Underwater range-based navigation, Single-beacon localization, Trajectory optimization, Fisher information matrix.

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## 1. INTRODUCTION

We study the problem of optimal motion planning for range-based marine vehicle positioning in the presence of unknown currents. This research work is motivated by the need to develop low cost, easy to deploy and operate AUV positioning systems. In this context, range-based positioning systems have recently emerged as a viable alternative to conventional acoustic based navigation methods such as LBL (Long Baseline) and USBL (Ultra Short Baseline) systems. In its essence, the problem of range-based positioning can be formulated as that of estimating the position of one or more vehicles based on the measurements of their ranges to fixed or moving acoustic beacons, the positions of which are known in advance.

In contrast to most range-based positioning algorithms, we address the case where the currents are unknown and seek to estimate them explicitly, as well. This increases the complexity of the problem at hand and raises interesting observability issues. The latter have been the subject of a large number of publications in the area, see for example (Gadre et al., 2005), (Crasta et al., 2013) and the references therein. However, the issue of optimal trajectory

computation so as to increase the range-based information available for vehicle positioning in the presence of ocean currents is still a subject of research. This is in contrast to the case where currents are absent, for which theoretical and experimental results are available. As a representative example, we cite the work in (Moreno-Salinas et al., 2016) where the authors address not only the problem of range-based positioning but also the dual problem of target localization in the absence of currents, with due account for system implementation and field testing.

The present paper extends the techniques proposed in (Moreno-Salinas et al., 2016) to deal explicitly with the presence of unknown currents. The final goal is to estimate the position of one or more vehicles from a sequence of range measurements to fixed or moving acoustic beacons with known locations. In this situation, the vehicles must undergo sufficiently exciting maneuvers so as to maximize the range-based information available for joint current /multiple vehicle position estimation. By adopting a classical estimation setting, the optimal vehicle trajectories are obtained by maximizing the determinant of a suitable defined Fisher information matrix (FIM), subject to collision avoidance and maneuvering constraints. A numerical solution is proposed for the general set-up of multiple vehicles and beacons. Analytical solutions are obtained for the case of one vehicle and one static beacon. The efficacy of the strategies proposed for vehicle trajectory optimization in the more general case of multiple vehicles and beacons is shown by numerical simulations.

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The paper is organized as follow. Section 2 introduces the basic notation required and contains some preliminary results. Section 3 summarizes the process and measurement models for the problem at hand and introduces the problem formulation. In Section 4 we derive the Fisher information matrix that is at the core of our work and show how to compute its optimal value. In Section 5 we provide an analytical solution to the single vehicle, single (stationary) beacon scenario. Section 6 contains illustrative numerical examples. Finally, Section 7 contains the main conclusions.

## 2. PRELIMINARIES

We find it convenient to identify  $\mathbb{R}^2$  with the complex plane  $\mathbb{C}$  by letting the complex number  $a + jb$  be described by  $[a \ b]^T \in \mathbb{R}^2$ , where  $j := \sqrt{-1}$ . For every  $z \in \mathbb{C}$ , we let  $\arg(z) \in [0, 2\pi)$  whenever  $z \neq 0$ ,  $\bar{z} \in \mathbb{C}$ , and  $|z| := \sqrt{z\bar{z}} \geq 0$  denote the argument, the complex conjugate, and the modulus of  $z$ , respectively. Using Euler's formula,  $z \in \mathbb{C}$  can be written in exponential (phasor) form  $z = |z|e^{j\arg(z)}$ .

For  $n \in \mathbb{N}$ , we let  $\mathbb{I}_n := \{1, \dots, n\}$ . In this paper, we use  $i \in \mathbb{I}_p$  to denote the  $i^{\text{th}}$  vehicle,  $\alpha \in \mathbb{I}_q$  to denote the  $\alpha^{\text{th}}$  beacon, and  $k \in \{0\} \cup \mathbb{I}_{m-2}$  to denote that  $k^{\text{th}}$  sample. We denote the Euclidean norm in  $\mathbb{R}^n$  by  $\|\cdot\|$  and the unit sphere in  $\mathbb{R}^n$  by  $S^n$ , that is,  $S^n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}$ . By  $\mathbb{T}^n$  we mean the  $n$ -torus, that is,  $\mathbb{T}^n := S^1 \times \dots \times S^1 = (S^1)^n$ . We define  $\mathbf{g} : [0, 2\pi) \rightarrow S^1$  and  $\mathbf{g}^\perp : [0, 2\pi) \rightarrow S^1$  by  $\mathbf{g}(\theta) = [\cos\theta \ \sin\theta]^T$  and  $\mathbf{g}^\perp(\theta) = [-\sin\theta \ \cos\theta]^T$ ,  $\theta \in [0, 2\pi)$ , respectively.

By  $I_n$  and  $\mathbf{0}_{m \times n}$  we mean the identity matrix of size  $n$  and the zero matrix of size  $m \times n$ , respectively. Given  $A \in \mathbb{R}^{m \times n}$ , we let  $\text{vec}(A) \in \mathbb{R}^{mn}$  denote the column vector obtained by stacking the columns of the matrix  $A$  on top of one another. Given  $\mathbf{w} \in \mathbb{R}^n$ ,  $\text{diag}(\mathbf{w})$  denotes the diagonal matrix whose diagonal elements are the elements of  $\mathbf{w}$ . Similarly, given  $A_j \in \mathbb{R}^{n \times n}$ ,  $j \in \mathbb{I}_s$ , we can define  $\text{diag}(A_1, \dots, A_s)$  and we define the direct sum of  $A_1, \dots, A_s$  by  $\bigoplus_{j \in \mathbb{I}_s} A_j := \text{diag}(A_1, \dots, A_s)$ .

Finally, we recall the following result (Popescu et al., 2004).

*Lemma 1.* Consider  $Q_{ij} \in \mathbb{R}^{n \times n}$ ,  $i, j \in \mathbb{I}_m$ , and  $Q_{ij} = Q_{ij}^T$  and  $Q_{ij} \succ 0$  with  $b_{ij} := \text{trace}(Q_{ij})$  is a constant. Define  $\mathcal{Q} \in \mathbb{R}^{nm \times nm}$  and  $\mathcal{M} \in \mathbb{R}^{m \times m}$  by

$$\mathcal{Q} := \begin{bmatrix} Q_{11} & \dots & Q_{1m} \\ \vdots & \ddots & \vdots \\ Q_{m1} & \dots & Q_{mm} \end{bmatrix} \text{ and } \mathcal{M} := \begin{bmatrix} b_{11} & \dots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mm} \end{bmatrix}.$$

Then, the  $\det(\mathcal{Q})$  is maximized when  $Q_{ij} = (n^{-1}b_{ij})I_n$ ,  $i \leq j$ ,  $i, j \in \mathbb{I}_m$ , and the maximum value of the  $\det(\mathcal{Q})$  is given by  $(n^{-m} \det(\mathcal{M}))^n$ . ■

## 3. PROBLEM FORMULATION

Consider  $p \geq 1$  vehicles and  $q \geq 1$  beacons. Referring to Fig. 1, the kinematic model of the  $i^{\text{th}}$  vehicle,  $i \in \mathbb{I}_p$ , (Margarida et al., 2015) is described by

$$\dot{\mathbf{p}}^{[i]} = \begin{bmatrix} \cos(\chi^{[i]}) & -\sin(\chi^{[i]}) \\ \sin(\chi^{[i]}) & \cos(\chi^{[i]}) \end{bmatrix} \begin{bmatrix} v^{[i]} \\ 0 \end{bmatrix} + \mathbf{v}_c, \quad (1)$$

$$\dot{\chi}^{[i]} = r^{[i]}, \quad (2)$$

$$\dot{\mathbf{v}}_c = \mathbf{0}, \quad (3)$$

where  $t \in [0, t_f]$ ,  $\mathbf{p}^{[i]} \in \mathbb{R}^2$  is the instantaneous inertial position vector of the  $i^{\text{th}}$  vehicle,  $v^{[i]} : [0, t_f] \rightarrow [0, \infty)$  is the relative speed with respect to the water, i.e.  $v^{[i]} \equiv \|\mathbf{v}^{[i]}\|$ ,  $\chi^{[i]} : [0, t_f] \rightarrow [0, 2\pi)$  is the relative (to the water) course angle that gives the orientation of the vehicle's flow-frame with respect to an inertial frame, and  $r^{[i]} : [0, t_f] \rightarrow \mathbb{R}$  is the relative course angle rate. The symbol  $\mathbf{v}_c \in \mathbb{R}^2$  denotes the constant ocean current vector. In what follows, with an obvious abuse of the notation, we will refer to relative course angle and relative course angle rate simply as course angle and course angle rate, respectively. Using a state-space formulation,  $\mathbf{x}^{[i]} := (\mathbf{p}^{[i]}, \chi^{[i]}, \mathbf{v}_c) \in \mathcal{M}_c := \mathbb{R}^2 \times [0, 2\pi) \times \mathbb{R}^2$  is the state vector and  $\mathbf{u}^{[i]} := (v^{[i]}, r^{[i]}) \in \mathcal{U} := [0, \infty) \times \mathbb{R}$  is the input. To avoid any collisions, for all  $t \in [0, t_f]$ , the vehicles must satisfy

$$\|\mathbf{p}^{[i]}(t) - \mathbf{p}^{[j]}(t)\| \geq R, \quad i, j \in \mathbb{I}_p, \quad i < j,$$

$$\|\mathbf{p}^{[i]}(t) - \mathbf{b}^{[\alpha]}(t)\| \geq R, \quad i \in \mathbb{I}_p, \quad \alpha \in \mathbb{I}_q,$$

for all  $t \in [0, t_f]$ , where  $R > 0$  is a safety radius.

Consider a finite set of fixed/moving beacons  $\mathcal{B} := \{\mathbf{b}^{[1]}, \dots, \mathbf{b}^{[q]}\} \subset \mathbb{R}^2$ , which we assume are known functions of time (see Fig. 1). Each vehicle is equipped with sensors that measure distances to these beacons. The instantaneous measurements of distances collected at time  $t \in [0, t_f]$ , denoted  $Y(t)$ , are corrupted by additive white Gaussian noise as follows:

$$Y(t) = D(t) + \eta(t),$$

where  $D \in \mathbb{R}^{q \times p}$  (the matrix of true instantaneous distances) and  $\eta \in \mathbb{R}^{q \times p}$  are given by

$$D := \begin{bmatrix} d_{11} & \dots & d_{1p} \\ \vdots & \ddots & \vdots \\ d_{q1} & \dots & d_{qp} \end{bmatrix}, \quad \eta := \begin{bmatrix} \eta_{11} & \dots & \eta_{1p} \\ \vdots & \ddots & \vdots \\ \eta_{q1} & \dots & \eta_{qp} \end{bmatrix},$$

and  $\eta_{\alpha i} \sim \mathcal{N}(0, \sigma_{\alpha i}^2)$ ,  $\alpha \in \mathbb{I}_q$ ,  $i \in \mathbb{I}_p$ . In the above, for  $i \in \mathbb{I}_p$  and  $\alpha \in \mathbb{I}_q$ , let  $\mathbf{d}^{[\alpha i]}$  denote the relative position vector of the AUV with respect to the beacon  $\mathbf{b}^{[\alpha]}$ , that is,  $\mathbf{d}^{[\alpha i]} := \mathbf{p}^{[i]} - \mathbf{b}^{[\alpha]}$  and let  $d_{\alpha i} := \|\mathbf{d}^{[\alpha i]}\|$  denote the corresponding relative distance.

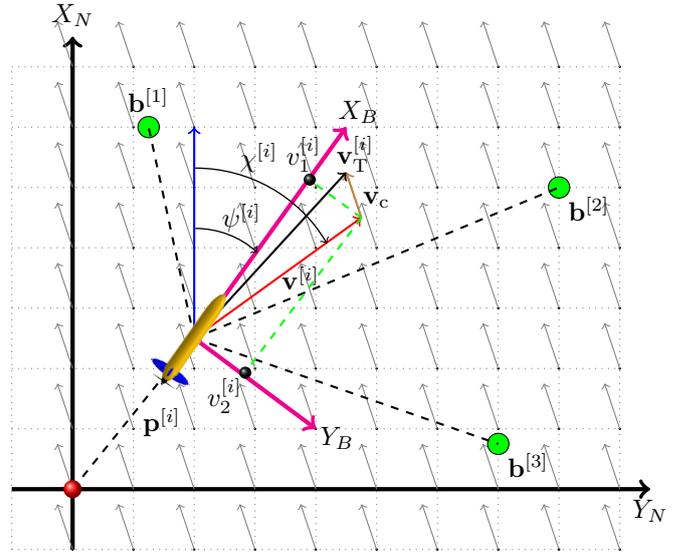


Fig. 1. Illustration of a single vehicle with three beacons in the presence of ocean currents.

The solution to (1)-(3) at time  $t \in [0, t_f]$  for the initial condition  $\mathbf{x}_0^{[i]} = (\mathbf{p}_0^{[i]}, \chi_0^{[i]}, \mathbf{v}_{c_0}) \in \mathcal{M}_c$  and the input  $\mathbf{u}^{[i]} = (v^{[i]}, r^{[i]})$  is given by

$$\mathbf{x}^{[i]}(t) = \mathbf{x}_0^{[i]} + \left( \int_0^t v^{[i]}(\tau) \mathbf{g}(\chi^{[i]}(\tau)) d\tau + t \mathbf{v}_{c_0}, \int_0^t r^{[i]}(\tau) d\tau, \mathbf{v}_{c_0} \right). \quad (4)$$

Let  $m \in \mathbb{N}$ ,  $m \geq 4$  and consider a strictly monotonically increasing time sequence  $\{t_k\}_{k=0}^{m-1} \subseteq [0, t_f]$  of length  $m$  with  $t_0 := 0$  and  $t_{m-1} := t_f$ . For the sake of simplicity, we let  $\mathbf{p}_k^{[i]} := \mathbf{p}^{[i]}(t_k)$ ,  $\mathbf{v}_{c_k} := \mathbf{v}_c(t_k)$ ,  $\chi_k^{[i]} := \chi^{[i]}(t_k)$ ,  $\mathbf{g}_k^{[i]} := \mathbf{g}(\chi^{[i]}(t_k))$ ,  $Y_k := Y(t_k)$ , and  $D_k := D(t_k)$ . Then, for  $t \in [t_k, t_{k+1})$ , (4) can be written as

$$\mathbf{x}^{[i]}(t) = \mathbf{x}_k^{[i]} + \left( \int_{t_k}^t v^{[i]}(\tau) \mathbf{g}(\chi^{[i]}(\tau)) d\tau + (t - t_k) \mathbf{v}_{c_0}, \int_{t_k}^t r^{[i]}(\tau) d\tau, \mathbf{v}_{c_0} \right). \quad (5)$$

*Assumption 1.* We assume that the inputs are piecewise constant and are bounded (above and below), that is, for all  $k \in \{0\} \cup \mathbb{I}_{m-2}$ ,

$$\begin{aligned} v^{[i]}(t) &\equiv \bar{v}_k^{[i]} \in [0, \bar{v}_{\text{ub}}], \quad t \in [t_k, t_{k+1}), \\ r^{[i]}(t) &\equiv \bar{r}_k^{[i]} \in [-\bar{r}_{\text{ub}}, \bar{r}_{\text{ub}}], \quad t \in [t_k, t_{k+1}). \end{aligned}$$

With these assumptions, from (5) we get

$$\chi^{[i]}(t) = \chi_k^{[i]} + \bar{r}_k^{[i]}(t - t_k), \quad t \in [t_k, t_{k+1}) \quad (6)$$

and using (6) in (5) we get

$$\begin{aligned} \mathbf{p}^{[i]}(t) = \mathbf{p}_k^{[i]} + \left( \frac{\bar{v}_k^{[i]}}{\bar{r}_k^{[i]}} \right) \left\{ \left( \mathbf{g}_k^{[i]} \right)^\perp - \left( \mathbf{g}(\chi^{[i]}(t)) \right)^\perp \right\} \\ + (t - t_k) \mathbf{v}_{c_0}, \quad t \in [t_k, t_{k+1}), \quad (7) \end{aligned}$$

whenever  $\bar{r}_k^{[i]} \neq 0$ .

With this background, we now formulate the following question: “Given the time-history of the beacon positions in a given time window, what is the best sequence of actions for each of the AUVs (in terms of vehicle speed and course rate histories) so as to collectively maximize the information available to compute their unknown initial positions and the unknown constant ocean current?”

In what follows, to answer this question we adopt the classical set-up of estimation theory exploited in (Moreno-Salinas et al., 2016) for range-based positioning in the absence of currents, to which we refer the reader for background information.

#### 4. FISHER INFORMATION MATRIX

In this section we derive the FIM for the model described before, consisting of the vehicle kinematics and measurement equations. The objective is to estimate the initial positions of the AUVs and the unknown current. As is well known, the inverse of the FIM is instrumental in computing a lower bound on the covariance of the estimates of a deterministic parameter that can be achieved with any unbiased estimator. This result yields the celebrated Cramér-Rao Lower Bound (Trees et al., 1968), which we seek to reduce by maximizing the determinant of the FIM, subjected to collision and vehicle maneuvering constraints; see Moreno-Salinas et al. (2016) for the simplified case where there are no currents.

Following a by now standard procedure, for a given number of samples  $m \geq 2$ , with input sequence  $\mathbf{U} := (u_0, \dots, u_{m-1})$  and unknown parameter  $\boldsymbol{\theta} \in \mathbb{R}^n$ , we denote the corresponding FIM by  $FIM_{\mathbf{U}}(\boldsymbol{\theta}) \in \mathbb{R}^{n \times n}$ . The FIM with respect to the unknown parameter of interest  $\boldsymbol{\theta}$  is given by

$$FIM_{\mathbf{U}}(\boldsymbol{\theta}) := \mathbb{E} \left\{ [\nabla_{\boldsymbol{\theta}} (\log \mathbb{L}_{\boldsymbol{\theta}}(\mathbf{y}))] [\nabla_{\boldsymbol{\theta}} (\log \mathbb{L}_{\boldsymbol{\theta}}(\mathbf{y}))]^\top \right\},$$

where  $\mathbf{y} \in \mathbb{R}^m$  is the measurement vector,  $\mathcal{L}_{\boldsymbol{\theta}}(\mathbf{y})$  is the likelihood function of the measurement with respect to the parameter  $\boldsymbol{\theta}$ , and  $\mathbb{E}$  and  $\nabla$  are the expectation and gradient operators, respectively. To maximize the Fisher information, we define a scalar function

$$J(\mathbf{U}) := \ln \det(FIM_{\mathbf{U}}(\boldsymbol{\theta})),$$

and minimize  $\ln \det((FIM_{\mathbf{U}}(\boldsymbol{\theta}))^{-1})$ , that is,  $-J(\mathbf{U})$ . We next derive the FIM for the problem under consideration.

For the sake of simplicity, we use the following compact notation:

$$\widehat{\mathcal{P}}_{\alpha}^{[i]} := \begin{bmatrix} \mathbf{d}_0^{[\alpha i]} & \dots & \mathbf{d}_{m-1}^{[\alpha i]} \\ d_{\alpha i, 0} & \dots & d_{\alpha i, m-1} \end{bmatrix} \in \mathbb{R}^{2 \times m}, \quad (8)$$

$$\Omega := \text{diag}(t_0, t_1, \dots, t_{m-1}) \in \mathbb{R}^{m \times m}, \quad (9)$$

$$\mathbf{v}^{[i]} := (\bar{v}_0^{[i]}, \dots, \bar{v}_{m-2}^{[i]}) \in \mathbb{R}_{\geq 0}^{m-1}, \quad (10)$$

$$\mathbf{r}^{[i]} := (\bar{r}_0^{[i]}, \dots, \bar{r}_{m-2}^{[i]}) \in \mathbb{R}^{m-1}, \quad (11)$$

$$\mathbf{U}_i := (\mathbf{v}^{[i]}, \mathbf{r}^{[i]}). \quad (12)$$

##### 4.1 FIM for a single vehicle

First we derive the FIM for a single vehicle ( $p = 1$ ) with  $q$  beacons. Consider the  $i^{\text{th}}$  AUV motion described by

$$\begin{aligned} \begin{bmatrix} \mathbf{p}_{k+1}^{[i]} \\ \mathbf{v}_{c_{k+1}}^{[i]} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_0^{[i]} \\ \mathbf{v}_{c_0}^{[i]} \end{bmatrix} + t_{k+1} \begin{bmatrix} \mathbf{v}_{c_0}^{[i]} \\ \mathbf{0} \end{bmatrix} + \\ \left[ \sum_{j=0}^k \begin{bmatrix} \bar{v}_k^{[i]} \\ \bar{r}_k^{[i]} \end{bmatrix} \left\{ \left( \mathbf{g}_k^{[i]} \right)^\perp - \left( \mathbf{g}(\chi^{[i]}(t)) \right)^\perp \right\} \right], \quad (13) \end{aligned}$$

$$\mathbf{y}_k^{[i]} = \mathbf{d}_k^{[i]} + \boldsymbol{\eta}_k^{[i]}, \quad (14)$$

where  $\mathbf{d}_k^{[i]}$  and  $\boldsymbol{\eta}_k^{[i]}$  are the  $i^{\text{th}}$  columns of  $D_k$  and  $\boldsymbol{\eta}_k$ , respectively. In this particular case, we have  $\boldsymbol{\theta}_i = (\mathbf{p}_0^{[i]}, \mathbf{v}_{c_0})$  and we let  $FIM_{\mathbf{U}_i}^{[i]}(\boldsymbol{\theta}_i) \in \mathbb{R}^{4 \times 4}$  denote the corresponding FIM for this set-up.

Straightforward computations show that

$$FIM_{\mathbf{U}_i}^{[i]}(\boldsymbol{\theta}_i) = \sum_{\alpha \in \mathbb{I}_q} \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} \sigma_{\alpha i}^{-2} (\nabla_{\boldsymbol{\theta}_i} d_{\alpha i, k}) (\nabla_{\boldsymbol{\theta}_i} d_{\alpha i, k})^\top, \quad (15)$$

where

$$\nabla_{\boldsymbol{\theta}_i} d_{\alpha i, k} = \frac{1}{d_{\alpha i, k}} \begin{bmatrix} \mathbf{d}_k^{[\alpha i]} \\ t_k \mathbf{d}_k^{[\alpha i]} \end{bmatrix} \in \mathbb{R}^4. \quad (16)$$

Substituting (16) into (15), and simplifying further yields

$$FIM_{\mathbf{U}_i}^{[i]}(\boldsymbol{\theta}_i) = \sum_{\alpha \in \mathbb{I}_q} \sigma_{\alpha i}^{-2} \begin{bmatrix} A_{\alpha}^{[i]} & B_{\alpha}^{[i]} \\ B_{\alpha}^{[i]} & C_{\alpha}^{[i]} \end{bmatrix},$$

where

$$\begin{aligned} A_{\alpha}^{[i]} &:= \widehat{\mathcal{P}}_{\alpha}^{[i]} (\widehat{\mathcal{P}}_{\alpha}^{[i]})^\top, \\ B_{\alpha}^{[i]} &:= \widehat{\mathcal{P}}_{\alpha}^{[i]} \Omega (\widehat{\mathcal{P}}_{\alpha}^{[i]})^\top, \\ C_{\alpha}^{[i]} &:= \widehat{\mathcal{P}}_{\alpha}^{[i]} \Omega^2 (\widehat{\mathcal{P}}_{\alpha}^{[i]})^\top. \end{aligned}$$

Notice that

$$\text{trace}(A_\alpha^{[i]}) = m, \quad (17)$$

$$\text{trace}(B_\alpha^{[i]}) = \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} t_k, \quad (18)$$

$$\text{trace}(C_\alpha^{[i]}) = \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} (t_k)^2. \quad (19)$$

Therefore,

$$\text{trace}(FIM_{\mathbf{U}_i}^{[i]}(\boldsymbol{\theta}_i)) = \left( \sum_{\alpha \in \mathbb{I}_q} \sigma_{\alpha i}^{-2} \right) \left( m + \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} (t_k)^2 \right).$$

#### 4.2 FIM for multiple vehicles

We next derive the FIM for multiple vehicles. We now have  $\mathbf{U} := (\mathbf{U}_1, \dots, \mathbf{U}_p)$ ,  $\boldsymbol{\theta} := (\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_p)$  and we let  $FIM_{\mathbf{U}}(\boldsymbol{\theta}) \in \mathbb{R}^{4p \times 4p}$  denote the FIM for this set-up. The FIM for the complete system is given by

$$FIM_{\mathbf{U}}(\boldsymbol{\theta}) = \sum_{i \in \mathbb{I}_p} \sum_{\alpha \in \mathbb{I}_q} \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} \sigma_{\alpha i}^{-2} (\nabla_{\boldsymbol{\theta}} d_{\alpha i, k}) (\nabla_{\boldsymbol{\theta}} d_{\alpha i, k})^T,$$

where

$$\nabla_{\boldsymbol{\theta}} d_{\alpha i, k} = \frac{1}{d_{\alpha i, k}} \begin{bmatrix} \mathbf{0}_{4(i-1) \times 1} \\ \nabla_{\boldsymbol{\theta}_i} d_{\alpha i, k} \\ \mathbf{0}_{4(p-i) \times 1} \end{bmatrix} \in \mathbb{R}^{4p}.$$

Simplifying further yields

$$FIM_{\mathbf{U}}(\boldsymbol{\theta}) = \bigoplus_{i \in \mathbb{I}_p} FIM_{\mathbf{U}_i}^{[i]}(\boldsymbol{\theta}_i).$$

Note that the overall FIM depends on  $(\mathbf{v}^{[i]}, \mathbf{r}^{[i]})$ ,  $i \in \mathbb{I}_p$ . In the following subsection we find the maximum value of the FIM determinant.

#### 4.3 Optimal Fisher determinant

Consider the cost functional described by

$$J(\mathbf{U}) = \log \det(FIM_{\mathbf{U}}(\boldsymbol{\theta})) = \sum_{i \in \mathbb{I}_p} \log \det(FIM_{\mathbf{U}_i}^{[i]}(\boldsymbol{\theta}_i)).$$

In the above equation, the second equality follows from the fact that

$$\det \left( \bigoplus_{i \in \mathbb{I}_p} FIM_{\mathbf{U}_i}^{[i]}(\boldsymbol{\theta}_i) \right) = \prod_{i \in \mathbb{I}_p} \det(FIM_{\mathbf{U}_i}^{[i]}(\boldsymbol{\theta}_i)). \quad (20)$$

Thus, it suffices to maximize the FIM associated with each of the vehicles to maximize the overall FIM. We have the following result.

*Proposition 2.* Consider  $i \in \mathbb{I}_p$  and assume that  $\sigma_{\alpha i} := \sigma$ ,  $\alpha \in \mathbb{I}_q$ . Denote

$$b_{11} := qm, \quad b_{12} := \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} qt_k, \quad \text{and} \quad b_{22} := \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} q(t_k)^2,$$

and, let

$$\sum_{\alpha \in \mathbb{I}_q} A_\alpha^{[i]} = \frac{b_{11}}{2} I_2, \quad \sum_{\alpha \in \mathbb{I}_q} B_\alpha^{[i]} = \frac{b_{12}}{2} I_2, \quad \text{and} \quad \sum_{\alpha \in \mathbb{I}_q} C_\alpha^{[i]} = \frac{b_{22}}{2} I_2. \quad (21)$$

Then,  $\det(FIM_{\mathbf{U}_i}^{[i]}(\boldsymbol{\theta}_i))$  is maximum and is given by

$$(2^{-2} \det(\mathcal{M}))^2,$$

where

$$\mathcal{M} := \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \mathbb{R}^{2 \times 2}.$$

In particular, for uniform sampling we have

$$\det(FIM_{\mathbf{U}_i}^{[i]}(\boldsymbol{\theta}_i)) = \left( \frac{T^4 q^4}{2304 \sigma^8} \right) m^4 (m^2 - 1)^2. \quad (22)$$

**Proof.** Let  $Q_{11} := \sum_{\alpha \in \mathbb{I}_q} A_\alpha^{[i]}$ ,  $Q_{12} := \sum_{\alpha \in \mathbb{I}_q} B_\alpha^{[i]}$ , and  $Q_{22} := \sum_{\alpha \in \mathbb{I}_q} C_\alpha^{[i]}$ . By the hypothesis of the Proposition,  $Q_{11} \succ 0$ ,  $Q_{12} \succ 0$ , and  $Q_{22} \succ 0$ . Define

$$\mathcal{Q} := \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} \in \mathbb{R}^{4 \times 4}. \quad (23)$$

By setting  $\mathcal{Q} = FIM_{\mathbf{U}_i}^{[i]}(\boldsymbol{\theta}_i)$  in Lemma 1 it follows that  $\det(FIM_{\mathbf{U}_i}^{[i]}(\boldsymbol{\theta}_i))$  is maximum and the maximum value of  $\det(FIM_{\mathbf{U}_i}^{[i]}(\boldsymbol{\theta}_i))$  is given by  $(2^{-2} \det(\mathcal{M}))^2$ .

In particular, if  $t_k = kT$ , then

$$\det(FIM_{\mathbf{U}_i}^{[i]}(\boldsymbol{\theta}_i)) = \left( \frac{T^4 q^4}{2304 \sigma^8} \right) m^4 (m^2 - 1)^2. \quad (24)$$

This completes the proof.  $\blacksquare$

The above result gives the maximum value of the cost functional with respect to a single vehicle. The following corollary of Proposition 2 follows immediately from (20).

*Corollary 3.* The optimum value of the  $\det(FIM_{\mathbf{U}}(\boldsymbol{\theta}))$  is given by

$$(2^{-2} \det(\mathcal{M}))^{2p}.$$

In particular, for uniform sampling

$$\det(FIM_{\mathbf{U}}(\boldsymbol{\theta})) = \left[ \left( \frac{T^4 q^4}{2304 \sigma^8} \right) m^4 (m^2 - 1)^2 \right]^p. \quad (25)$$

In this section we derived the optimal value for the cost functional adopted (FIM determinant), but we gave no insight into the optimal trajectories of the vehicles. In general, it is not possible to characterize these trajectories analytically. For this reason, we will resort to numerical optimization methods to compute the optimal vehicle trajectories that maximize the determinant of the FIM, subject to collision and vehicle maneuvering constraints. Before we explore numerical solutions, however, in the next section we characterize analytically the solutions obtained for the tractable case of a single vehicle and a single static beacon.

## 5. OPTIMAL TRAJECTORIES: SINGLE-BEACON (STATIC) SCENARIO

We consider the case of single-beacon navigation, with a single vehicle and a static beacon. We make  $p = q = 1$ , and we drop all super and sub scripts. Without loss of generality, we assume that the beacon is fixed at the origin. Let  $\delta_k$  denote the direction of vector  $d_k^{-1} \mathbf{p}_k$ , that is,  $d_k^{-1} \mathbf{p}_k = \mathbf{g}(\delta_k)$ . Then,

$$A = \frac{1}{2} \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} \begin{bmatrix} (1 + \cos(2\delta_k)) & \sin(2\delta_k) \\ \sin(2\delta_k) & (1 - \cos(2\delta_k)) \end{bmatrix},$$

$$B = \frac{1}{2} \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} \begin{bmatrix} t_k(1 + \cos(2\delta_k)) & t_k \sin(2\delta_k) \\ t_k \sin(2\delta_k) & t_k(1 - \cos(2\delta_k)) \end{bmatrix},$$

$$C = \frac{1}{2} \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} \begin{bmatrix} (t_k)^2(1 + \cos(2\delta_k)) & (t_k)^2 \sin(2\delta_k) \\ (t_k)^2 \sin(2\delta_k) & (t_k)^2(1 - \cos(2\delta_k)) \end{bmatrix}.$$

The optimality conditions given in (21) imply that

$$\begin{aligned}
\sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} \cos(2\delta_k) &= 0 = \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} \sin(2\delta_k), \\
\sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} t_k \cos(2\delta_k) &= 0 = \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} t_k \sin(2\delta_k), \quad (26) \\
\sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} (t_k)^2 \cos(2\delta_k) &= 0 = \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} (t_k)^2 \sin(2\delta_k).
\end{aligned}$$

Using complex number notation, the above optimality conditions can be written in compact form as

$$\sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} e^{j2\delta_k} = \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} t_k e^{j2\delta_k} = \sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} (t_k)^2 e^{j2\delta_k} = 0.$$

Further, the above three complex equations can be expressed by one single equation as

$$\sum_{k \in \{0\} \cup \mathbb{I}_{m-1}} a_k e^{j2\delta_k} = 0. \quad (27)$$

In (27), with  $a_k \equiv 1$  we can recover the first set of equations in (26). Similarly, in (27) setting  $a_k \equiv t_k$  and  $a_k \equiv (t_k)^2$  we can recover the second and third sets of equations, respectively, in (26).

We next prove the main results of this section.

*Theorem 4.* Let  $m \geq 8$  be an even integer and  $\delta_0 \in [0, 2\pi)$ . Denote  $\mathcal{I}_1 := \{1, \dots, (m/2) - 1\}$ ,  $\mathcal{I}_2 = \{m/2, \dots, m - 2\}$  and

$$\bar{\gamma} := (\gamma_1, \dots, \gamma_{\frac{m}{2}-1}) \in \mathbb{T}^{\frac{m}{2}-1},$$

$$\gamma_{1, \dots, j} := \sum_{s=1}^j \gamma_s, \quad j \in \mathcal{I}_1,$$

$$\mathbf{G}_e(\bar{\gamma}) := \sum_{s \in \mathcal{I}_1} \sin(\gamma_s),$$

$$\mathbf{F}_e(\bar{\gamma}) := (m-1) + \sum_{s \in \mathcal{I}_1} (m-2s-1) \cos(\gamma_{1, \dots, s}),$$

$$\mathbf{H}_e(\bar{\gamma}) := \sum_{s \in \mathcal{I}_1} (s^2 + (m-s-1)^2) \sin(\gamma_{1, \dots, s}),$$

$$\mathcal{G}_e := \{\mathbf{x} \in \mathbb{T}^{\frac{m}{2}-1} : \mathbf{F}_e(\mathbf{x}) = \mathbf{G}_e(\mathbf{x}) = \mathbf{H}_e(\mathbf{x}) = 0\}.$$

Consider  $\bar{\gamma} \in \mathcal{G}_e$  and define

$$\delta_k := \begin{cases} 0.5(2\delta_0 + \gamma_{1, \dots, k}) & k \in \mathcal{I}_1, \\ 0.5(\pi + 2\delta_0 - \gamma_{1, \dots, m-k-1}) & k \in \mathcal{I}_2, \end{cases}$$

with  $\delta_{m-1} - \delta_0 = \pi/2$ . Then, (27) holds.

**Proof.** Under the hypotheses of the Theorem, the optimality conditions in (21) yield

$$\begin{aligned}
-2 \sin(2\delta_0) \mathbf{G}_e(\bar{\gamma}) &= 0, \\
2 \cos(2\delta_0) \mathbf{G}_e(\bar{\gamma}) &= 0, \\
-\cos(2\delta_0) \mathbf{F}_e(\bar{\gamma}) - (m-1) \sin(2\delta_0) \mathbf{G}_e(\bar{\gamma}) &= 0, \\
-\sin(2\delta_0) \mathbf{F}_e(\bar{\gamma}) + (m-1) \cos(2\delta_0) \mathbf{G}_e(\bar{\gamma}) &= 0, \\
-(m-1) \cos(2\delta_0) \mathbf{F}_e(\bar{\gamma}) - \sin(2\delta_0) \mathbf{H}_e(\bar{\gamma}) &= 0, \\
-(m-1) \sin(2\delta_0) \mathbf{F}_e(\bar{\gamma}) + \cos(2\delta_0) \mathbf{H}_e(\bar{\gamma}) &= 0.
\end{aligned}$$

Clearly, the above equations can be re-written as

$$\begin{aligned}
e^{2j\delta_0} \mathbf{G}_e(\bar{\gamma}) &= 0, \\
e^{2j\delta_0} \left[ \mathbf{F}_e(\bar{\gamma}) + (m-1)e^{j\pi/2} \mathbf{G}_e(\bar{\gamma}) \right] &= 0, \\
e^{-2j\delta_0} \left[ e^{4j\delta_0} (m-1) \mathbf{F}_e(\bar{\gamma}) + e^{j\pi/2} \mathbf{H}_e(\bar{\gamma}) \right] &= 0.
\end{aligned}$$

Since  $\bar{\gamma} \in \mathcal{G}_e$ , it follows that  $\mathbf{F}_e(\bar{\gamma}) = \mathbf{G}_e(\bar{\gamma}) = \mathbf{H}_e(\bar{\gamma}) = 0$ . Thus, the above equations are satisfied and this completes the proof.  $\blacksquare$

*Theorem 5.* Let  $m \geq 8$  be an odd integer and let  $\delta_0 \in [0, 2\pi)$ . Denote  $\mathcal{J}_1 := \{1, \dots, (m-3)/2\}$ ,  $\mathcal{J}_2 := \{(m+1)/2, \dots, m-2\}$ , and

$$\bar{\gamma} := (\gamma_1, \dots, \gamma_{\frac{m-3}{2}}) \in \mathbb{T}^{\frac{m-3}{2}},$$

$$\gamma_{1, \dots, j} := \sum_{s=1}^j \gamma_s, \quad j \in \mathcal{J}_1,$$

$$\mathbf{G}_o(\bar{\gamma}) := \sum_{s \in \mathcal{J}_1} \sin(\gamma_s),$$

$$\mathbf{F}_o(\bar{\gamma}) := (m-1) + \sum_{s \in \mathcal{J}_1} (m-2s-1) \cos(\gamma_{1, \dots, s}),$$

$$\mathbf{H}_o(\bar{\gamma}) := \sum_{s \in \mathcal{J}_1} (s^2 + (m-s-1)^2) \sin(\gamma_{1, \dots, s}),$$

$$\mathcal{G}_o := \left\{ \mathbf{x} \in \mathbb{T}^{\frac{m-3}{2}} : \mathbf{F}_o(\mathbf{x}) = \mathbf{G}_o(\mathbf{x}) = \mathbf{H}_o(\mathbf{x}) = 0 \right\}.$$

Consider  $\bar{\gamma} \in \mathcal{G}_o$  and let  $\delta_0 \in [0, 2\pi)$  and  $\gamma_s \in [0, 2\pi)$ ,  $s \in \{1, \dots, \frac{m}{2} - 1\}$  be such that

$$2\delta_{m-1} - 2\delta_0 = \pi = 2 \sum_{s=1}^{\frac{m}{2}-1} \gamma_s.$$

Define

$$\delta_k := \begin{cases} 0.5(2\delta_0 + \gamma_{1, \dots, k}) & k \in \mathcal{J}_1, \\ 0.5(2\delta_0 + \gamma_{1, \dots, k}) & k = (m-1)/2, \\ 0.5(\pi + 2\delta_0 - \gamma_{1, \dots, m-k-1}) & k \in \mathcal{J}_2. \end{cases}$$

Then, (27) holds.

**Proof.** Let

$$\hat{A} = \cos\left(2\delta_0 + \gamma_{1, \dots, \frac{m-1}{2}}\right) \text{ and } \hat{B} = \sin\left(2\delta_0 + \gamma_{1, \dots, \frac{m-1}{2}}\right).$$

Then the optimality conditions in (21) yield

$$\begin{aligned}
-2 \sin(2\delta_0) \mathbf{G}_o(\bar{\gamma}) + \hat{A} &= 0 \\
2 \cos(2\delta_0) \mathbf{G}_o(\bar{\gamma}) + \hat{B} &= 0 \\
-\cos(2\delta_0) \mathbf{F}_o(\bar{\gamma}) - (m-1) \sin(2\delta_0) \mathbf{G}_o(\bar{\gamma}) + \left(\frac{m-1}{2}\right) \hat{A} &= 0 \\
-\sin(2\delta_0) \mathbf{F}_o(\bar{\gamma}) + (m-1) \cos(2\delta_0) \mathbf{G}_o(\bar{\gamma}) + \left(\frac{m-1}{2}\right) \hat{B} &= 0 \\
-(m-1) \cos(2\delta_0) \mathbf{F}_o(\bar{\gamma}) - \sin(2\delta_0) \mathbf{H}_o(\bar{\gamma}) + \left(\frac{m-1}{2}\right)^2 \hat{A} &= 0, \\
-(m-1) \sin(2\delta_0) \mathbf{F}_o(\bar{\gamma}) + \cos(2\delta_0) \mathbf{H}_o(\bar{\gamma}) + \left(\frac{m-1}{2}\right)^2 \hat{B} &= 0.
\end{aligned}$$

In complex form, the above equations can be re-written as

$$\begin{aligned}
e^{2j\delta_0} \mathbf{G}_o(\bar{\gamma}) &= 0, \\
e^{2j\delta_0} \left[ \mathbf{F}_o(\bar{\gamma}) + (m-1)e^{j\pi/2} \mathbf{G}_o(\bar{\gamma}) \right] &= 0, \\
e^{-2j\delta_0} \left[ e^{4j\delta_0} (m-1) \mathbf{F}_o(\bar{\gamma}) + e^{j\pi/2} \mathbf{H}_o(\bar{\gamma}) \right] &= 0.
\end{aligned}$$

Since  $\bar{\gamma} \in \mathcal{G}_o$ , it follows that  $\mathbf{F}_o(\bar{\gamma}) = \mathbf{G}_o(\bar{\gamma}) = \mathbf{H}_o(\bar{\gamma}) = 0$ . Thus, the above equations are satisfied.  $\blacksquare$

The significance of the above two results is that they allow for the construction of optimal vehicle trajectories in simplified situations, without resorting to numerical optimization methods. The discussion of this issue is eschewed, due to space limitations.

## 6. NUMERICAL EXAMPLES

In what follows, we study three different scenarios: i) single vehicle with a single beacon, ii) single vehicle with multiple beacons, and iii) multiple vehicles with a single static beacon. To solve these problems, we resort to numerical methods to maximize the determinant of the FIMs associated with each of the vehicles using the *Simulated Annealing technique*<sup>1</sup>. In all of the following examples, we assume constant linear vehicle speed  $v^{[1]} = v^{[2]} = 2$  m/s, sampling time  $T = 4$  s, and  $\sigma_1 = \sigma_2 = 0.1$  m. The optimization variable is the course rate  $r$ , which is piecewise constant, and bounded by  $\pi/6$  in magnitude. For computational efficiency, in the case of a large number of sampling points the optimal trajectory is approximated using a sequential procedure that involves a number of optimization steps. At each step, an optimal path with a small number  $m$  samples is computed, where the first point of the current trajectory is the last point of the trajectory computed in the previous step. The final path is thus obtained as the concatenation of optimal paths of  $m$  samples each.

### 6.1 Single vehicle with one beacon

We consider a single vehicle in the presence of an unknown ocean current, with the vehicle measuring ranges with respect to a single beacon. The initial position of the vehicle is unknown. Two cases are studied: a) static beacon and b) moving beacon.

*Static beacon* In the first case, the beacon is static at  $\mathbf{b}^{[1]} = \mathbf{0}$ ; we take ten samples, i.e.  $m = 10$ , and compute the optimal vehicle trajectory in a single step. With these values, from (24) for  $q = 1$ , the optimal value of  $\ln \det(\text{FIM}_{\mathbf{U}}(\boldsymbol{\theta}))$  is given by 34.6240. Fig. 2 shows the optimal trajectory starting from the initial position  $\mathbf{p}_0^{[1]} = [-2.1313 \ -3.1195]^T$  [m] with a current  $\mathbf{v}_{c_0} = [0.4 \ 0]^T$  [m/s]. Although we have resorted to numerical optimization, the optimal positions of the vehicle along the computed trajectory are consistent with the analytical result of Theorem 4.

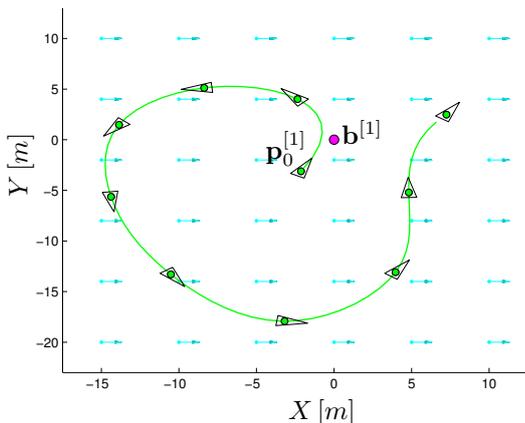


Fig. 2. Optimal vehicle trajectory for 10 range measurement points (samples) and a current of 0.4 m/s along the X-axis.

<sup>1</sup> Global Optimization Toolbox: Simulated Annealing, <http://es.mathworks.com/help/gads/index.html>, The MathWorks, Inc., 1994-2016.

*Moving beacon* We consider two different cases. We start by addressing the case where the beacon moves in a straight line starting from the origin of the inertial frame along the positive X-axis with a constant inertial speed of 0.5 m/s, i.e.  $\mathbf{b}_k^{[1]} = 0.5kT[1 \ 0]^T$  [m]. We take five samples ( $m = 5$ ) for each optimization step. From (24), with  $q = 1$ , the maximum  $\ln \det(\text{FIM}_{\mathbf{U}}(\boldsymbol{\theta}))$  is 29.0173. We assume there is a current of 0.3 m/s along the Y-axis, i.e.  $\mathbf{v}_{c_0} = [0 \ 0.3]^T$  [m/s]. Fig. 3 shows the optimal trajectory for this particular case; the computed average value of the cost function adopted is 28.4066, which is quite close to the optimal value of 29.1073.

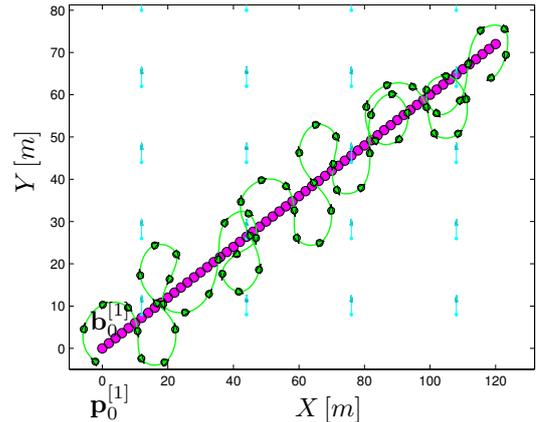


Fig. 3. Optimal vehicle trajectory for a current of 0.3 m/s along the Y-axis and the beacon moving in a straight line with a speed of 0.5 m/s along the X-axis.

We now consider the case where the beacon moves along a circumference of radius 100 m with a linear speed of 0.25 m/s, or equivalently, with an angular speed of 0.0025 rad/s. The initial position for the beacon is  $\mathbf{b}_0^{[1]} = [110 \ 0]^T$  [m]. In this case the current is of 0.4 m/s along the X-axis. The optimal trajectory is shown in Fig. 4.

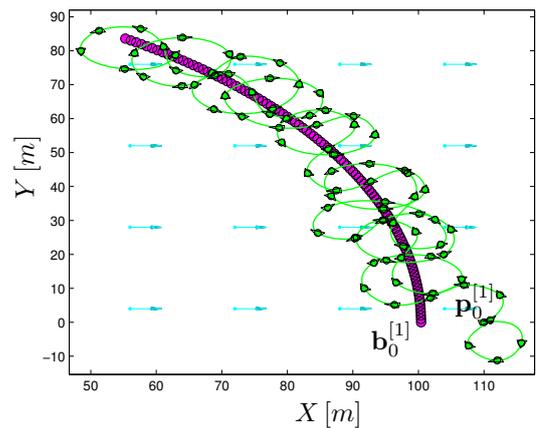


Fig. 4. Optimal vehicle trajectory for a current of 0.4 m/s along the X-axis and the beacon moving in a circular path.

### 6.2 Single vehicle with multiple fixed beacons

In this second scenario we consider the special case where the vehicle measures ranges to two known static beacons.

Interestingly enough, the solution to this problem can be found analytically by seeking inspiration from the solution to the problem of target localization in 2D using two range measuring sensors (Moreno-Salinas et al., 2013). Exploiting this concept, it can be shown that the optimal vehicle positions are such that the relative position vectors of the vehicle with respect to the two beacons are orthogonal. In other words, the optimal positions to acquire range measurements are on the circumference that passes through the two fixed beacons and is centered at the mid point of the line joining them. Thus, in this example, we will see that once the vehicle converges to the circumference thus defined, it continues to move along this circumference, yielding the optimal FIM.

The numerically computed trajectory shown in Fig. 5 is composed of individual trajectories of seven points each (computed at each iteration step), and the beacons are separated by 100 [m]. In this case, the optimal  $\ln \det(FIM_U(\theta))$ , considering two beacons and seven range measurements, is 34.5221. For a current of 0.5 m/s along the X-axis we can notice in Fig. 5 how the vehicle keeps moving in the circumference defined by the two beacons, thus yielding the maximum FIM determinant.

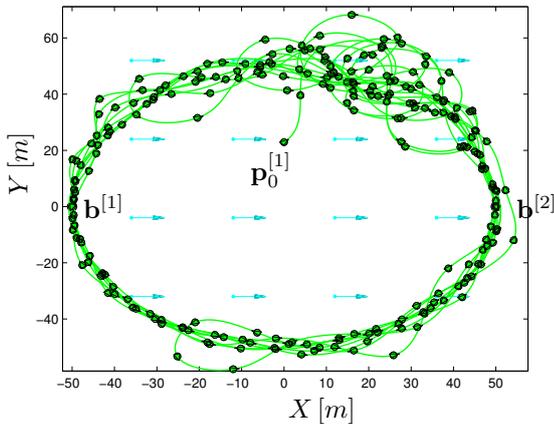


Fig. 5. Optimal vehicle trajectory for a current of 0.5 m/s along the X-axis and two fixed beacons. Here  $\mathbf{p}_0^{[1]} = [0 \ 20]^T$  [m],  $\mathbf{b}^{[1]} = [-50 \ 0]^T$  [m] and  $\mathbf{b}^{[2]} = [50 \ 0]^T$  [m].

### 6.3 Multiple vehicles and one beacon

In this last example we consider two vehicles and a single static beacon. This case is similar to the one with a single vehicle, but we must now into account collision avoidance between vehicles, so that the optimal trajectories are limited by the requirement to maintain a safety distance between vehicles. In this example the current is again of 0.5 m/s along the Y-axis of the inertial coordinate frame. The optimal trajectories obtained with the optimization algorithm are shown in Fig. 6, considering  $m = 10$  measurement points (one single step in the iteration procedure). The optimal  $\ln \det(FIM_U(\theta))$  for the first and second vehicle is given by 33.4725 and 32.3285, respectively, both of which are close to the optimal value of 34.6240.

## 7. CONCLUSIONS

In this paper we explored trajectories for the planar range-based positioning problem of single and multiple AUVs

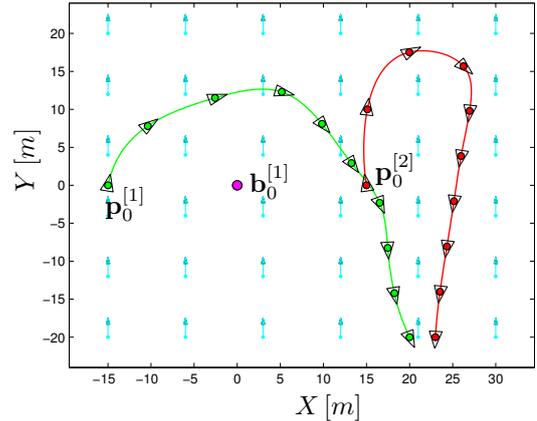


Fig. 6. Optimal vehicle trajectory for 10 range measurement points (samples), with a current of 0.5 m/s along the Y-axis and with the final point of the trajectory given as an additional constraint. Here  $\mathbf{p}_0^{[1]} = [-15 \ 0]^T$  [m],  $\mathbf{p}_0^{[2]} = [15 \ 0]^T$  [m] and  $\mathbf{b}^{[1]} = \mathbf{0}$  [m].

in the presence of constant unknown ocean currents. The trajectories were obtained by maximizing the determinant of a suitably defined Fisher information matrix with the initial vehicle positions and constant ocean current playing the roles of parameters to be estimated. Under some mild assumptions, we derived the optimal value of the Fisher determinant and a set of analytical solutions. Finally, via numerical simulations we presented different scenarios with multiple vehicles and beacons.

## REFERENCES

- A. Gadre and D. J. Stilwell, Underwater navigation in the presence of unknown currents based on range measurements from a single location. *Proc. American Control Conference, Portland, OR, USA, June 8-10, 2005*.
- N. Crasta, M. Bayat, A. P. Aguiar, and A. M. Pascoal, Observability analysis of 2D single-beacon navigation in the presence of constant currents for two classes of maneuvers. *Proceedings of the 9th IFAC CAMS Conference, Osaka, Japan, Sept. 17-20, 2013*.
- D. Moreno-Salinas, N. Crasta, M. Ribeiro, B. Bayat, A. M. Pascoal, and J. Aranda, Integrated motion planning, control, and estimation for range-based marine vehicle positioning and target localization. *Proceedings of the 10th IFAC CAMS Conference, Trondheim, Norway, Sept. 13-16, 2016*.
- O. Popescu, C. Rose, and D. C. Popescu, Maximizing the determinant for a special class of block-partitioned matrices. *Mathematical Problems in Engineering, 49-61, 2004*.
- M. Pedro, D. Moreno-Salinas, N. Crasta, and A. M. Pascoal, Underwater single-beacon localization: Optimal trajectory planning and minimum-energy estimation. *4th IFAC Workshop on NGCUV, Girona, Spain, Apr. 28-30, 2015*.
- Van Trees and L Harry. *Detection, Estimation and Modulation Theory*. NY: John Wiley and Sons Inc., 1968.
- D. Moreno-Salinas, A. M. Pascoal and J. Aranda, Optimal sensor placement for multiple target positioning with range-only measurements in two-dimensional scenarios. *Sensors, 13, 10674-10710, 2013*.