OBSERVABILITY ANALYSIS OF THE SIMULTANEOUS UNDERWATER VEHICLE LOCALIZATION AND MAPPING BASED ON RANGING MEASUREMENTS

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Abstract: The localization of an autonomous underwater vehicle (AUV) is a challenging and important problem in marine robotics. In this paper we investigate the observability problem of the process of Simultaneous Localization and Mapping (SLAM) of an AUV equipped with inertial sensors, a depth sensor, and an acoustic ranging device that provides relative range measurements to stationary beacons. For the case that the motion of the AUV corresponds to constant linear and angular velocities (expressed in the body-frame), also known as trimming or steady-state trajectories, we provide conditions under which it is possible to reconstruct the initial state of the resulting SLAM system (and in particular the position of the AUV). We show that the unobservable subspace $\mathcal{UO}$ restricted to the assumption that the position of one of the beacons or the initial position of the AUV is known, contains only the zero vector with exception of a particular case where the $\mathcal{UO}$ is composed by a finite set of isolated points. Numerical tests that illustrate the results derived are presented and discussed.

Keywords: Observability analysis, SLAM, range measurements, AUV.

1. INTRODUCTION

The localization of an autonomous underwater vehicle (AUV) is a challenging and important problem in marine robotics. Since electromagnetic signals do not propagate well below the sea surface, a variety of different approaches has been developed that make use of acoustic signals. Ultra Short BaseLine (USBL), Long BaseLine (LBL), and GPS Intelligent Buoy (GIB) are examples of underwater navigation and positioning systems where all use the concept of beacons, transponders, and range measurements taken from relative/absolute time of flight of acoustic signals.

Another promising and interesting approach is to use only one beacon for localization. One of the first works on single beacon acoustic navigation is reported in (Larsen, 2000), where the author describes a synthetic long base-line (SLBL) navigation algorithm, which makes use of a single LBL in combination with a a high performance dead-reckoning navigation system. In (Casey and Hu, 2007) the authors describe an extended Kalman filter (EKF) for localization of an AUV using a single beacon, and in (Saude and Aguiar, 2009) combining the dead-reckoning information with multiple range measurements taken at different instants of time from the vehicle to a single beacon, a robust estimation algorithm was proposed for vehicle localization in the presence of unknown ocean currents. In (Olson *et al.*, 2004), the authors described a range only beacon localization, which assumes no prior knowledge of beacons' locations. A pure range only sub-sea SLAM has been designed in (Newman and Leonard, 2003). Cooperative AUV navigation using a single maneuvering surface craft has been studied in (Fallon *et al.*, 2009). Regarding observability studies in the single beacon AUV localization, one of the first results are described in (Gadre and Stilwell, 2005; Gadre, 2007), where the authors have investigated the observability of the linearized single beacon navigation system. Another study that

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reformulates the problem to a linear time varying (LTV) system is reported in (Batista *et al.*, 2010). In (Arrichiello *et al.*, 2011), the authors investigated the nonlinear observability concepts of a nonlinear inter-vehicle ranging system using observability rank conditions. The results obtained are validated experimentally in an equivalent single beacon navigation scenario.

This work addresses the single/multiple beacon observability analysis of the simultaneous localization and mapping (SLAM) for AUV navigation using range measurements to stationary beacons. To this effect, we first apply a coordinate transformation similar to the one presented in (Aguiar and Hespanha, 2006) and then a time-scaling transformation to obtain a LTV system. Then, we investigate for the case that the motion of the AUV corresponds to constant linear and angular velocities expressed in the body-frame, (also known as trimming or steady-state trajectories), under which conditions it is possible to reconstruct the initial state of the resulting SLAM system (and in particular the position of the AUV). Comparing to the work mentioned above, there are two main differences: i) we focus on obtaining conditions expressed in the body frame of the AUV, which usually are more meaningful, and ii) we address the SLAM problem (multiple beacon case). We show that the unobservable subspace $\mathcal{UO}$ restricted to the assumption that the position of one of the beacons or the initial position of the AUV is known, contains only the zero vector with exception of a particular case where the $\mathcal{UO}$ is composed by a finite set of isolated points.

The paper is organized as follows: Section 2 formulates the process model of single/multiple beacon system. The observability analysis of the proposed system is investigated in Section 3. Section 4 presents simulation results that illustrates the observability results and the concluding remarks are given in Section 5.

2. PROCESS MODEL

In this section we investigate the observability of the problem of computing in real time an estimate of the position of an AUV and simultaneously constructing a map of its surrounding. The map whose building process is based on ranging measurements obtained from stationary acoustic modems (beacons) contains an estimate of the location of the beacons. To formulate the process model we consider two coordinate frames: fixed earth or inertial coordinate frame $\{\mathcal{I}\}$, and body fixed coordinate frame $\{\mathcal{B}\}$ that is attached to the AUV, which moves with respect to the coordinate frame $\{\mathcal{I}\}$. Let $({}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}}, {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}) \in SE(3)$ be the configuration of the frame $\{\mathcal{B}\}$ with respect to $\{\mathcal{I}\}$, where ${}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}}$ indicates the position of the AUV in frame $\{\mathcal{I}\}$, and ${}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}$ its rotation matrix that describes the rotation matrix from $\{\mathcal{B}\}$ to $\{\mathcal{I}\}$. The equations of motion can be written as

$$ {}^{\mathcal{I}}\dot{\boldsymbol{p}}_{\mathcal{B}} = {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,\boldsymbol{\nu} \tag{1}$$

$$ {}^{\mathcal{I}}_{\mathcal{B}}\dot{\mathcal{R}} = {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,S(\boldsymbol{\omega}) \tag{2}$$

where the linear and angular velocities ($\boldsymbol{\nu}, \boldsymbol{\omega} : [0,\infty) \to \mathbb{R}^3$) are viewed as input signals to the system (1)-(2). In (2), $S(\cdot)$ is a function from $\mathbb{R}^3$ to the space of skew-symmetric matrices $\mathbb{S} := \{M \in \mathbb{R}^{3\times 3} : M = -M'\}$ defined by

$$ S(\boldsymbol{a}) := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} $$

In what follows we will use the Euler angles $\boldsymbol{\eta} = [\phi, \theta, \psi]$ to parameterize the rotation matrix. Consider now n stationary beacons located at unknown positions ${}^{\mathcal{I}}\boldsymbol{q}_i$, $i = 1, 2, ..., n$, that is

$$ {}^{\mathcal{I}}\dot{\boldsymbol{q}}_i = 0 \tag{3}$$

For each $i \in \{1, 2, ..., n\}$ let $r_i(t)$ be an acoustic ranging measurement acquired at time t from the i^{th} beacon. In this case, the measurement or output model is given by

$$ r_i = \|{}^{\mathcal{I}}\boldsymbol{q}_i - {}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}}\| \tag{4}$$

$$ z_i = [0, 0, 1]{}^{\mathcal{I}}\boldsymbol{q}_i \tag{5}$$

$$ z_0 = [0, 0, 1]{}^{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{B}} \tag{6}$$

where z_0 is the depth of the AUV that is assumed to be available (we consider the practical situation that the AUV is equipped with a depth sensor). We also consider that the location of the beacons q_i are only unknown in the horizontal plane, that is, we assume that we know the depth z_i. This is a reasonable assumption if each beacon is attached to a buoy that is at the surface.

Equations (1)-(6) represent the process model of the problem of Simultaneous Localization and Mapping (SLAM) of the AUV. In this paper we address the observability problem. To that effect, for the nonlinear system (1)-(6) we construct a new linear time varying system (LTV) and derive under what conditions the new system is *equivalent* to (1)-(6). Note that once we have an LTV system we can apply the powerful tools of linear estimation theory for observability analysis. The strategy to obtain an LTV system does not follow the ones described in (Krener and Isidori, 1983; Plestan and Glumineau, 1997) but it is specific tailored for our application. The idea is to view the beacons q_i in body frame $\{\mathcal{B}\}$ and introduce a *virtual* beacon, $\boldsymbol{q}_0$, located at the origin of $\{\mathcal{I}\}$ (see Figure 1). Following this strategy and resorting to some of the ideas in (Aguiar and Hespanha, 2006), we first express $\boldsymbol{q}_0$ in $\{\mathcal{B}\}$ as

$$ {}^{\mathcal{B}}\boldsymbol{q}_0 = {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}'\,{}^{\mathcal{I}}\boldsymbol{q}_0 - {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}'\,{}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}} $$

and compute its dynamic equation given by

$$ \begin{aligned} {}^{\mathcal{B}}\dot{\boldsymbol{q}}_0 &= {}^{\mathcal{I}}_{\mathcal{B}}\dot{\mathcal{R}}'\,({}^{\mathcal{I}}\boldsymbol{q}_0 - {}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}}) + {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}'\,{}^{\mathcal{I}}\dot{\boldsymbol{q}}_0 - {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}'\,{}^{\mathcal{I}}\dot{\boldsymbol{p}}_{\mathcal{B}} \\ &= -S(\boldsymbol{\omega}){}^{\mathcal{B}}\boldsymbol{q}_0 - \boldsymbol{\nu} \end{aligned} $$

where we have used (3). To obtain the dynamics of the position of the other beacons $\boldsymbol{q}_i$ in body frame, we introduce the vector $\boldsymbol{p}_i$ that connects the virtual beacon $\boldsymbol{q}_0$ to $\boldsymbol{q}_i$. Note that ${}^{\mathcal{I}}\boldsymbol{p}_i$ is a stationary vector while ${}^{\mathcal{B}}\boldsymbol{p}_i$ is in general a time dependent vector (with same magnitude of ${}^{\mathcal{I}}\boldsymbol{p}_i$ but rotated by ${}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}'$). Therefore,

$$ {}^{\mathcal{B}}\boldsymbol{p}_i = {}^{\mathcal{B}}\boldsymbol{q}_i - {}^{\mathcal{B}}\boldsymbol{q}_0 \tag{7}$$

$$ {}^{\mathcal{B}}\dot{\boldsymbol{p}}_i = -S(\boldsymbol{\omega}){}^{\mathcal{B}}\boldsymbol{p}_i $$

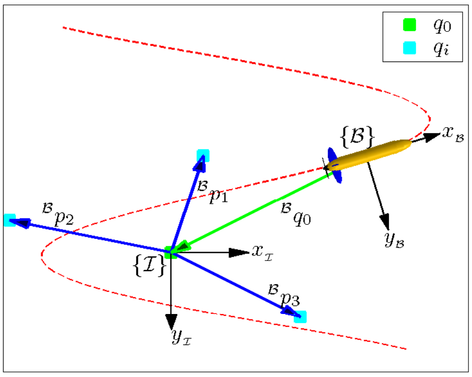

Fig. 1. Illustration of representative state vectors in 2D space.

From (4), (7), and using the fact that ${}^{\mathcal{I}}\boldsymbol{q}_i = {}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}} + {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,{}^{\mathcal{B}}\boldsymbol{q}_i$ the measurement model can be written as

$$r_i = \|{}^{\mathcal{I}}\boldsymbol{q}_i - {}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}}\| = \|{}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,{}^{\mathcal{B}}\boldsymbol{q}_i\| = \|{}^{\mathcal{B}}\boldsymbol{p}_i + {}^{\mathcal{B}}\boldsymbol{q}_0\|$$

where we have used, in the last equality, the fact that $\|\mathcal{R}\| = 1$. Defining the scalar state variable $\chi_i = \|{}^{\mathcal{B}}\boldsymbol{p}_i + {}^{\mathcal{B}}\boldsymbol{q}_0\|$, $i = 1, 2, ..., n$, the output equation (4) becomes $r_i = \chi_i$ where χ_i satisfies

$$\dot{\chi}_i = \frac{({}^{\mathcal{B}}\boldsymbol{p}_i + {}^{\mathcal{B}}\boldsymbol{q}_0)' S(\boldsymbol{\omega})({}^{\mathcal{B}}\boldsymbol{p}_i + {}^{\mathcal{B}}\boldsymbol{q}_0) - \boldsymbol{\nu}'({}^{\mathcal{B}}\boldsymbol{p}_i + {}^{\mathcal{B}}\boldsymbol{q}_0)}{\chi_i}$$
$$= \frac{-\boldsymbol{\nu}'({}^{\mathcal{B}}\boldsymbol{p}_i + {}^{\mathcal{B}}\boldsymbol{q}_0)}{r_i}$$

Using the equalities ${}^{\mathcal{I}}\boldsymbol{q}_i = {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,{}^{\mathcal{B}}\boldsymbol{p}_i$ and ${}^{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{B}} = -{}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,{}^{\mathcal{B}}\boldsymbol{q}_0$ we can write the output equation (5) and (6) as

$$z_i = [0, 0, 1]{}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,{}^{\mathcal{B}}\boldsymbol{p}_i$$
$$z_0 = -[0, 0, 1]{}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,{}^{\mathcal{B}}\boldsymbol{q}_0$$

In summary we obtain an LTV system described by

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= A(\boldsymbol{u}(t), \boldsymbol{y}(t))\,\boldsymbol{x}(t) + \boldsymbol{b}(\boldsymbol{u}(t))\\ \boldsymbol{y}(t) &= C(\boldsymbol{u}(t))\,\boldsymbol{x}(t)\end{aligned} \tag{8}$$

where

$$\boldsymbol{x} := \left[\,{}^{\mathcal{B}}\boldsymbol{q}_0', {}^{\mathcal{B}}\boldsymbol{p}_1' \, {}^{\mathcal{B}}\boldsymbol{p}_2' \ldots {}^{\mathcal{B}}\boldsymbol{p}_n', \chi_1 \chi_2 \ldots \chi_n\,\right]'$$
$$\boldsymbol{y} := \left[r_1 \; \ldots \; r_n, \; z_0, \; z_1 \; \ldots \; z_n\right]'$$
$$\boldsymbol{u} := \left[\,\boldsymbol{\nu}' \;\; \boldsymbol{\omega}' \;\; \boldsymbol{\eta}'\,\right]'$$
$$\boldsymbol{s} := \left[\frac{1}{r_1} \;\; \frac{1}{r_2} \; \ldots \; \frac{1}{r_n}\right]'$$
$$A(\boldsymbol{u}, \boldsymbol{y}) := \begin{bmatrix} -S(\boldsymbol{\omega}) & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -I_n \otimes S(\boldsymbol{\omega}) & \boldsymbol{0} \\ -\boldsymbol{s} \otimes \boldsymbol{\nu}' & -I_n \otimes (\boldsymbol{s} \otimes \boldsymbol{\nu}') & \boldsymbol{0} \end{bmatrix}$$
$$\boldsymbol{b}(\boldsymbol{u}) := \left[\,-\boldsymbol{\nu}' \;\; \boldsymbol{0} \;\; \boldsymbol{0}\,\right]'$$
$$C(\boldsymbol{u}) := \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\; I_n \\ -[0, 0, 1]\,{}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}(\boldsymbol{\eta}) & \boldsymbol{0} & \boldsymbol{0}\;\boldsymbol{0} \\ \boldsymbol{0} & I_n \otimes \left([0, 0, 1]\,{}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}(\boldsymbol{\eta})\right) & \boldsymbol{0}\;\boldsymbol{0} \end{bmatrix}$$

We remark that (8) is not defined when $r_i = 0$, which corresponds to the particular case that the position of the AUV coincides with the location of the i^{th} beacon.

3. OBSERVABILITY ANALYSIS

Simultaneous localization and mapping (SLAM) is a technique to build an estimate of the environment map (within a complete unknown environment or with some a priori knowledge of the environment), while simultaneously compute an estimate of the position of the vehicle. Unless there is an *anchor* that relates the relative localization with the global (inertial) position, the SLAM process in (8), where ${}^{\mathcal{B}}\boldsymbol{q}_0$ can be viewed as the position of the AUV and the rest of the states correspond to beacons positions, is not observable. In fact the idea is to use a priori knowledge of one of the states and estimate the other unknown ones. For example, one assumption is to consider that the initial condition of the location of the AUV is known (which can be done in practice if the AUV starts at the surface and there is GPS). Then, since ${}^{\mathcal{B}}\boldsymbol{q}_0(t_0)$ and the input $\boldsymbol{u}(t)$ are known, it follows that ${}^{\mathcal{B}}\boldsymbol{q}_0(t)$ will also be known for all $t \geq 0$. In this case, we can remove ${}^{\mathcal{B}}\boldsymbol{q}_0$ from the state vector and instead be viewed as an input signal.

Using the above argument and considering at this stage only one beacon, $n = 1$, from (8) we obtain the single beacon system with squared range state χ_1^2 and output r_1^2, as follows

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \nu(t)\left(A(\boldsymbol{u}(t))\,\boldsymbol{x}(t) + \boldsymbol{b}(\boldsymbol{u}(t))\right)\\ \boldsymbol{y}(t) &= C(\boldsymbol{u}(t))\boldsymbol{x}(t)\end{aligned} \tag{9}$$

where

$$\boldsymbol{x} := \left[\,{}^{\mathcal{B}}\boldsymbol{p}_1' \;\; \chi_1^2\,\right]' = \left[\,{}^{\mathcal{B}}p_{1,x} \;\; {}^{\mathcal{B}}p_{1,y} \;\; {}^{\mathcal{B}}p_{1,z} \;\; \chi_1^2\,\right]'$$
$$\boldsymbol{y} := \left[r_1^2, \; z_1\right]'$$
$$\boldsymbol{u} := \left[\,\boldsymbol{\nu}' \;\; \boldsymbol{\omega}' \;\; \boldsymbol{\eta}' \;\; {}^{\mathcal{B}}\boldsymbol{q}_0'\,\right]'$$
$$A(\boldsymbol{u}, \boldsymbol{y}) := \begin{bmatrix} 0 & \frac{\omega_3(t)}{\nu(t)} & -\frac{\omega_2(t)}{\nu(t)} & 0 \\ -\frac{\omega_3(t)}{\nu(t)} & 0 & \frac{\omega_1(t)}{\nu(t)} & 0 \\ \frac{\omega_2(t)}{\nu(t)} & -\frac{\omega_1(t)}{\nu(t)} & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}$$
$$\boldsymbol{b}(\boldsymbol{u}) := \left[\,0 \;\; 0 \;\; 0 \;\; -2\,{}^{\mathcal{B}}q_{0,x}(t)\,\right]'$$
$$C(\boldsymbol{u}) := \begin{bmatrix} 0 & 0 & 0 & 1 \\ -s\theta(t) & s\phi(t)c\theta(t) & c\phi(t)c\theta(t) & 0 \end{bmatrix}$$

with $\boldsymbol{\omega}(t) = [\omega_1(t), \omega_2(t), \omega_3(t)]'$, $\boldsymbol{\nu}(t) = [\nu(t), 0, 0]'$, and ${}^{\mathcal{B}}\boldsymbol{q}_0 = [{}^{\mathcal{B}}q_{0,x}, {}^{\mathcal{B}}q_{0,y}, {}^{\mathcal{B}}q_{0,z}]'$.

Remark 1. In (9) we have considered squared ranging measurement to simplify the observability analysis. This does not change the observability results. In fact if we conclude that (9) is observable with a given input $\boldsymbol{u}^*$ in the sense that for every pair of distinct initial conditions $(\boldsymbol{x}_0, \boldsymbol{z}_0)$ there exists a time interval $t \in [t^*, t_f]$, $t^* \geq t_0$ such that the corresponding squared range outputs are different

$$y(t; \boldsymbol{u}^*, \boldsymbol{x}_0) := r_1^2(t; \boldsymbol{u}^*, \boldsymbol{x}_0) \neq y(t; \boldsymbol{u}^*, \boldsymbol{z}_0) := r_1^2(t; \boldsymbol{u}^*, \boldsymbol{z}_0)$$

Then it also follows that $r_1(t; \boldsymbol{u}^*, \boldsymbol{x}_0) \neq r_1(t; \boldsymbol{u}^*, \boldsymbol{z}_0)$, which implies that the initial conditions $(\boldsymbol{x}_0, \boldsymbol{z}_0)$ for the original system (using ranges) will produce different outputs. $\square$

Remark 2. For observability analysis we can consider without loss of generality that the linear velocity $\boldsymbol{\nu}$ does not include any non null term in y and z components. If this is not the case (e.g., there is sideslip in steady state), then for observability analysis, instead of using the body fixed frame $\{\mathcal{B}\}$ to obtain the process model (9) we use the *flow frame* $\{\mathcal{F}\}$, which its origin coincides with $\{\mathcal{B}\}$ but the orientation is such that the linear velocity expressed in $\{\mathcal{F}\}$ is $\boldsymbol{\nu}(t) = [\nu(t), 0, 0]'$. Note that in this case the orientation $\boldsymbol{\eta}$ is with respect to $\{\mathcal{F}\}$. □

To re-write (9) in a standard LTV system we further apply a time scale transformation with $\dot{\tau} = \nu(t)$ and assume that $\nu(t) \neq 0$. In this case, we obtain

$$\begin{aligned} \frac{d\boldsymbol{x}}{d\tau} &= A(\tau)\boldsymbol{x}(\tau) + b(\tau) \\ \boldsymbol{y}(\tau) &= C(\tau)\boldsymbol{x}(\tau) \end{aligned} \tag{10}$$

Since (10) is a time scaled version of (9), then both systems are equivalent in observability sense. Furthermore without loss of generality we can set $\nu(t) = 1$ in (9).

We are now ready to study the observability of the single beacon case. We will consider the case that the vehicle is in steady state motion, that is, the linear and angular velocities are constant.

Theorem 1. Consider system (9) without depth measurement, that is $C = [0, 0, 0, 1]$, and suppose that

$$\begin{cases} \omega_1 \neq 0 \\ \omega_2 \neq 0 \text{ or } \omega_3 \neq 0 \end{cases} \tag{11}$$

Then the system is observable.

PROOF. Since we are assuming that the linear and angular velocities are constant, it turns out that the system (9) without depth measurement is an LTI system. In this case the observability matrix is computed as

$$\mathcal{O}_4 = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = 2 \begin{bmatrix} 0 & 0 & 0 & 0.5 \\ -1 & 0 & 0 & 0 \\ 0 & -\omega_3 & \omega_2 & 0 \\ \omega_2^2 + \omega_3^2 & -\omega_1\omega_2 & -\omega_1\omega_3 & 0 \end{bmatrix}$$

If the conditions of theorem hold, it follows that $\mathcal{O}$ is a full rank matrix and so the system is observable. ■

We remark that the fact of introducing a new output to the system (in this case the depth measurement) will not change the observability results of Theorem 1 but as it will be seen it is possible to obtain less conservative results.

Theorem 2. Consider system (9) and suppose that

$$\omega_3 \neq -\omega_2 \tan(\phi) \tag{13}$$

Then, the system is observable.

PROOF. Using the observability rank condition we can conclude that if the following matrix

$$\mathcal{O}_3 = \begin{bmatrix} C \\ CA + \dot{C} \\ CA^2 + 2\dot{C}A + C\dot{A} + \ddot{C} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -s\theta & c\theta s\phi & c\phi c\theta & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2\omega_3 & 2\omega_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is full rank then the system (9) is observable. Computing the determinant (taking out the nonzero rows), it follows that $\mathcal{O}_3$ is full rank if (13) holds. ■

From the above results we can conclude that the condition

$$\omega_3 = -\omega_2 \tan(\phi) \tag{14}$$

is a particular case. In fact, in this case the Euler angles rates satisfy

$$\begin{bmatrix} \dot{\phi} & \dot{\theta} & \dot{\psi} \end{bmatrix} = \begin{bmatrix} \omega_1 & \dfrac{\omega_2}{\cos\phi} & 0 \end{bmatrix} \tag{15}$$

Notice also that from the assumption of $\boldsymbol{\omega}$ being constant, using (14) we can conclude that ϕ has to be constant, which implies that $\dot{\phi} = 0$ and consequently from (15) that $\omega_1 = 0$.

Theorem 3. Consider system (9) and suppose that (14) holds. Then, it is possible to reconstruct the initial condition of the state $\boldsymbol{x}(0)$ from the input and output signals on the interval $[0, t_f]$ with the exception that the pair $({}^{\mathcal{B}}p_{1,y}, {}^{\mathcal{B}}p_{1,z})(0)$ can have two possible solutions.

PROOF. Suppose first that $\phi_0 = 0$. Since $\phi(t) = \phi_0 = 0$ for all $t \geq 0$ it is possible to compute explicitly the Gramian of observability

$$\mathcal{W}(0,t) = \int_0^t \Phi'(s,0)C'(s)C(s)\Phi(s,0)ds$$

which is given in (12). Note that the second row and column are zero, which means that it is not possible using the Gramian to reconstruct the initial condition $\boldsymbol{x}(0)$. Now consider a reduced order system with state $\boldsymbol{x}_r = [{}^{\mathcal{B}}p_{1,x}, {}^{\mathcal{B}}p_{1,z}, \chi_1^2]$. Clearly this system is observable because the observability Gramian matrix is full rank (rank 3). Furthermore, in this case it is possible to reconstruct the initial condition $\boldsymbol{x}_r(0)$. Using now the fact that $\chi_1^2 = {}^{\mathcal{B}}q_{1,x}^2 + {}^{\mathcal{B}}q_{1,y}^2 + {}^{\mathcal{B}}q_{1,z}^2$ it follows that we can obtain two possible solutions for ${}^{\mathcal{B}}p_{1,y}$.

Consider now the case $\phi_0 \neq 0$ and let $\boldsymbol{\eta}_0 = [\phi_0, 0, 0]'$. Note that $\phi(t) = \phi_0$ for all $t \geq 0$. By performing a change of coordinates

$$\begin{aligned} \bar{\boldsymbol{\eta}} &= \boldsymbol{\eta} - \boldsymbol{\eta}_0 \\ {}^{\mathcal{B}}\bar{\boldsymbol{p}}_1 &= {}^{\mathcal{I}}_{\mathcal{B}}R(-\boldsymbol{\eta}_0)\,{}^{\mathcal{B}}\boldsymbol{p}_1 \\ {}^{\mathcal{B}}\bar{\boldsymbol{q}}_0 &= {}^{\mathcal{I}}_{\mathcal{B}}R(-\boldsymbol{\eta}_0)\,{}^{\mathcal{B}}\boldsymbol{q}_0 \end{aligned}$$

we obtain an equivalent system to the previous one but with $\bar{\phi}_0 = 0$ and $\chi_1^2 = {}^{\mathcal{B}}\bar{q}_{1,x}^2 + {}^{\mathcal{B}}\bar{q}_{1,y}^2 + {}^{\mathcal{B}}\bar{q}_{1,z}^2$. Using the same reasoning as before, it follows that we can obtain two possible solutions for ${}^{\mathcal{B}}\bar{p}_{1,y}$. Rotating back the solution to the original coordinate ${}^{\mathcal{B}}\boldsymbol{p}_1$, we can conclude that the pair $({}^{\mathcal{B}}p_{1,y}, {}^{\mathcal{B}}p_{1,z})$ can have two possible solutions. This concludes the result. ■

$$\mathcal{W}(0,t) = \begin{bmatrix} \frac{t\omega_2^3 \sin^2\theta_0 - \sin(2t\omega_2) - 2t\omega_2}{\omega_2^3} & 0 & \frac{-t\omega_2^3 \sin(2\theta_0) - 16\sin(\frac{t\omega_2}{2})^4}{2\omega_2^3} & \frac{2\cos t\omega_2 - 2}{\omega_2^2} \\ 0 & 0 & 0 & 0 \\ -s\theta c\phi_0 c\theta & 0 & \frac{t\omega_2^3\cos^2(\theta_0) + \sin(2t\omega_2) - 8\sin(t\omega_2) + 6t\omega_2}{\omega_2^3} & \frac{-2\sin(t\omega_2) - 2t\omega_2}{\omega_2^2} \\ \frac{2\cos t\omega_2 - 2}{\omega_2^2} & 0 & \frac{-2\sin(t\omega_2) - 2t\omega_2}{\omega_2^2} & t \end{bmatrix} \tag{12}$$

From Theorems 1-3 we can now characterize the Unobservable Space $\mathcal{UO}$ of (9) that corresponds to the subspace given by the kernel of $\mathcal{W}(t_0,t)$, $\forall t \in [0,t_f]$. Note that the importance of the Unobservable subspace $\mathcal{UO}$ stems from the fact that if x_0 is a given initial condition of (9) for a compatible input/output pair, then the initial condition $z_0 = x_0 + x_{uo}$, $x_{uo} \in \mathcal{UO}$ is also compatible with the same input/output pair.

Corollary 1. The $\mathcal{UO}$ of system (9) is given by

$$\mathcal{UO} = \begin{cases} \{\alpha[0, c\phi_0, s\phi_0, 0]'\} & ,\omega_3 = -\omega_2\tan(\phi) \\ \{\mathbf{0}\} & ,\omega_3 \neq -\omega_2\tan(\phi) \end{cases}$$

where $\alpha \in \{0, -2(\bar{y}_1 c\phi_0 + \bar{z}_1 s\phi_0)\}$ with $[\bar{x}_1, \bar{y}_1, \bar{z}_1]' = {}^{\mathcal{B}}\boldsymbol{q}_0(0) + {}^{\mathcal{B}}\boldsymbol{p}_1(0)$.

PROOF. Suppose that $\omega_3 \neq -\omega_2\tan(\phi)$, which implies that system (9) is observable and therefor the unobservable set $\mathcal{UO}$ contains only the origin. Otherwise, if $\omega_3 = -\omega_2\tan(\phi)$, by computing the kernel of $\mathcal{W}(t_0,t)$ we get

$$\ker(\mathcal{W}(t_0,t)) = \{\alpha[0,1,0,0]' : \alpha \in \mathbb{R}\}$$

where we have used the assumption $\phi_0 = 0$. If this is not the case, we just have to perform a rotation of ${}^{\mathcal{I}}_{\mathcal{B}}R(\boldsymbol{\eta}_0)$ to obtain

$$\ker(\mathcal{W}(t_0,t,\phi_0)) = \{\alpha[0, c\phi_0, s\phi_0, 0]' : \alpha \in \mathbb{R}\}$$

We now have to consider the effect of the constraint $\chi_1^2 = {}^{\mathcal{B}}\bar{q}_{1,x}^2 + {}^{\mathcal{B}}\bar{q}_{1,y}^2 + {}^{\mathcal{B}}\bar{q}_{1,z}^2$. To this end, consider two initial conditions $\boldsymbol{x}_0$ and $\boldsymbol{z}_0 = \boldsymbol{x}_0 + \alpha[0, c\phi, s\phi, 0]'$, $\alpha \in \mathbb{R}$ and apply the constraint to obtain

$$\begin{aligned} \chi_1(t_0)^2 &= \bar{x}_1^2 + (\bar{y}_1 + \alpha c\phi_0)^2 + (\bar{z}_1 + \alpha s\phi_0)^2 \\ \chi_1(t_0)^2 &= \bar{x}_1^2 + \bar{y}_1^2 + \bar{z}_1^2 \end{aligned}$$

Solving the above equation yields $\alpha^2 = -2\alpha(\bar{y}_1 c\phi_0 + \bar{z}_1 s\phi_0)$ which results into two possible solutions

$$\alpha = 0 \text{ or } \alpha = -2(\bar{y}_1 c\phi_0 + \bar{z}_1 s\phi_0)$$

This completes the proof. □

Until this point we have investigated the case where it is assumed that the initial location of the AUV ${}^{\mathcal{B}}q_0(t_0)$, is known and the position of the beacon ${}^{\mathcal{B}}p_1$, is unknown. We now consider the dual case, that is, ${}^{\mathcal{B}}p_1$ is known but not the position of the AUV. In this case, we obtain the system

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \nu(t)\left(A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t)\right) \\ \boldsymbol{y}(t) &= C(t)\boldsymbol{x}(t) \end{aligned} \tag{16}$$

but with

$$\begin{aligned} \boldsymbol{x} &:= \left[\,{}^{\mathcal{B}}\boldsymbol{q}_0' \;\; \chi_1^2\,\right]' = \left[\,{}^{\mathcal{B}}q_{0,x} \;\; {}^{\mathcal{B}}q_{0,y} \;\; {}^{\mathcal{B}}q_{0,z} \;\; \chi_1^2\,\right]' \\ \boldsymbol{y} &:= \left[r_1^2,\; z_0\right]' \\ \boldsymbol{u} &:= \left[\,\boldsymbol{\nu}' \;\; \boldsymbol{\omega}' \;\; \phi \;\; \theta \;\; \psi \;\; {}^{\mathcal{B}}\boldsymbol{p}_1'\,\right]' \\ A(t) &:= \begin{bmatrix} 0 & \frac{\omega_3(t)}{\nu(t)} & -\frac{\omega_2(t)}{\nu(t)} & 0 \\ -\frac{\omega_3(t)}{\nu(t)} & 0 & \frac{\omega_1(t)}{\nu(t)} & 0 \\ \frac{\omega_2(t)}{\nu(t)} & -\frac{\omega_1(t)}{\nu(t)} & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \\ \boldsymbol{b}(t) &:= \left[\,-1 \;\; 0 \;\; 0 \;\; -2{}^{\mathcal{B}}p_{1,x}\,\right]' \\ C(t) &:= \begin{bmatrix} 0 & 0 & 0 & 1 \\ s\theta(t) & -s\phi(t)c\theta(t) & -c\phi(t)c\theta(t) & 0 \end{bmatrix} \end{aligned}$$

The following result holds.

Theorem 4. The dual system (16) has the same observability properties as the system (9), that is, Theorems 1-3 hold.

PROOF. Comparing (16) with (9) it can be seen that the state matrix A is the same and the output matrix C differs by a negative sign in the second row. Clearly this makes no difference in the observability analysis of the dual system and the results obtained for system (9) hold. ■

We are now ready to state the main result for the SLAM system (8) which extends to more than one beacon.

Theorem 5. Consider the SLAM system (8) with constant linear velocity $\boldsymbol{\nu} \neq 0$ and angular velocity $\boldsymbol{\omega}$. Suppose that there is an anchor, that is, the initial condition ${}^{\mathcal{B}}\boldsymbol{q}_0(t_0)$ or the position of one of the beacons ${}^{\mathcal{B}}\boldsymbol{p}_i(t_0)$ is known. Then the initial condition of other states can be reconstructed from the observed input and output pair $\{\boldsymbol{u}(t), \boldsymbol{y}(t)\}$, $t \in [t_0, t_f)$, provided that (13) holds. If (13) does not hold then the initial condition of each unknown vector in $\{{}^{\mathcal{B}}\boldsymbol{q}_0, {}^{\mathcal{B}}\boldsymbol{p}_i; i = 1, 2, ..., n\}$ has two possible solutions.

PROOF. For only one beacon the result follows from Theorems 1-4. Consider now more than one beacon. In this case it can be concluded from (8) that the dynamic equations of each pair $\{{}^{\mathcal{B}}\boldsymbol{p}_i, \chi_i\}$ does not depend on the other pairs. This means that the observability of the multiple beacon system can be investigated by analyzing the observability of each single beacon system and therefore the result follows from Theorems 1-4. ■

4. SIMULATION RESULTS

To illustrate the observability results derived in the previous section we have simulated two scenarios:

- A square type trajectory where the condition $\omega_3 = -\omega_2\tan(\phi)$ holds except when the AUV is turning;
- A helix type trajectory given by a constant pitch $\theta = (\frac{-\pi}{10})$ and turning rate $\boldsymbol{\omega} = [\tan(\frac{-\pi}{10})\frac{\pi}{50}, 0, \frac{\pi}{50}]'$, which means that conditions (11) and (13) hold.

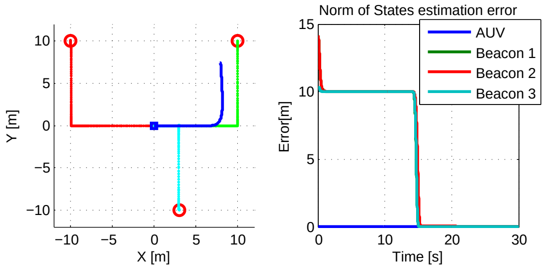

Fig. 2. Evolution of the estimated position of the AUV and the beacons (left), and the state estimation error (right) for the square type motion.

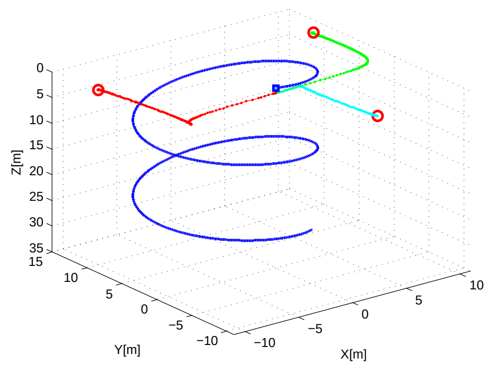

Fig. 3. Evolution of the estimated position of the AUV and the beacons for the helix type motion.

In both scenarios, we consider three stationary beacons as shown in Figures 2 and 3 with the symbol (o). The AUV is moving with a constant forward speed of $0.5[m/s]$ starting from initial condition (□). To estimate the states of the model introduced in (8) we use a continuous time constrained estimator (Aguiar and Hespanha, 2003) combined with a multiple model approach. The models in the estimator are initialized with the same initial condition but different up to a sign in initial condition ${}^{\mathcal{B}}p_{i,y}(t_0)$.

As expected for the helix type trajectory the system is observable and therefore the estimate converges to the true value. In Figure 4 it can be seen that the convergence time would decrease significantly when the depth measurement is used. For the square type trajectory there are two possible solutions for each beacon, as long as the AUV is moving on a straight line. As soon as the AUV turns, the observability condition $\omega_3 \neq -\omega_2 \tan(\phi)$ holds and the set of possible solutions for the initial condition will shrink to the true value.

5. CONCLUSIONS

We addressed the observability problem of SLAM for an AUV equipped with inertial sensors, depth sensor, and an acoustic ranging device to obtain relative range measurements to stationary beacons. We provided conditions under which it is possible to reconstruct the initial state of the resulting SLAM system (and in particular the position of the AUV). From the derived results it can be concluded that the observability and more precisely the unobservable subspace $\mathcal{UO}$ is independent of the location of the beacons and it only depends on the motion of the AUV.

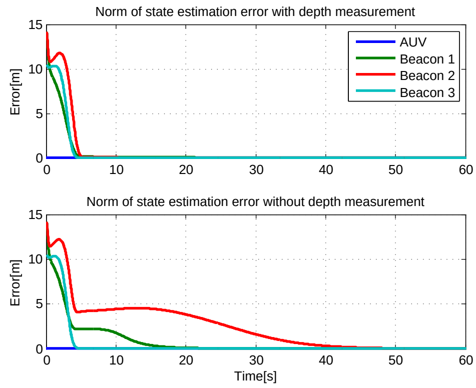

Fig. 4. Evolution of the position estimation error for helix type motion with (top) and without (bottom) depth measurement.

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