Low SNR analysis of the non-coherent MIMO channel under arbitrary channel and noise correlation structures

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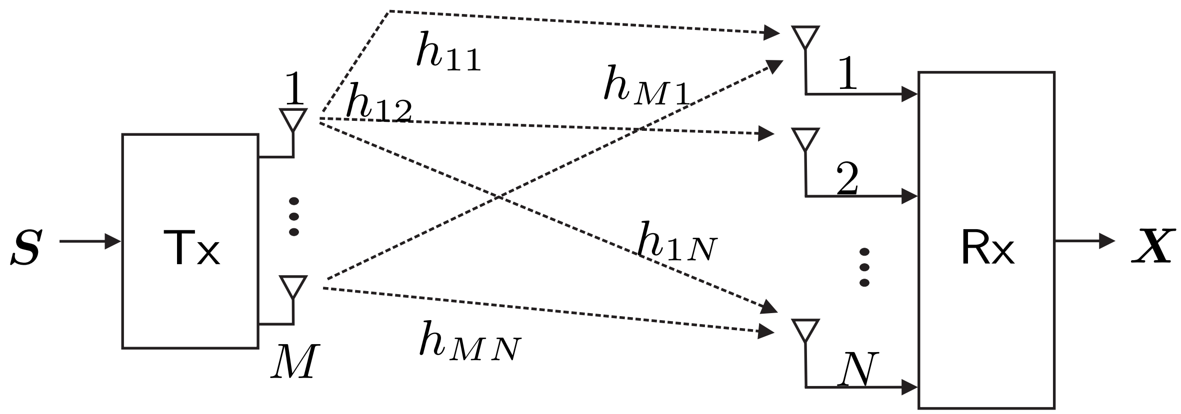
Problem Formulation

▷ Data model: $\boldsymbol{X} = \boldsymbol{S}\boldsymbol{H} + \boldsymbol{E}$



▷ $\boldsymbol{X}, \boldsymbol{E}$: $T \times N$, $\boldsymbol{S}$: $T \times M$, $\boldsymbol{H}$: $M \times N$

▷ Contribution: mutual information analysis for on-off and Gaussian signaling when $\boldsymbol{H} = \sqrt{\frac{\rho}{M}}\boldsymbol{K}^{\frac{1}{2}}\boldsymbol{H}_w$ and vec$\left(\boldsymbol{E}\right) \sim \mathcal{CN}\left(\mathbf{0}, \boldsymbol{\Upsilon}\right)$ (colored noise)

Mutual information: on-off signaling

▷ The on-off signaling: for any $\epsilon > 1$,

$$\boldsymbol{S} = \boldsymbol{S}_{on}\rho^{-\frac{\epsilon}{2}} \text{ w.p. } \rho^{\epsilon}; \qquad\qquad \boldsymbol{S} = \boldsymbol{0} \text{ w.p. } 1 - \rho^{\epsilon}$$

▷ At sufficiently low SNR

$$I(\boldsymbol{X};\boldsymbol{S}) = \frac{\rho}{M} \operatorname{tr}\left(\boldsymbol{\Upsilon}^{-1}\left(\boldsymbol{I}_N \otimes \boldsymbol{S}_{on}\right)\boldsymbol{K}\left(\boldsymbol{I}_N \otimes \boldsymbol{S}_{on}^H\right)\right) + o(\rho), \tag{1}$$

Mutual information: on-off signaling

▷ We maximize $I(\boldsymbol{X};\boldsymbol{S})$ in (1) w.r.t $\boldsymbol{S}_{on}$ and $\boldsymbol{K}$

$$\max_{\substack{\mathrm{tr}\left(\boldsymbol{S}_{on}\boldsymbol{S}_{on}^H\right) \leq TM \\ \boldsymbol{K} \in \mathcal{P}_{MN}}} \mathrm{tr}\left(\boldsymbol{\Upsilon}^{-1}\overline{\boldsymbol{S}}_{on}\boldsymbol{K}\overline{\boldsymbol{S}}_{on}^H\right) \tag{2}$$

where $\mathcal{P}_n = \{\boldsymbol{Y} : n \times n$ matrix such that $\boldsymbol{Y} \succeq \boldsymbol{0}$, $\mathrm{tr}\left(\boldsymbol{Y}\right) = n\}$ and $\overline{\boldsymbol{S}}_{on} = \boldsymbol{I}_N \otimes \boldsymbol{S}_{on}$.

▷ The maximum in (1) is attained by

$$\widehat{\boldsymbol{S}}_{on} = \mathsf{ivec}(\widehat{\boldsymbol{s}}_{on}),\ \widehat{\boldsymbol{K}} = MN\widehat{\boldsymbol{u}}\widehat{\boldsymbol{u}}^H, \tag{3}$$

where

$$(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{s}}_{on}) = \underset{\substack{||\boldsymbol{u}||=1,\\ ||\boldsymbol{s}_{on}|| \leq \sqrt{TM}}}{\arg\max} \boldsymbol{s}_{on}^H \boldsymbol{M}_{on} \boldsymbol{s}_{on} \tag{4}$$

with $\boldsymbol{M}_{on} = \boldsymbol{K}_{on}^H \left(\left(\boldsymbol{u}\boldsymbol{u}^H\right)^T \otimes \boldsymbol{\Upsilon}^{-1} \right) \boldsymbol{K}_{on}$, $\boldsymbol{u} \in \mathbb{C}^{MN}$ and $\boldsymbol{s}_{on} = \mathsf{vec}\left(\boldsymbol{S}_{on}\right) \in \mathbb{C}^{TM}$. $TMN^2 \times TM$ matrix $\boldsymbol{K}_{on}$ is such that

$$\mathsf{vec}\left(\boldsymbol{I}_N \otimes \boldsymbol{S}_{on}\right) = \boldsymbol{K}_{on} \mathsf{vec}\left(\boldsymbol{S}_{on}\right)$$

Mutual information: on-off signaling

- ▷ The optimization problem in (4) always admits a solution (maximization of a continuous function over a compact set)
- ▷ For the choice in (3), the maximal mutual information (p.c.u) is equal to

$$\frac{1}{T} I(\boldsymbol{X};\boldsymbol{S}) = \rho\, N\, M \hat{\lambda} + o(\rho).$$

where

$$\hat{\lambda} = \underline{\widehat{\boldsymbol{s}}}^H \boldsymbol{K}_{on}^H \left(\left(\widehat{\boldsymbol{u}} \widehat{\boldsymbol{u}}^H \right)^T \otimes \boldsymbol{\Upsilon}^{-1} \right) \boldsymbol{K}_{on} \underline{\widehat{\boldsymbol{s}}} \qquad (5)$$

and $\underline{\widehat{\boldsymbol{s}}} = 1/\sqrt{TM}\, \widehat{\boldsymbol{s}}_{on}$.

Mutual information: on-off signaling

Fixed eigenvectors of $\boldsymbol{K}$

▷ In practice, by changing the antenna separation one can control the eigenvalues of $\boldsymbol{K}$, but not their eigenvectors.

▷ In this case, the optimization problem defined in (2) becomes

$$\left(\widehat{\boldsymbol{S}}_{on}, \widehat{\boldsymbol{\Lambda}}\right) = \underset{\substack{||\boldsymbol{S}_{on}|| \leq \sqrt{TM} \\ \boldsymbol{\Lambda} \in \mathcal{D}_{MN}}}{\arg\max} \ \mathrm{tr}\left(\boldsymbol{\Upsilon}^{-1}\overline{\boldsymbol{S}}_{on}\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{H}\overline{\boldsymbol{S}}_{on}^{H}\right) \tag{6}$$

where $\mathcal{D}_n = \{\boldsymbol{Y} : n \times n$ diagonal matrix such that $\boldsymbol{Y} \succeq \mathbf{0}$ and $\mathrm{tr}\,(\boldsymbol{Y}) = n\}$, and $\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{H}$ is the EVD of $\boldsymbol{K}$.

Mutual information: on-off signaling

Fixed eigenvectors of $\boldsymbol{K}$

▷ The maximum in (6) is attained by

$$\widehat{\boldsymbol{S}}_{on} = \sqrt{TM}\,\text{ivec}(\boldsymbol{s}_{\text{max}}),\ \widehat{\boldsymbol{K}} = MN\boldsymbol{u}_{i^*}\boldsymbol{u}_{i^*}^H, \tag{7}$$

where $\boldsymbol{u}_i$'s are the eigenvectors of $\boldsymbol{K}$, $\boldsymbol{s}_{\text{max}}$ is an unit-norm eigenvector associated to the $\lambda_{\text{max}}\left(\boldsymbol{M}_{on}^{i^*}\right)$ with

$$\boldsymbol{M}_{on}^{i^*} = \boldsymbol{K}_{on}^H\left(\left(\boldsymbol{u}_{i^*}\boldsymbol{u}_{i^*}^H\right)^T \otimes \boldsymbol{\Upsilon}^{-1}\right)\boldsymbol{K}_{on} \tag{8}$$

and

$$i^* = \underset{i=1,2,...,MN}{\arg\max}\ \lambda_{\text{max}}\left(\boldsymbol{M}_{on}^{i}\right). \tag{9}$$

Mutual information: on-off signaling

Fixed eigenvectors of $\boldsymbol{K}$

- For the choice in (7), the maximal mutual information (p.c.u.) is given by

$$\frac{1}{T} I(\boldsymbol{X}; \boldsymbol{S}) = \rho \, N \, M \lambda_{\mathsf{max}} \left(\boldsymbol{M}_{on}^{i^*} \right) + o(\rho).$$

Mutual information: on-off signaling

▷ Conclusions:

- From (3) and (7) we see that $\boldsymbol{K}$ should be of rank one
- Correlated Rayleigh fading channel is beneficial from capacity viewpoint. Gain of order M with respect to uncorrelated Rayleigh fading channel
- On-off signaling attains the known channel capacity
- Correlation in noise is beneficial too, $\hat{\lambda} \geq 1$

Mutual information: Gaussian modulation

- ▷ On-off signaling is unpracticable due to large peakiness of the input signal
- ▷ Let $\boldsymbol{s} = \text{vec}(\boldsymbol{S}) \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{P})$. At sufficiently low SNR

$$I(\boldsymbol{X}; \boldsymbol{S}) = \frac{\rho^2}{2M^2} \operatorname{tr}\left(\mathsf{E}[\boldsymbol{Z}^2] - (\mathsf{E}[\boldsymbol{Z}])^2\right) + o(\rho^2) \tag{10}$$

where $\boldsymbol{Z} = \boldsymbol{\Upsilon}^{-\frac{1}{2}} \overline{\boldsymbol{S}}_{on} \boldsymbol{K} \overline{\boldsymbol{S}}_{on}^{H} \boldsymbol{\Upsilon}^{-\frac{1}{2}}$

Mutual information: Gaussian modulation

▷ We maximize $I(\boldsymbol{X};\boldsymbol{S})$ in (10) w.r.t $\boldsymbol{P}$ and $\boldsymbol{K}$

$$\max_{\boldsymbol{P} \in \mathcal{H}_{TM}, \boldsymbol{K} \in \mathcal{P}_{MN}} \quad \mathsf{tr}\left(\mathsf{E}[\boldsymbol{Z}^2] - (\mathsf{E}[\boldsymbol{Z}])^2\right) \tag{11}$$

where $\mathcal{H}_n = \{\boldsymbol{P} : n \times n$ matrices such that $\boldsymbol{P} = \boldsymbol{P}^H \succeq \mathbf{0}$ and $\mathsf{tr}\,(\boldsymbol{P}) \leq n\}$.

Mutual information: Gaussian modulation

▷ The maximum in (11) is attained by

$$\widehat{\boldsymbol{P}} = \widehat{\boldsymbol{s}}_{on}\widehat{\boldsymbol{s}}_{on}^H, \ \widehat{\boldsymbol{K}} = MN\widehat{\boldsymbol{u}}\widehat{\boldsymbol{u}}^H, \tag{12}$$

where

$$(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{s}}_{on}) = \underset{\substack{||\boldsymbol{u}|| = 1, \\ ||\boldsymbol{s}_{on}|| \leq \sqrt{TM}}}{\arg\max} \ \boldsymbol{s}_{on}^H \boldsymbol{M}_{on} \boldsymbol{s}_{on} \tag{13}$$

Mutual information: Gaussian modulation

▷ For the choice in (12), the maximal mutual information (p.c.u) is equal to

$$\frac{1}{T} I(\boldsymbol{X}; \boldsymbol{S}) = \frac{\rho^2}{2} \, N^2 \, T \, M^2 \, \hat{\lambda}^2 + o(\rho^2)$$

where

$$\hat{\lambda} = \underline{\widehat{\boldsymbol{s}}}^H \boldsymbol{K}_{on}^H \left(\left(\widehat{\boldsymbol{u}} \widehat{\boldsymbol{u}}^H \right)^T \otimes \boldsymbol{\Upsilon}^{-1} \right) \boldsymbol{K}_{on} \underline{\widehat{\boldsymbol{s}}}$$

and $\underline{\widehat{\boldsymbol{s}}} = 1/\sqrt{TM} \, \widehat{\boldsymbol{s}}_{on}$.

Mutual information: Gaussian modulation

Fixed eigenvectors of $\boldsymbol{K}$

▷ When the eigenvalues of $\boldsymbol{K}$ are fixed then the maximum of (11) is attained by

$$\widehat{\boldsymbol{P}} = TM\, \boldsymbol{s}_{\mathsf{max}} \boldsymbol{s}_{\mathsf{max}}^H,\ \widehat{\boldsymbol{K}} = MN \boldsymbol{u}_{i^*} \boldsymbol{u}_{i^*}^H, \tag{14}$$

where $\boldsymbol{u}_i$'s are the eigenvectors of $\boldsymbol{K}$, $\boldsymbol{s}_{\mathsf{max}}$ is an unit-norm eigenvector associated to the $\lambda_{\mathsf{max}}\left(\boldsymbol{M}_{on}^{i^*}\right)$ with

$$\boldsymbol{M}_{on}^{i^*} = \boldsymbol{K}_{on}^H \left(\left(\boldsymbol{u}_{i^*} \boldsymbol{u}_{i^*}^H\right)^T \otimes \boldsymbol{\Upsilon}^{-1}\right) \boldsymbol{K}_{on} \tag{15}$$

and

$$i^* = \underset{i=1,2,\ldots,MN}{\arg\max}\ \lambda_{\mathsf{max}}\left(\boldsymbol{M}_{on}^{i}\right). \tag{16}$$

Mutual information: Gaussian modulation

Fixed eigenvectors of K

▷ For the choice in (14), the maximal mutual information (p.c.u.) is given by

$$\frac{1}{T} I(\boldsymbol{X}; \boldsymbol{S}) = \frac{\rho^2}{2} N^2 T M^2 \lambda^2_{\mathsf{max}} \left(\boldsymbol{M}^{i^*}_{on} \right) + o(\rho^2).$$

Mutual information: Gaussian modulation

▷ Conclusions:

- From (12) and (14) we see that $\boldsymbol{K}$ should be of rank one
- Correlated Rayleigh fading channel is beneficial from capacity viewpoint. Gain of order M^2N with respect to uncorrelated Rayleigh fading channel
- Correlation in noise is beneficial too, $\hat{\lambda} \geq 1$

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