Codebook design for the non-coherent GLRT receiver and low SNR MIMO block fading channel

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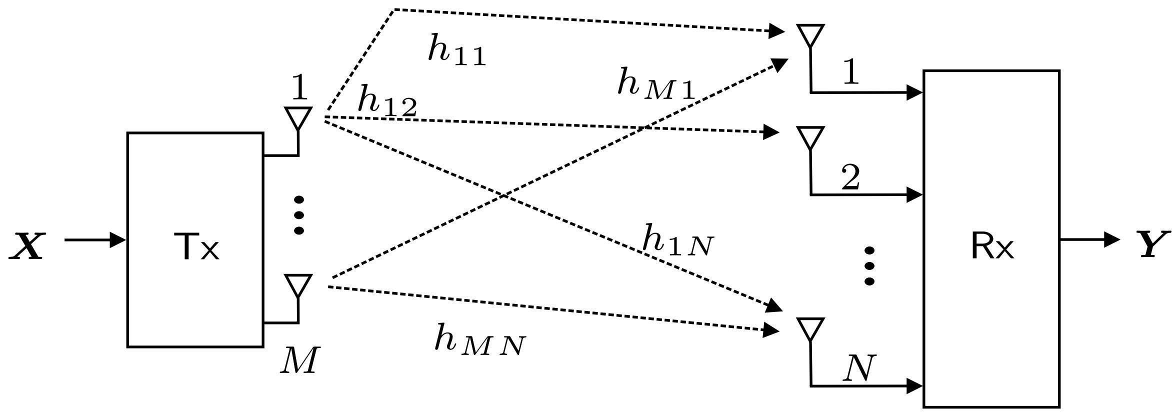
Problem Formulation

▷ Data model: $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{H}^H + \boldsymbol{E}$



▷ $\boldsymbol{Y}, \boldsymbol{E}$: $T \times N$, $\boldsymbol{x}$: $T \times 1$, $\boldsymbol{h}$: $N \times 1$ for $M = 1$

▷ Codebook : $\mathcal{X} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_K\}$ is a point in the manifold

$$\mathcal{M} = \{(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K) : \boldsymbol{x}_k^H \boldsymbol{x}_k = 1\}$$

▷ Contribution: PEP analysis and codebook design for $M = 1$ in low SNR regime when $\boldsymbol{H}$ is deterministic and unknown

Problem Formulation

▷ GLRT receiver:

$$\widehat{k} = \underset{k=1,2,\ldots,K}{\text{argmax}} \; p(\boldsymbol{y}|\boldsymbol{x}_k, \widehat{\boldsymbol{g}}_k) = \underset{k=1,2,\ldots,K}{\text{argmin}} \; ||\boldsymbol{y} - \widetilde{\boldsymbol{X}_k}\widehat{\boldsymbol{g}}_k||$$

$\widetilde{\boldsymbol{X}_k} = \boldsymbol{I}_N \otimes \boldsymbol{x}_k$, $\widehat{\boldsymbol{g}}_k = \widetilde{\boldsymbol{X}_k}^H \boldsymbol{y}$ (ML channel estimate), $\boldsymbol{y} = \text{vec}(\boldsymbol{Y})$

▷ PEP analysis: it can be shown that at sufficiently low SNR

$$P_{\boldsymbol{x}_i \to \boldsymbol{x}_j} = P\left(\sum_{n=1}^{N} \left(|a_n|^2 - |b_n|^2\right) > ||\boldsymbol{h}||^2 \sin \alpha_{ij}\right) \tag{1}$$

where $a_n, b_n \overset{iid}{\sim} \mathcal{CN}(0,1)$ and the angle α_{ij} is the acute angle between the codewords $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$

Problem Formulation

- ▷ In our work [5] the expression for the PEP in the high SNR regime and $M = 1$ is given by

$$P_{\boldsymbol{x}_i \to \boldsymbol{x}_j} = \mathcal{Q}\left(\frac{1}{\sqrt{2}}||\boldsymbol{h}|| \sin \alpha_{ij}\right) \tag{2}$$

  where $\mathcal{Q}(x) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt$

- ▷ Equations (1)-(2) confirm that the codewords $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ should be constructed as separate as possible

- ▷ The problem of constructing good codes corresponds to the very well known packing problem in the complex projective space [4]

Problem Formulation

▷ Optimization problem: result (1) suggests the codebook design problem

$$\mathcal{X}^* = \underset{\mathcal{X} \in \mathcal{M}}{\arg\max} \ \min\{\boldsymbol{L}_{ij}(\mathcal{X}) : 1 \leq i \neq j \leq K\} \tag{3}$$

where $\boldsymbol{L}_{ij}(\mathcal{X}) = \boldsymbol{x}_i^H \boldsymbol{\Pi}_j^{\perp} \boldsymbol{x}_i = \sin^2 \alpha_{ij}$ with $\boldsymbol{\Pi}_j^{\perp} = \boldsymbol{I}_T - \underbrace{\boldsymbol{x}_j \boldsymbol{x}_j^H}_{\boldsymbol{\Pi}_j}$

▷ The problem in (3) is a high-dimensional, non-linear and non-smooth optimization problem

Example: for $K = 256$, $T = 8$, $M = 1$: $K(K-1) = 65280$ $\boldsymbol{L}_{ij}(\mathcal{X})$ functions and $2KTM = 4096$ real variables to optimize

Codebook Construction

- ▷ Two-phase methodology to tackle the optimization problem in (3) (see [5]-[6])
- ▷ Phase I: solves a convex semi-definite programming (SDP) relaxation
- ▷ Incremental approach: Let $\mathcal{X}^*_{k-1} = \{\boldsymbol{x}^*_1, ..., \boldsymbol{x}^*_{k-1}\}$ be the codebook at the $k-1^{\underline{th}}$ stage. The new codeword is found by solving

$$\boldsymbol{x}^*_k = \underset{\boldsymbol{x}_k^H \boldsymbol{x}_k = 1}{\arg\max} \; \min_{1 \le i \le k-1} \{\lambda_{\min}(\boldsymbol{L}_{ik}),\, \lambda_{\min}(\boldsymbol{L}_{ki})\} \tag{4}$$

for $k = 2, ..., K$

Codebook Construction - Phase I

▷ The optimization problem (4) is equivalent to (see [6])

$$(\widehat{\boldsymbol{Y}}^*, \widetilde{\boldsymbol{X}}^*, t^*) = \arg\max \quad t \tag{5}$$

with the following constraints

$$\begin{bmatrix} \mathrm{tr}(\boldsymbol{N}_i \boldsymbol{A}_1 \widehat{\boldsymbol{Y}} \boldsymbol{B}_1) - t & \cdots & \mathrm{tr}(\boldsymbol{N}_i \boldsymbol{A}_{MN} \widehat{\boldsymbol{Y}} \boldsymbol{B}_1) \\ \vdots & & \vdots \\ \mathrm{tr}(\boldsymbol{N}_i \boldsymbol{A}_1 \widehat{\boldsymbol{Y}} \boldsymbol{B}_{MN}) & \cdots & \mathrm{tr}(\boldsymbol{N}_i \boldsymbol{A}_{MN} \widehat{\boldsymbol{Y}} \boldsymbol{B}_{MN}) - t \end{bmatrix} \succeq \mathbf{0},$$

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{Z}_i \\ \boldsymbol{Z}_i^H & \boldsymbol{P}_i - t\boldsymbol{I}_{MN} \end{bmatrix} \succeq \mathbf{0} \, \forall_{1 \le i \le k-1}, \boldsymbol{K} \widehat{\boldsymbol{Y}} \boldsymbol{K}^H = \widetilde{\boldsymbol{X}}, \mathrm{tr}(\widetilde{\boldsymbol{X}}) = 1,$$

$$\boldsymbol{f} \, \widehat{\boldsymbol{Y}} \, \boldsymbol{f}^H = 1, \widehat{\boldsymbol{Y}} = \widehat{\boldsymbol{Y}}^H, \widehat{\boldsymbol{Y}} \succeq \mathbf{0}, \mathrm{rank}(\widehat{\boldsymbol{Y}}) = 1$$

and $\widetilde{\boldsymbol{X}} = \boldsymbol{x}_k \boldsymbol{x}_k^H, b^2 = 1$, $\widehat{\boldsymbol{Y}} = \boldsymbol{z}\boldsymbol{z}^H$, $\boldsymbol{z} = \left[\mathrm{vec}^T(\widetilde{\boldsymbol{X}_k}) \; b\right]^T$,
$\widetilde{\boldsymbol{X}_k} = \boldsymbol{I}_N \otimes \boldsymbol{x}_k$.

▷ The matrices $\boldsymbol{M}$, $\boldsymbol{Z}_i$ — linear in $\widehat{\boldsymbol{Y}}$

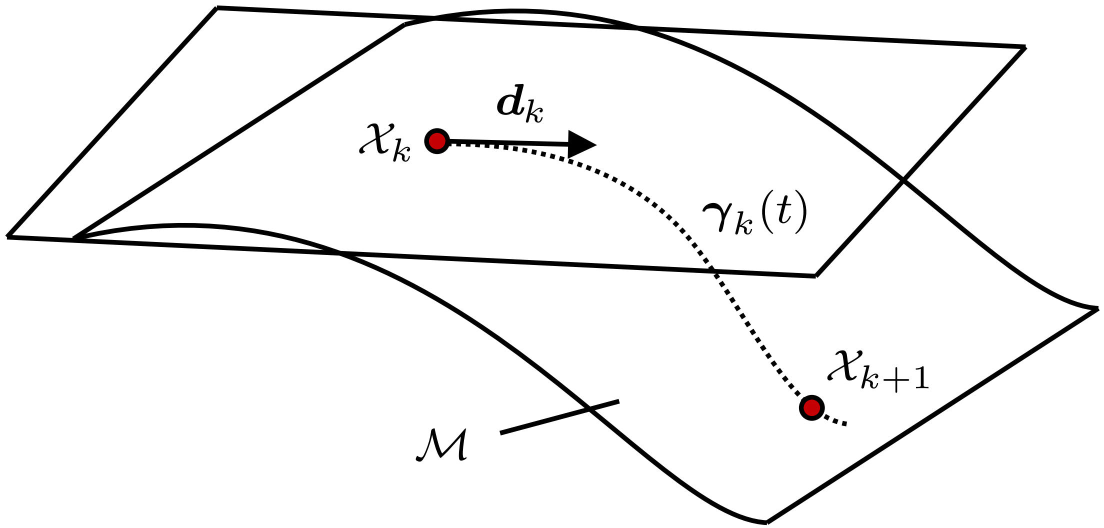
▷ The matrices $\boldsymbol{N}_i$, $\boldsymbol{P}_i$, $\boldsymbol{K}$, $\boldsymbol{f}$, $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ — constants

Codebook Construction - Phase 1

- ▷ Design of the codewords: high-dimensional difficult nonlinear optimization problem (rank condition in (5))
- ▷ Relaxing the rank constraint leads to an SDP [7]
- ▷ The $k^{\underline{th}}$ codeword is extracted from the output variable $\widetilde{\boldsymbol{X}}$ with a technique similar to [8]
- ▷ Initialization $\boldsymbol{x}_1^*$: randomly generated

Codebook Construction - Phase 2

▷ Phase II: optimizes a non-smooth function on a manifold



Codebook Construction - Phase 2

- ▷ Iterative algorithm, called GDA (geodesic descent algorithm)
- ▷ Identify "active" pairs (i, j) that attain minimum in (3)
- ▷ Check if there is an ascent direction $d_k \in T_{\mathcal{X}_k}\mathcal{M}$ for all active (i, j) (consists of solving LP)
- ▷ When d_k is found, perform Armijo rule along geodesic $\boldsymbol{\gamma}_k(t)$
- ▷ If no d_k is found, the algorithm stops

Computer Simulations

☐ **Example: Constellations with equal priors**

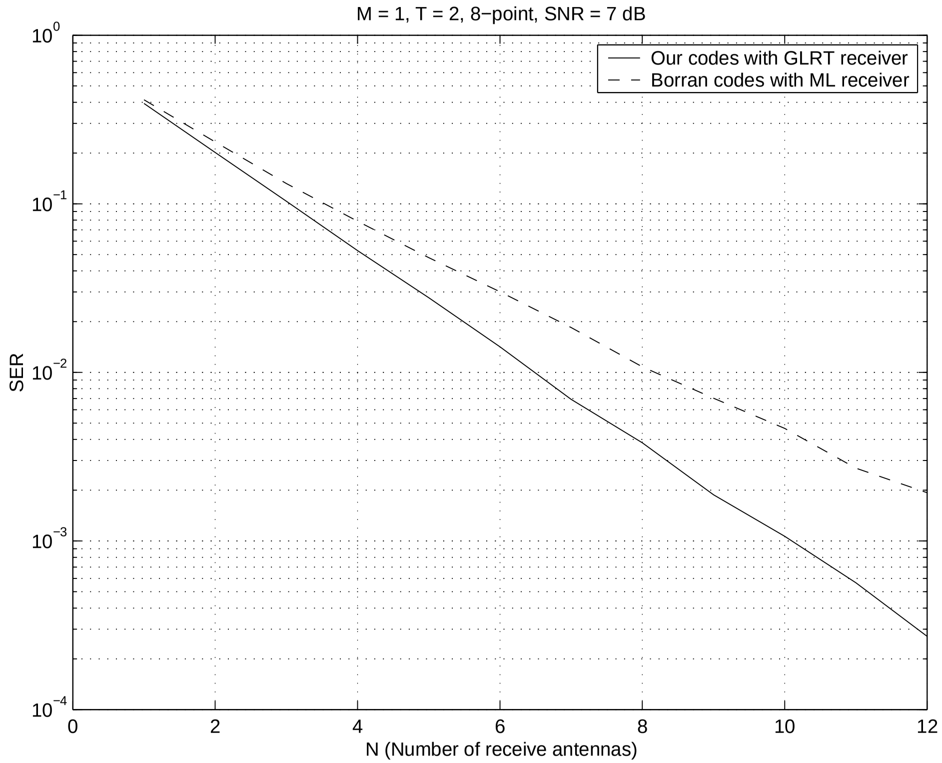


Figure 1: M=1, T=2, K=8, SNR = 7 dB. Solid curve:our codes with our GLRT receiver. Dashed curve:Borran codes designed for SNR = 7dB with ML receiver [1].

□ Example: Constellations with equal priors

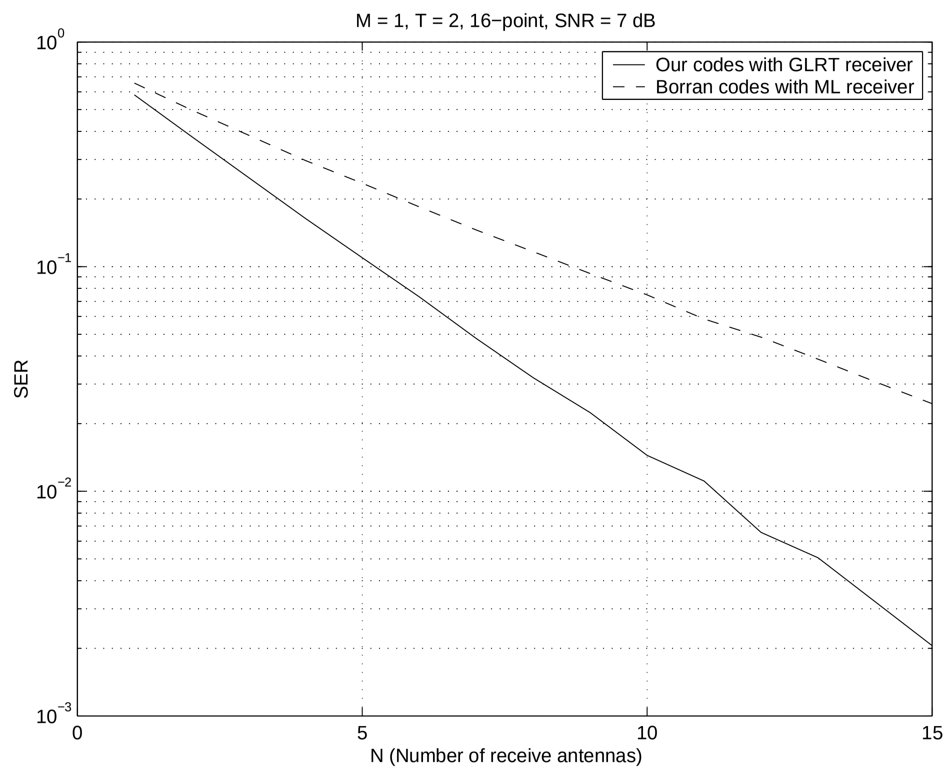


Figure 2: M=1, T=2, K=16, SNR = 7 dB. Solid curve:our codes with our GLRT receiver. Dashed curve:Borran codes designed for SNR = 7dB with ML receiver [1].

☐ Example: Constellations with unequal priors

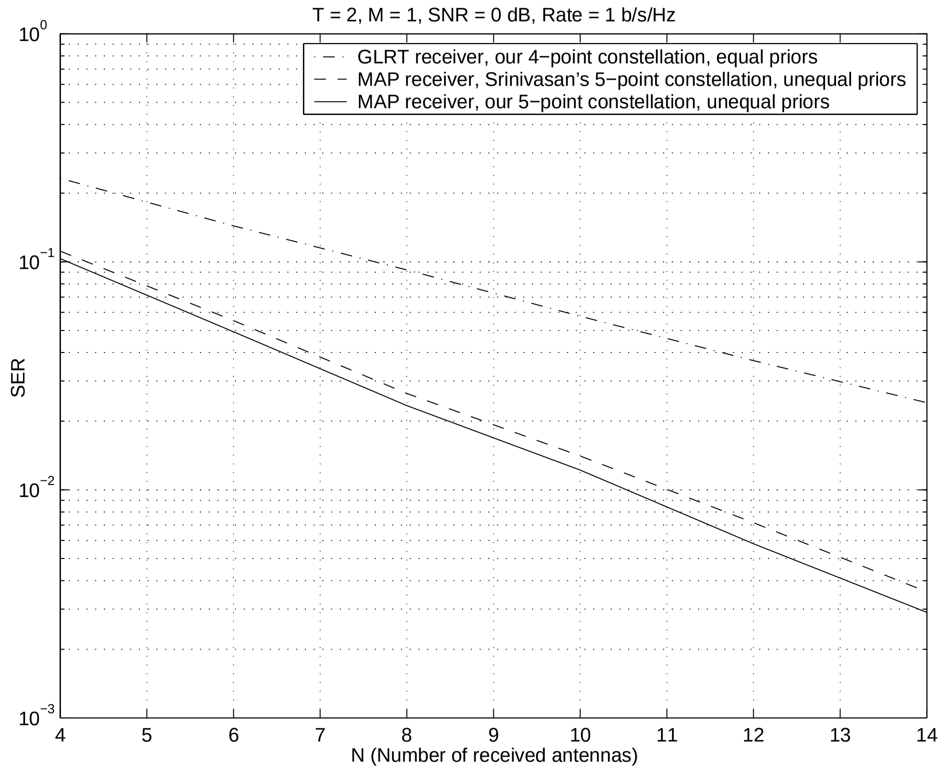


Figure 3: $T=2$, $M=1$, SNR $= 0$ dB, Rate $= 1$ b/s/Hz. Solid curve-our 5 point constellation with unequal priors, dashed curve-Srinivasan's 5 point constellation with unequal priors [2], dash-dotted curve-our 4 point constellation with equal priors. Our and Srinivasan's 5 point constellations use *maximum a-posteriori* (MAP) receiver, our 4 point constellation uses GLRT receiver.

□ Example: Constellations with equal priors and $M \geq 1$

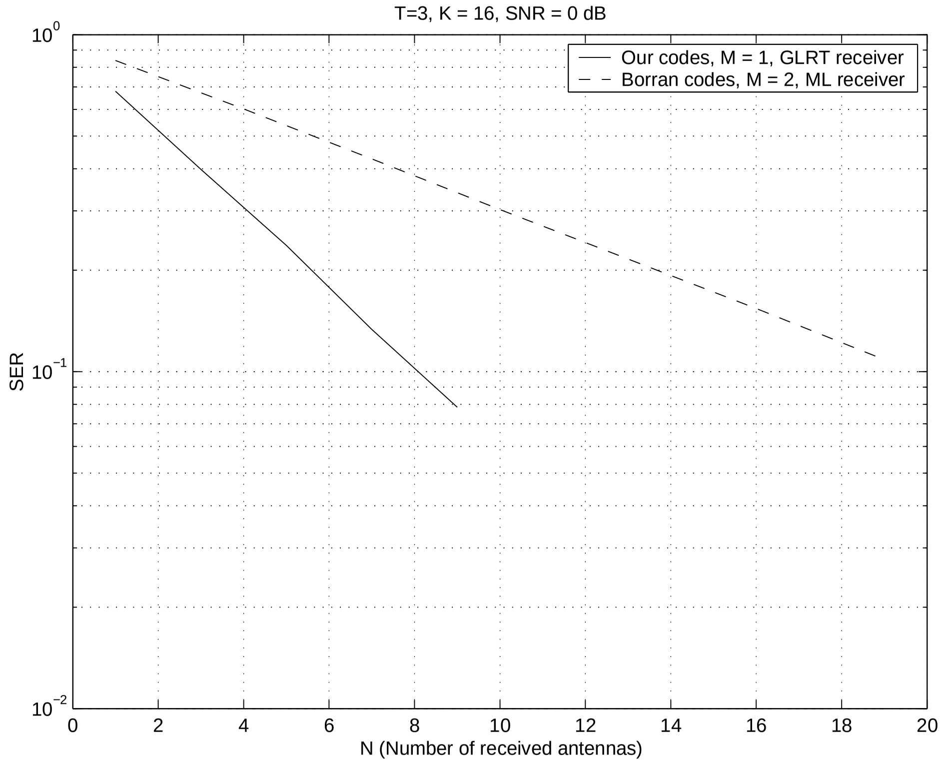


Figure 4: Solid curve-our codes for $K = 16$, $T = 3$, $M = 1$, dashed curve-Borran codes for $K = 16$, $T = 3$, $M = 2$.

☐ **Example: Constellations with equal priors and $M \geq 1$**

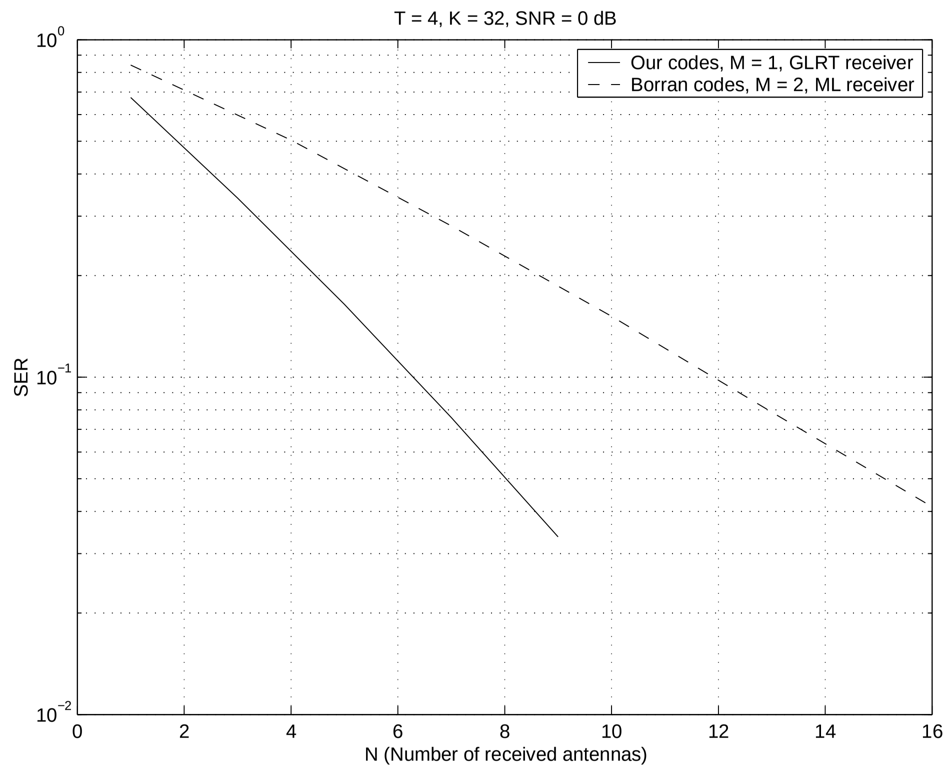


Figure 5: Solid curve-our codes for $K = 32$, $T = 4$, $M = 1$, dashed curve-Borran codes for $K = 32$, $T = 4$, $M = 2$.

☐ **Example: Constellations with equal priors and** $M \geq 1$

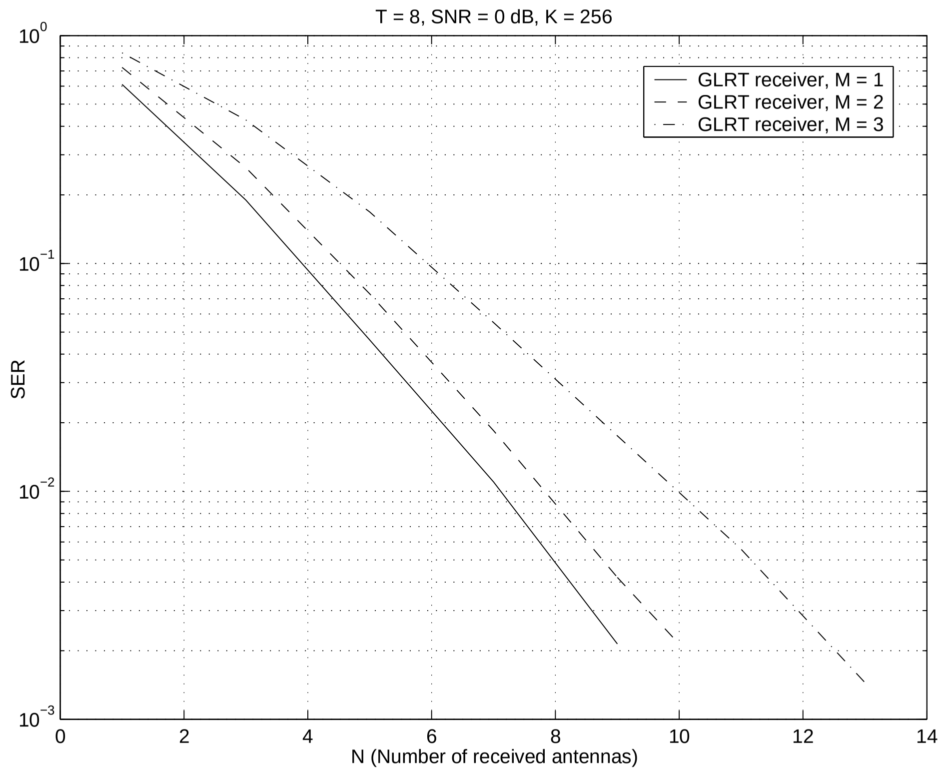


Figure 6: T=8, K=256, SNR = 0 dB. Solid curve-our codes for M = 1, dashed curve-our codes for M = 2, dash-dotted curve-our codes for M = 3. All codes use GLRT receiver.

Conclusions

- ▷ PEP analysis and codebook design for $M = 1$ in low SNR regime when $\boldsymbol{H}$ is deterministic and unknown

- ▷ Results
  - outperform significantly state-of-art known solutions which assume equal prior probabilities
  - also of interest for the constellations with unequal priors

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