Capacity and error probability analysis of non-coherent MIMO systems in the low SNR regime

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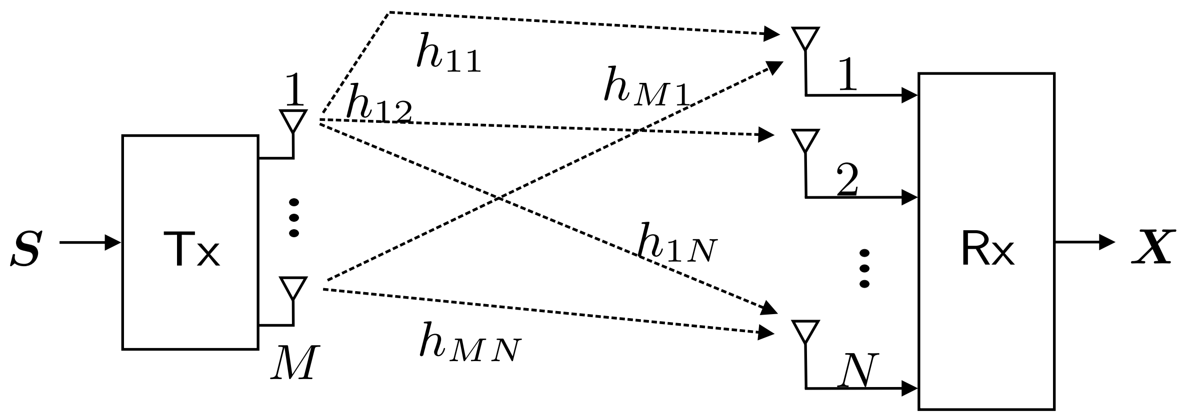
Part 1: Low SNR regime – random fading channel

Problem Formulation

- Data model: $\boldsymbol{X} = \boldsymbol{S}\boldsymbol{H} + \boldsymbol{E}$



- $\boldsymbol{X}, \boldsymbol{E}$: $T \times N$, $\boldsymbol{S}$: $T \times M$, $\boldsymbol{H}$: $M \times N$
- Contribution: mutual information analysis for on-off and Gaussian signaling when $\boldsymbol{H} = \sqrt{\frac{\rho}{M}} \boldsymbol{K}_t^{\frac{1}{2}} \boldsymbol{H}_w \boldsymbol{K}_r^{\frac{1}{2}}$ and $\text{vec}(\boldsymbol{E}) \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{\Upsilon})$ (colored noise)

Mutual information: on-off signaling

- The on-off signaling: for any $\epsilon > 1$, $\boldsymbol{S} = \boldsymbol{S}_{on}\rho^{-\frac{\epsilon}{2}}$ w.p. ρ^{ϵ}; $\boldsymbol{S} = \mathbf{0}$ w.p. $1 - \rho^{\epsilon}$

- At sufficiently low SNR

$$I(\boldsymbol{X};\boldsymbol{S}) = \frac{\rho}{M}\,\mathrm{tr}\left(\boldsymbol{\Upsilon}^{-1}\left(\boldsymbol{K}_r \otimes \boldsymbol{S}_{on}\boldsymbol{K}_t\boldsymbol{S}_{on}^H\right)\right) + o(\rho), \tag{1}$$

- We maximize $I(\boldsymbol{X};\boldsymbol{S})$ in (1) w.r.t $\boldsymbol{S}_{on}$, $\boldsymbol{K}_t$ and $\boldsymbol{K}_r$

- The maximum in (1) is attained by

$$\widehat{\boldsymbol{S}}_{on} = \sqrt{TM}\begin{bmatrix}\hat{\boldsymbol{s}} & \mathbf{0}_{T\times(M-1)}\end{bmatrix},\ \widehat{\boldsymbol{K}}_r = N\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}^H,\ \widehat{\boldsymbol{K}}_t(i,i) = M\delta_{i1} \tag{2}$$

where

$$(\hat{\boldsymbol{u}}, \hat{\boldsymbol{s}}) = \underset{\substack{\boldsymbol{u} \in \mathbb{C}^N, ||\boldsymbol{u}|| = 1 \\ \boldsymbol{s} \in \mathbb{C}^T, ||\boldsymbol{s}|| = 1}}{\arg\max} (\boldsymbol{u} \otimes \boldsymbol{s})^H \, \boldsymbol{\Upsilon}^{-1} (\boldsymbol{u} \otimes \boldsymbol{s}) \tag{3}$$

Mutual information: on-off signaling

- ▷ The optimization problem in (3) always admits a solution (maximization of a continuous function over a compact set)
- ▷ For the choice in (2), the maximal mutual information (p.c.u) is equal to

$$\frac{1}{T} I(\boldsymbol{X};\boldsymbol{S}) = \rho\, N\, M \hat{\lambda} + o(\rho).$$

where $\hat{\lambda} = (\hat{\boldsymbol{u}} \otimes \hat{\boldsymbol{s}})^H\, \boldsymbol{\Upsilon}^{-1}\, (\hat{\boldsymbol{u}} \otimes \hat{\boldsymbol{s}})$

▷ Conclusions:

- From (2) we see that both $\boldsymbol{K}_t$ and $\boldsymbol{K}_r$ should be of rank one
- Correlated Rayleigh fading channel is beneficial from capacity viewpoint. Gain of order M with respect to uncorrelated Rayleigh fading channel
- On-off signaling attains the known channel capacity
- Correlation in noise is beneficial too, $\hat{\lambda} \geq 1$

Mutual information: Gaussian modulation

▷ On-off signaling is unpracticable due to large peakiness of the input signal

▷ Let $\boldsymbol{s} = \text{vec}(\boldsymbol{S}) \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{P})$. At sufficiently low SNR

$$I(\boldsymbol{X};\boldsymbol{S}) = \frac{\rho^2}{2M^2}\,\text{tr}\left(\mathsf{E}[\boldsymbol{Z}^2] - (\mathsf{E}[\boldsymbol{Z}])^2\right) + o(\rho^2) \tag{4}$$

where $\boldsymbol{Z} = \boldsymbol{\Upsilon}^{-\frac{1}{2}}\left(\boldsymbol{K}_r \otimes \boldsymbol{S}\boldsymbol{K}_t\boldsymbol{S}^H\right)\boldsymbol{\Upsilon}^{-\frac{1}{2}}$

▷ We maximize $I(\boldsymbol{X};\boldsymbol{S})$ in (4) w.r.t $\boldsymbol{P}$, $\boldsymbol{K}_t$ and $\boldsymbol{K}_r$

▷ The maximum in (4) is attained by

$$\widehat{\boldsymbol{P}} = TM\boldsymbol{F}_1 \otimes \hat{\boldsymbol{s}}\hat{\boldsymbol{s}}^H,\ \widehat{\boldsymbol{K}}_r = N\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}^H,\ \widehat{\boldsymbol{K}}_t(i,i) = M\delta_{i1} \tag{5}$$

where

$$(\hat{\boldsymbol{u}}, \hat{\boldsymbol{s}}) = \underset{\substack{\boldsymbol{u} \in \mathbb{C}^N, ||\boldsymbol{u}|| = 1 \\ \boldsymbol{s} \in \mathbb{C}^T, ||\boldsymbol{s}|| = 1}}{\arg\max} (\boldsymbol{u} \otimes \boldsymbol{s})^H \boldsymbol{\Upsilon}^{-1} (\boldsymbol{u} \otimes \boldsymbol{s})$$

Mutual information: Gaussian modulation

- ▷ The $M \times M$ matrix $\boldsymbol{F}_1$ has all the entries equal to zero except the entry (1,1) which is one
- ▷ For the choice in (5), the maximal mutual information (p.c.u) is equal to

$$\frac{1}{T} I(\boldsymbol{X};\boldsymbol{S}) = \frac{\rho^2}{2} N^2 T M^2 \hat{\lambda}^2 + o(\rho^2).$$

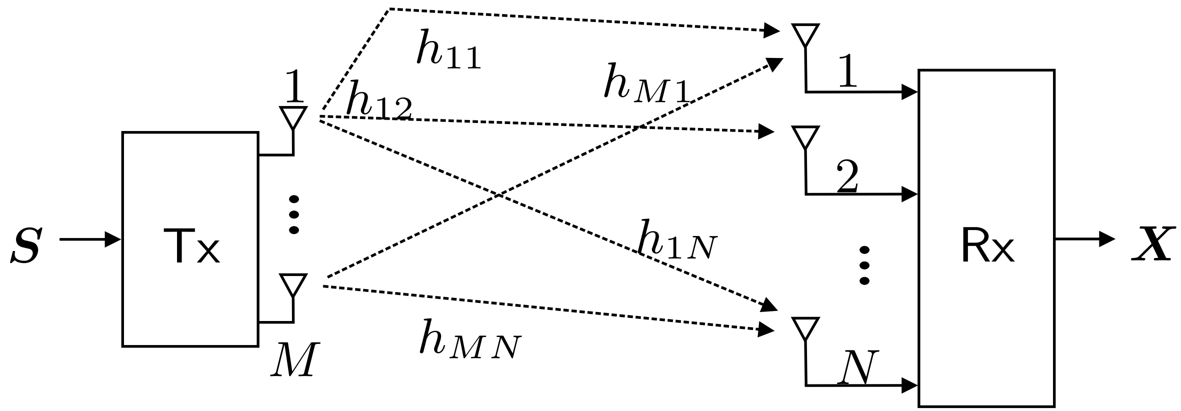
▷ Conclusions:

- From (5) we see that both $\boldsymbol{K}_t$ and $\boldsymbol{K}_r$ should be of rank one
- Correlated Rayleigh fading channel is beneficial from capacity viewpoint. Gain of order M^2N with respect to uncorrelated Rayleigh fading channel
- Correlation in noise is beneficial too, $\hat{\lambda} \geq 1$

Part 2: Low SNR regime – deterministic fading channel

Problem Formulation

▷ Data model: $\boldsymbol{X} = \boldsymbol{S}\boldsymbol{H} + \boldsymbol{E}$



▷ $\boldsymbol{X}, \boldsymbol{E}$: $T \times N$, $\boldsymbol{S}$: $T \times M$, $\boldsymbol{H}$: $M \times N$

▷ Codebook : $\mathcal{S} = \{\boldsymbol{S}_1, \boldsymbol{S}_2, ..., \boldsymbol{S}_K\}$ is a point in the manifold

$$\mathcal{M} = \{(\boldsymbol{S}_1, \ldots, \boldsymbol{S}_K) : \text{tr}(\boldsymbol{S}_k^H \boldsymbol{S}_k) = 1\}$$

▷ Contribution: design codebook when $\boldsymbol{H}$ deterministic, unknown and $\text{vec}\,(\boldsymbol{E}) \sim \mathcal{CN}\,(\mathbf{0}, \boldsymbol{\Upsilon})$ (colored noise)

▷ GLRT receiver:

$$\widehat{k} = \underset{k=1,2,\ldots,K}{\text{argmax}} \; p(\boldsymbol{x}|\boldsymbol{S}_k, \widehat{\boldsymbol{g}}_k)$$
$$= \underset{k=1,2,\ldots,K}{\text{argmin}} \; ||\boldsymbol{x} - \widetilde{\boldsymbol{S}_k}\widehat{\boldsymbol{g}}_k||^2_{\boldsymbol{\Upsilon}^{-1}}$$

$\widetilde{\boldsymbol{S}_k} = \boldsymbol{I}_N \otimes \boldsymbol{S}_k$, $\widehat{\boldsymbol{g}}_k = (\widehat{\boldsymbol{S}_k}^H \widehat{\boldsymbol{S}_k})^{-1} \widehat{\boldsymbol{S}_k}^H \boldsymbol{\Upsilon}^{-\frac{1}{2}} \boldsymbol{y}$ (ML channel estimate), $\widehat{\boldsymbol{S}_k} = \boldsymbol{\Upsilon}^{-\frac{1}{2}} \widetilde{\boldsymbol{S}_k}$, $||\boldsymbol{z}||^2_{\boldsymbol{A}} = \boldsymbol{z}^H \boldsymbol{A} \boldsymbol{z}$, $\boldsymbol{x} = \text{vec}\,(\boldsymbol{X})$

▷ PEP analysis: it can be shown that at low SNR and $T \geq 2M$

$$P_{\boldsymbol{S}_i \to \boldsymbol{S}_j} \approx \text{Prob}\left(Y > \boldsymbol{g}^H \boldsymbol{L}_{ij} \boldsymbol{g}\right), \tag{6}$$

with $\boldsymbol{L}_{ij} = \widehat{\boldsymbol{S}_i}^H \boldsymbol{\Pi}_j^{\perp} \widehat{\boldsymbol{S}_i}$, $\boldsymbol{\Pi}_j^{\perp} = \boldsymbol{I}_{TN} - \widehat{\boldsymbol{S}_j} \left(\widehat{\boldsymbol{S}_j}^H \widehat{\boldsymbol{S}_j} \right)^{-1} \widehat{\boldsymbol{S}_j}^H$, and $Y = \sum_{m=1}^{MN} \sin \alpha_m \left(|a_m|^2 - |b_m|^2 \right)$ where a_m, $b_m \overset{iid}{\sim} \mathcal{CN}(0,1)$ for $m = 1, \ldots, MN$. The angles α_m are the *principal angles* between the subspaces spanned by $\widehat{\boldsymbol{S}_i}$ and $\widehat{\boldsymbol{S}_j}$

Problem Formulation

- ▷ PEP analysis: for $M = 1$ and $\boldsymbol{\Upsilon} = \boldsymbol{I}_{TN}$, (6) becomes

$$P_{\boldsymbol{s}_i \to \boldsymbol{s}_j} = P\left(\sum_{n=1}^{N} \left(|a_n|^2 - |b_n|^2\right) > ||\boldsymbol{h}||^2 \sin \alpha_{ij}\right) \tag{7}$$

  where $a_n, b_n \overset{iid}{\sim} \mathcal{CN}(0,1)$ and the angle α_{ij} is the acute angle between the codewords $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$

- ▷ In our work [5, ?] the expression for the PEP in the high SNR regime, $M = 1$ and $\boldsymbol{\Upsilon} = \boldsymbol{I}_{TN}$ is given by

$$P_{\boldsymbol{s}_i \to \boldsymbol{s}_j} = \mathcal{Q}\left(\frac{1}{\sqrt{2}} ||\boldsymbol{h}|| \sin \alpha_{ij}\right) \tag{8}$$

  where $\mathcal{Q}(x) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt$

- ▷ Equations (7)-(8) confirm that the codewords $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$ should be constructed as separate as possible
- ▷ The problem of constructing good codes corresponds to the very well known packing problem in the complex projective space [4, 7]

Problem Formulation

▷ From (6), an upper bound on the PEP is readily found

$$P_{\boldsymbol{S}_i \rightarrow \boldsymbol{S}_j} \leq \text{Prob}\left(Z > ||\boldsymbol{g}||^2 \lambda_{\min}\left(\boldsymbol{L}_{ij}\right)\right), \tag{9}$$

where $Z = \sum_{m=1}^{MN} |a_m|^2$, $a_m \overset{iid}{\sim} \mathcal{CN}(0,1)$

▷ The codebook design criterion in (9) is equivalent to the one for the high SNR regime [7]

$$\mathcal{S}^* = \underset{\mathcal{S} \in \mathcal{M}}{\arg\max} \ \min\{\lambda_{\min}(\boldsymbol{L}_{ij}(\mathcal{X})) : 1 \leq i \neq j \leq K\}$$

Computer Simulations

□ Category 1 - spatio-temporally white observation noise: Constellations with equal priors

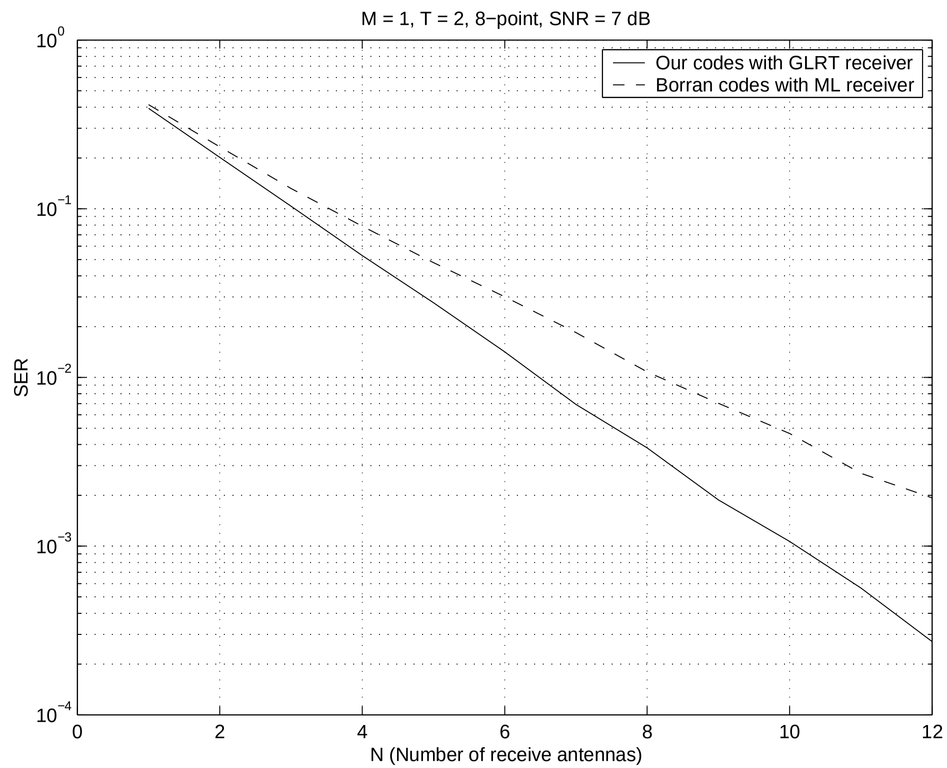


Figure 1: M=1, T=2, K=8, SNR = 7 dB. Solid curve:our codes with our GLRT receiver. Dashed curve:Borran codes designed for SNR = 7dB with ML receiver [1].

☐ Category 1 - spatio-temporally white observation noise: Constellations with equal priors

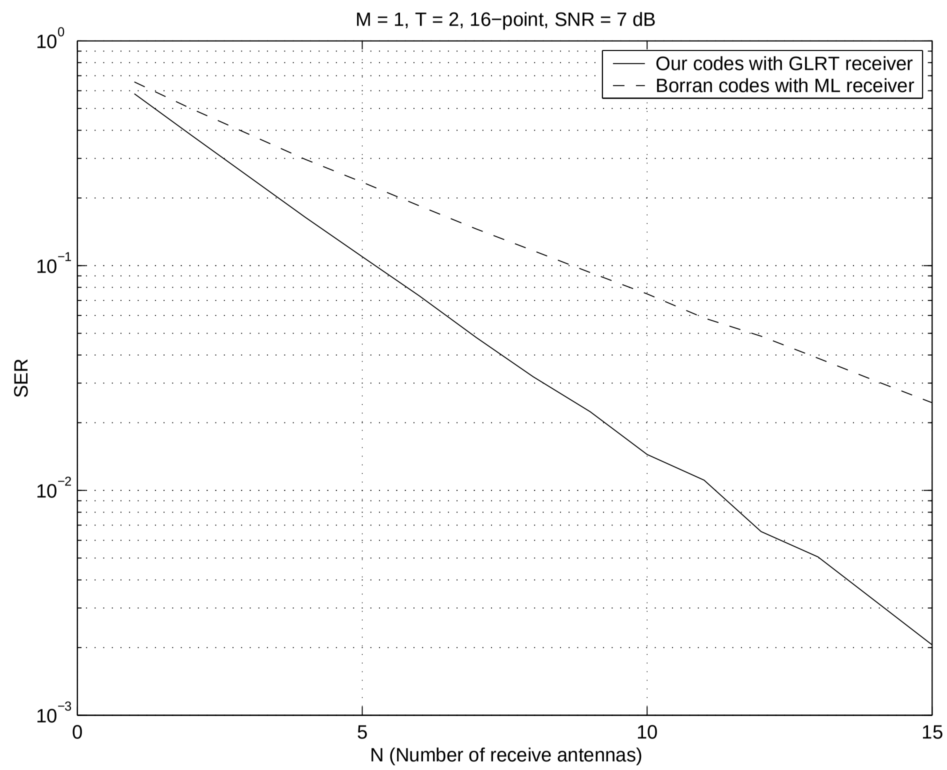


Figure 2: M=1, T=2, K=16, SNR = 7 dB. Solid curve:our codes with our GLRT receiver. Dashed curve:Borran codes designed for SNR = 7dB with ML receiver [1].

□ Category 1 - spatio-temporally white observation noise: Constellations with equal priors

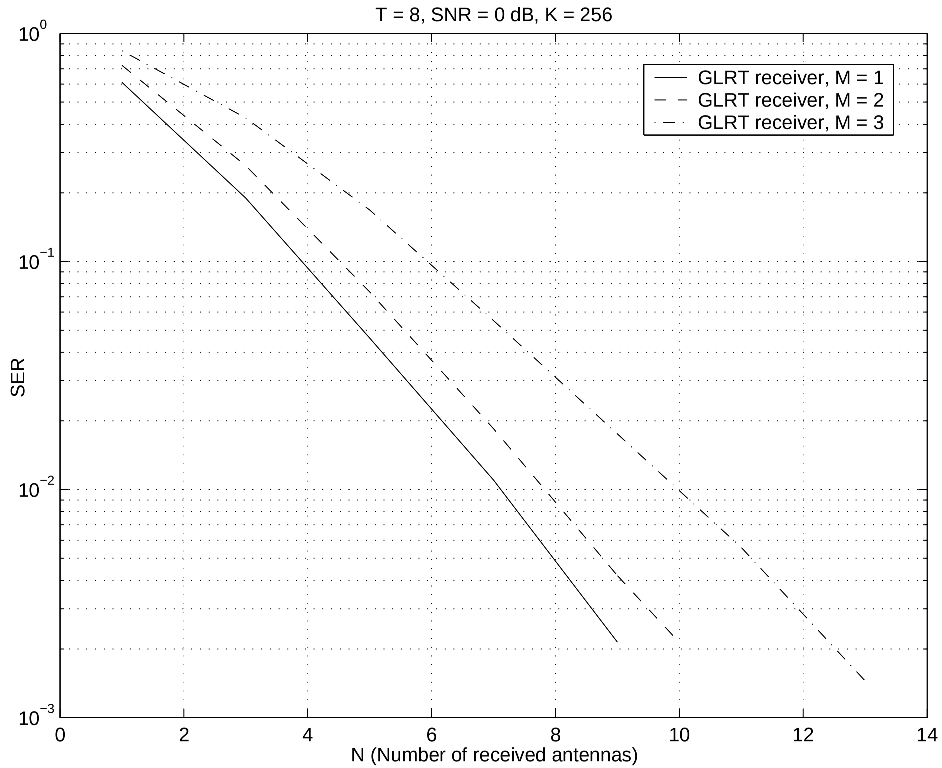


Figure 3: Category 1 - spatio-temporal white observation noise: $T = 8$, $K = 256$, SNR $= 0$ dB. Solid curve-our codes for $M = 1$, dashed curve-our codes for $M = 2$, dash-dotted curve-our codes for $M = 3$. All codes use GLRT receiver.

☐ **Category 1 - spatio-temporally white observation noise: Constellations with equal priors**

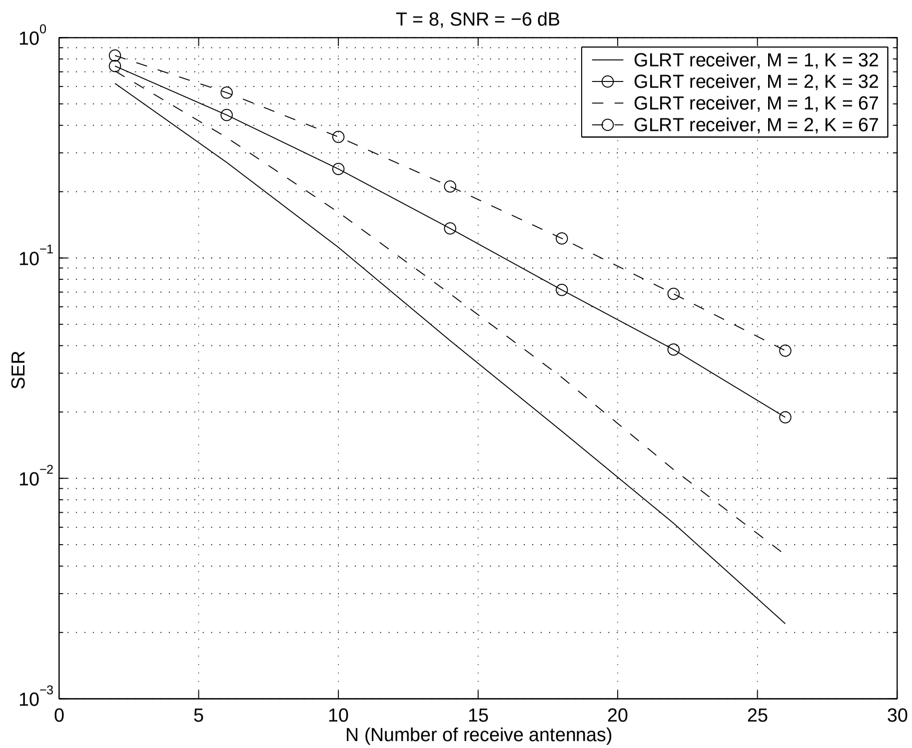


Figure 4: Category 1 - spatio-temporal white observation noise: $T = 8$, SNR = -6 dB. Solid curve-our codes for $M = 1$ and $K = 32$, dashed curve-our codes for $M = 1$ and $K = 67$, solid-circled curve-our codes for $M = 2$ and $K = 32$, dashed-circled curve-our codes for $M = 2$ and $K = 67$. All codes use GLRT receiver.

☐ Category 1 - spatio-temporally white observation noise: Constellations with equal priors

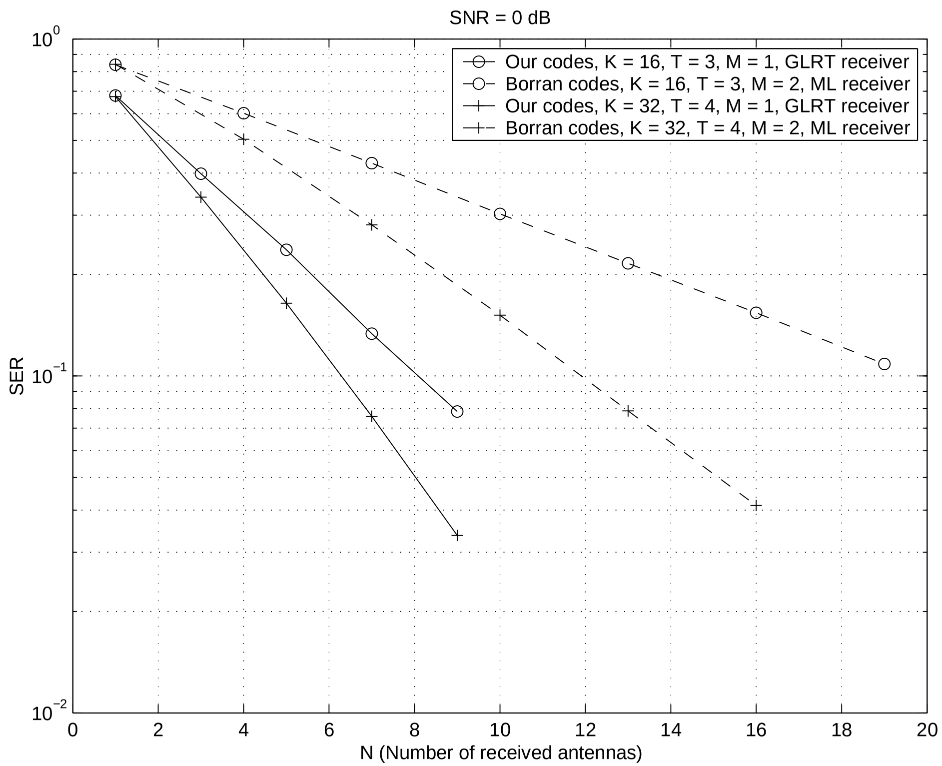


Figure 5: Category 1 - spatio-temporal white observation noise: Solid signed curve-our codes for $K = 32$, $T = 4$, $M = 1$, dashed signed curve-Borran's codes for $K = 32$, $T = 4$, $M = 2$, solid circled curve-our codes for $K = 16$, $T = 3$, $M = 1$, dashed circled curve-Borran's codes for $K = 16$, $T = 3$, $M = 2$.

☐ Category 1 - spatio-temporally white observation noise: Constellations with unequal priors

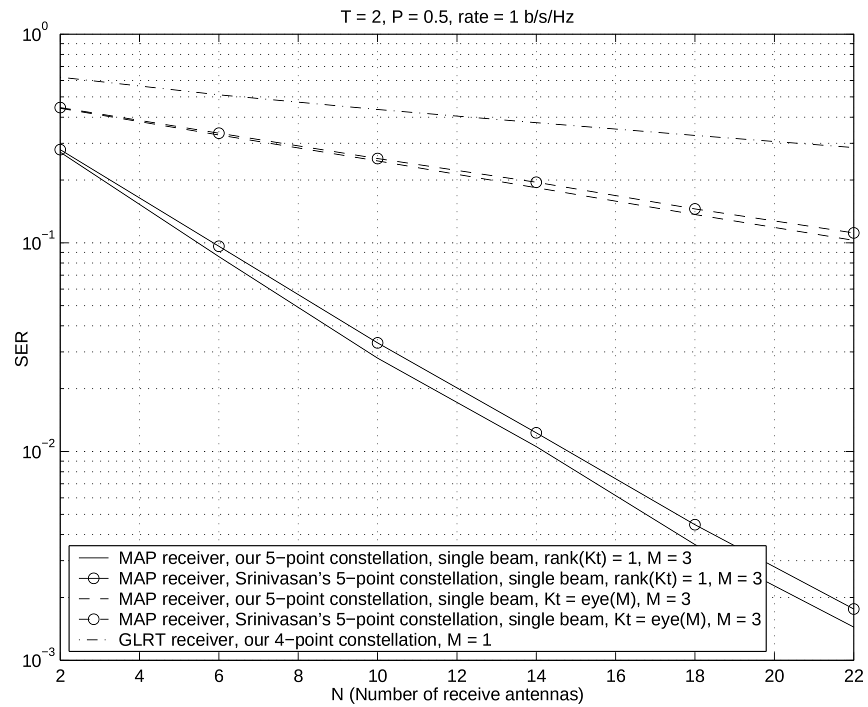


Figure 6: $\boldsymbol{K}_r = \boldsymbol{I}_N$. Correlated Rayleigh fading: solid curve-our 5 point single beam constellation with unequal priors for $M = 3$ and rank($\boldsymbol{K}_t$)=1, solid circled curve-Srinivasan's 5 point single beam constellation with unequal priors [2] for $M = 3$ and rank($\boldsymbol{K}_t$)=1. Uncorrelated Rayleigh fading: dashed curve-our 5 point single beam constellation with unequal priors for $M = 3$ and $\boldsymbol{K}_t = \boldsymbol{I}_M$, dashed circled curve-Srinivasan's 5 point single beam constellation with unequal priors for $M = 3$ and $\boldsymbol{K}_t = \boldsymbol{I}_M$. Dash-dotted curve-our 4 point constellation for $M = 1$.

□ **Category 2 - spatially white - temporally colored: $\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT} \otimes \Sigma(\boldsymbol{\rho})$**

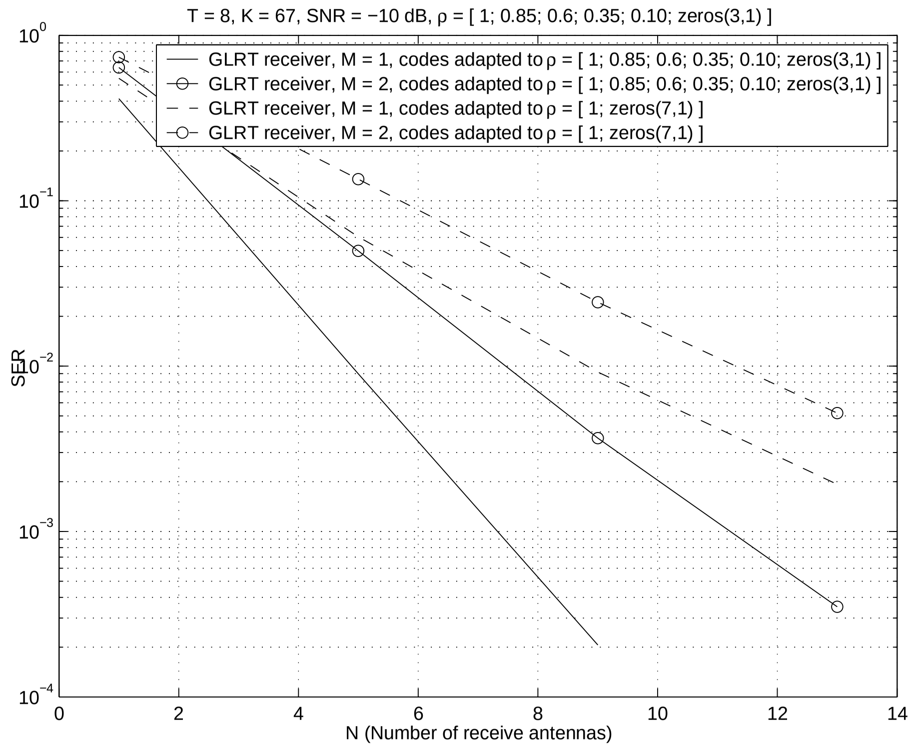


Figure 7: Category 2 - spatially white - temporally colored: $T = 8$, $K = 67$, SNR = -10 dB, $\boldsymbol{\rho}$=[1; 0.85; 0.6; 0.35; 0.1; zeros(3,1)]. Solid curve-our codes for $M = 1$ adapted to $\boldsymbol{\rho}$=[1; 0.85; 0.6; 0.35; 0.1; zeros(3,1)], solid-circled curve-our codes for $M = 2$ adapted to $\boldsymbol{\rho}$=[1; 0.85; 0.6; 0.35; 0.1; zeros(3,1)], dashed curve-our codes for $M = 1$ adapted to $\boldsymbol{\rho}$=[1; zeros(7,1)], dashed-circled curve-our codes for $M = 2$ adapted to $\boldsymbol{\rho}$=[1; zeros(7,1)].

□ **Category 2 - spatially white - temporally colored: $\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT} \otimes \Sigma(\boldsymbol{\rho})$**

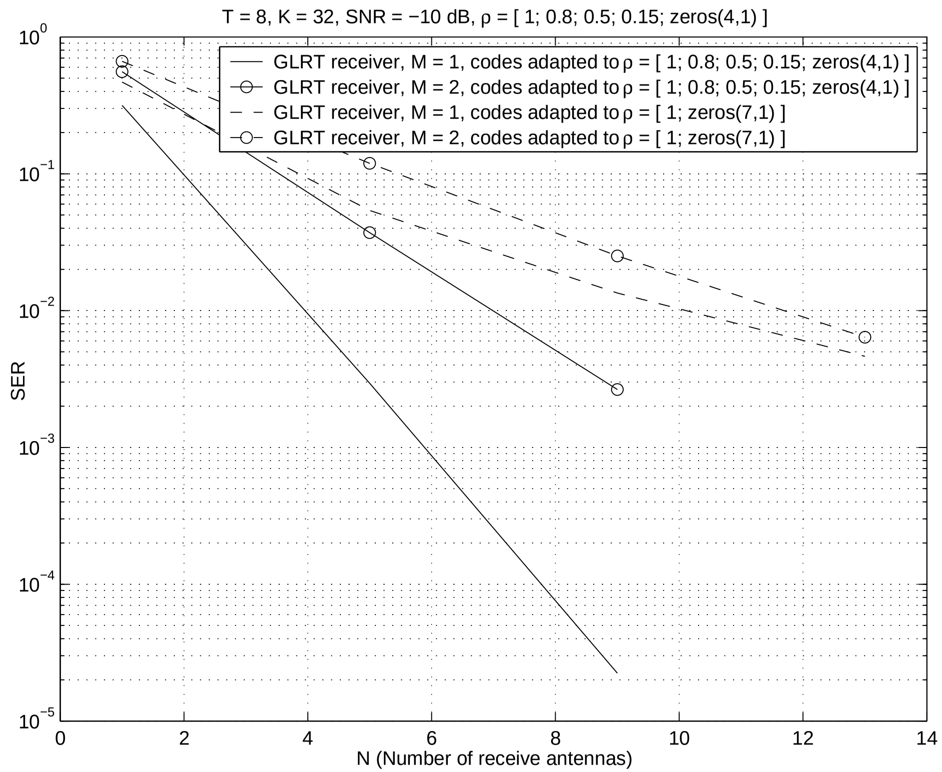


Figure 8: $T = 8$, $K = 32$, SNR = -10 dB, $\boldsymbol{\rho}$=[1; 0.8; 0.5; 0.15; zeros(4,1)]. Solid curve-our codes for $M = 1$ adapted to $\boldsymbol{\rho}$=[1; 0.8; 0.5; 0.15; zeros(4,1)], solid-circled curve-our codes for $M = 2$ adapted to $\boldsymbol{\rho}$=[1; 0.8; 0.5; 0.15; zeros(4,1)], dashed curve-our codes for $M = 1$ adapted to $\boldsymbol{\rho}$=[1; zeros(7,1)], dashed-circled curve-our codes for $M = 2$ adapted to $\boldsymbol{\rho}$=[1; zeros(7,1)].

□ Category 2 - spatially white - temporally colored:

$\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT} \otimes \Sigma(\boldsymbol{\rho})$

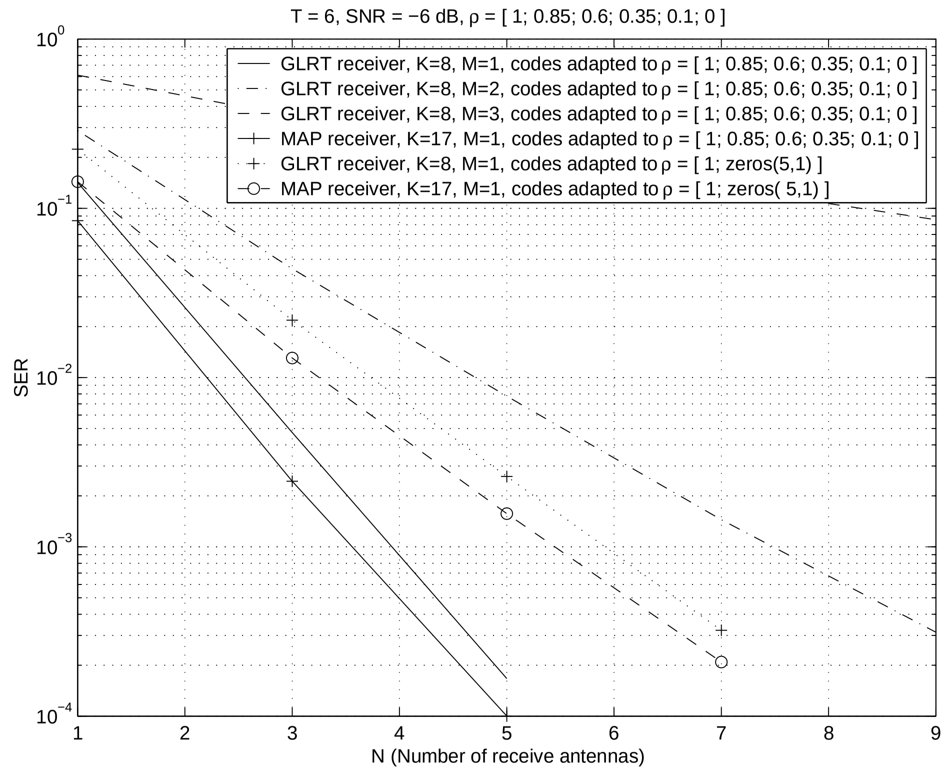


Figure 9: T=6, SNR=-6dB, $\boldsymbol{\rho}$=[1; 0.85; 0.6; 0.35; 0.1; 0].

□ **Category 2 - spatially white - temporally colored:** $\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT} \otimes \Sigma(\boldsymbol{\rho})$

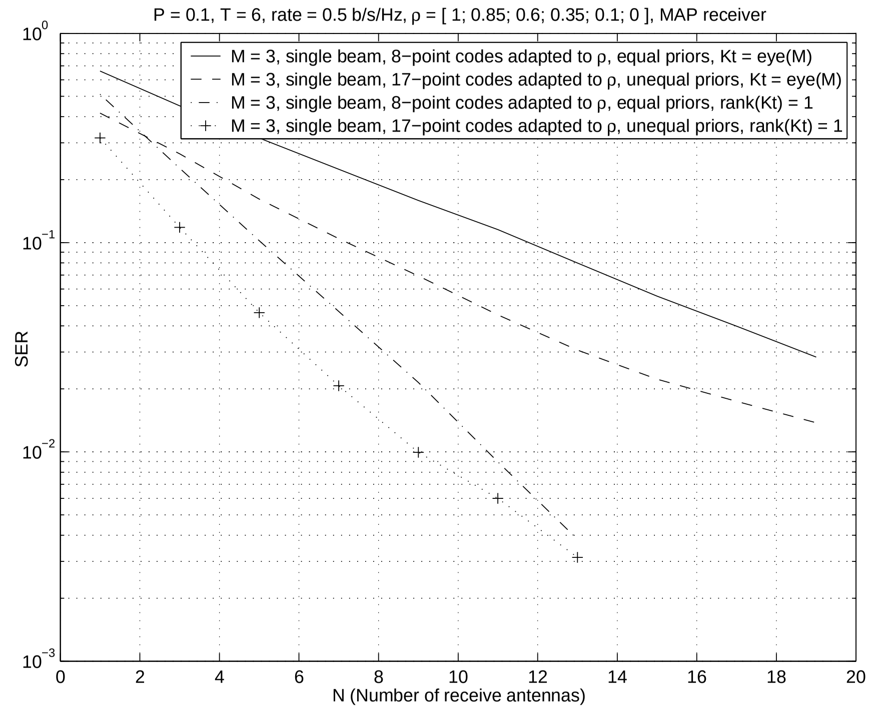


Figure 10: Correlated Rayleigh fading: dotted plus-signed curve-our 17 point single beam constellation with unequal priors for $M = 3$ and rank($\boldsymbol{K}_t$)=1, dash-dotted curve-our 8 point single beam constellation with equal priors for $M = 3$ and rank($\boldsymbol{K}_t$)=1. Uncorrelated Rayleigh fading: dashed curve-our 17 point single beam constellation with unequal priors for $M = 3$ and $\boldsymbol{K}_t = \boldsymbol{I}_M$, solid curve-our 8 point single beam constellation with equal priors for $M = 3$ and $\boldsymbol{K}_t = \boldsymbol{I}_M$. All codes use MAP receiver.

□ Category 3 - $\boldsymbol{E} = \boldsymbol{s}\,\boldsymbol{\alpha}^T + \boldsymbol{E}_{\text{temp}}$

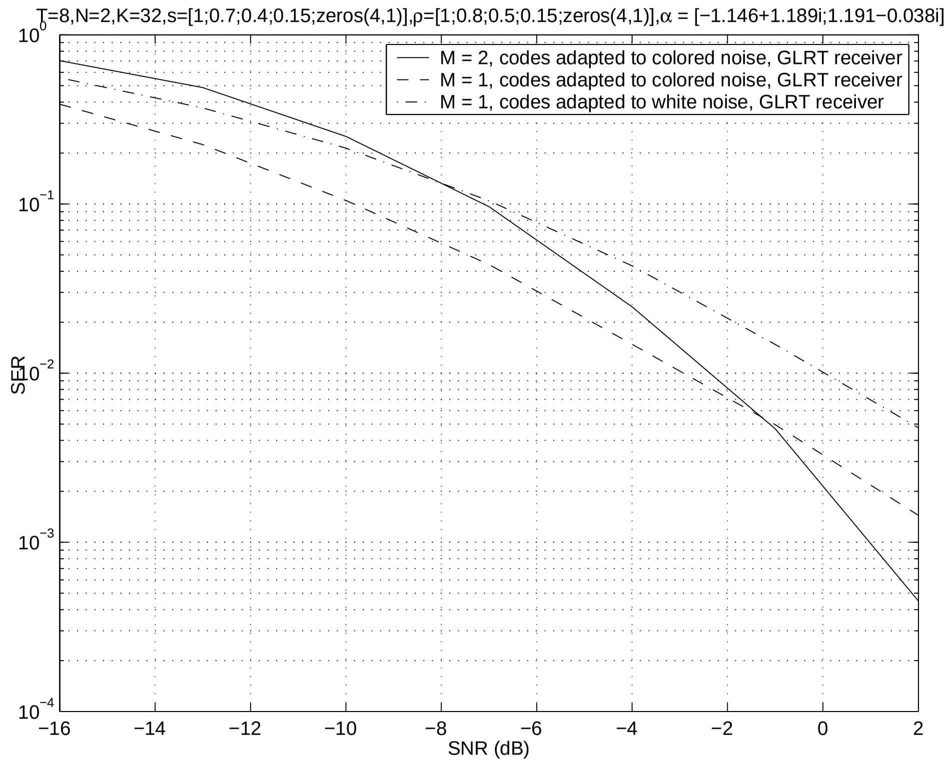


Figure 11: $T = 8$, $N = 2$, $K = 32$, $\boldsymbol{s}$ = [1;0.7;0.4;0.15;zeros(4,1)], $\boldsymbol{\rho}$ = [1;0.8;0.5;0.15;zeros(4,1)], $\boldsymbol{\alpha}$ = [-1.146 + 1.189i;1.191- 0.038i]. Solid curve-our codes for $M = 2$ adapted to colored noise, dashed curve-our codes for $M = 1$ adapted to colored noise, dash-dotted curve-our codes for $M = 1$ adapted to white noise. All codes use GLRT receiver.

□ **Category 3 -** $\boldsymbol{E} = \boldsymbol{s}\,\boldsymbol{\alpha}^T + \boldsymbol{E}_{\text{temp}}$

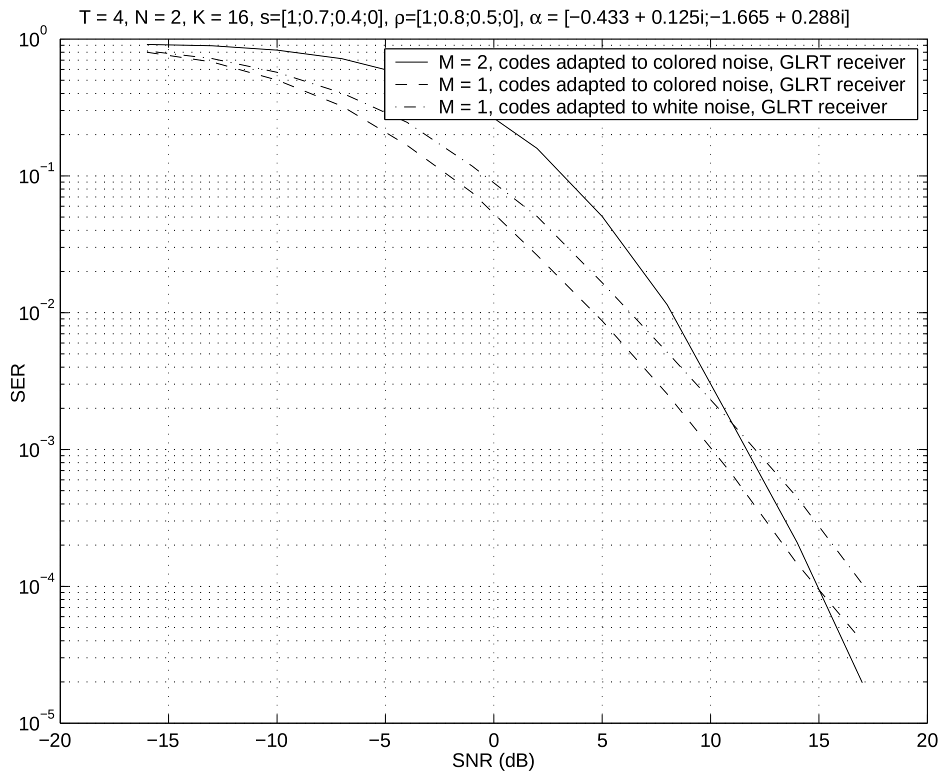


Figure 12: $T = 4$, $N = 2$, $K = 16$, $\boldsymbol{s}$ = [1;0.7;0.4;0], $\boldsymbol{\rho}$ = [1;0.8;0.5;0], $\boldsymbol{\alpha}$ = [-0.433 + 0.125i;-1.665 + 0.288i]. Solid curve-our codes for $M = 2$ adapted to colored noise, dashed curve-our codes for $M = 1$ adapted to colored noise, dash-dotted curve-our codes for $M = 1$ adapted to white noise. All codes use GLRT receiver.

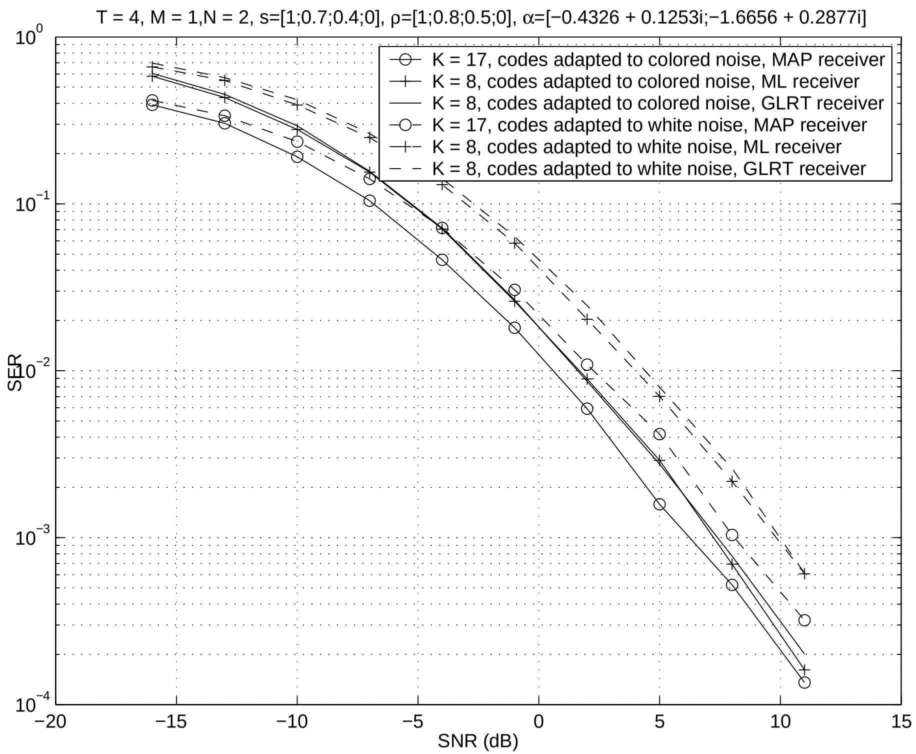


Figure 13: Solid-circled curve-our 17 point codes with unequal priors [2] adapted to colored noise (use MAP receiver), plus-signed solid curve-our 8 point codes with equal priors adapted to colored noise (use ML receiver), solid curve-our 8 point codes with equal priors adapted to colored noise (use GLRT receiver), dashed-circled curve-our 17 point codes with unequal priors adapted to white noise (use MAP receiver), plus-signed dashed curve-our 8 point codes with equal priors adapted to white noise (use ML receiver), dashed curve-our 8 point codes with equal priors adapted to white noise (use GLRT receiver).

Conclusions

▷ PEP analysis and codebook design in low SNR regime when $\boldsymbol{H}$ is deterministic and unknown

▷ Results

- outperform significantly state-of-art known solutions which assume equal prior probabilities
- also of interest for the constellations with unequal priors

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