Codebook Design for Non-coherent Communication in Multiple-antenna Systems

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Problem Formulation

▷ Data model: $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{H}^H + \boldsymbol{E}$

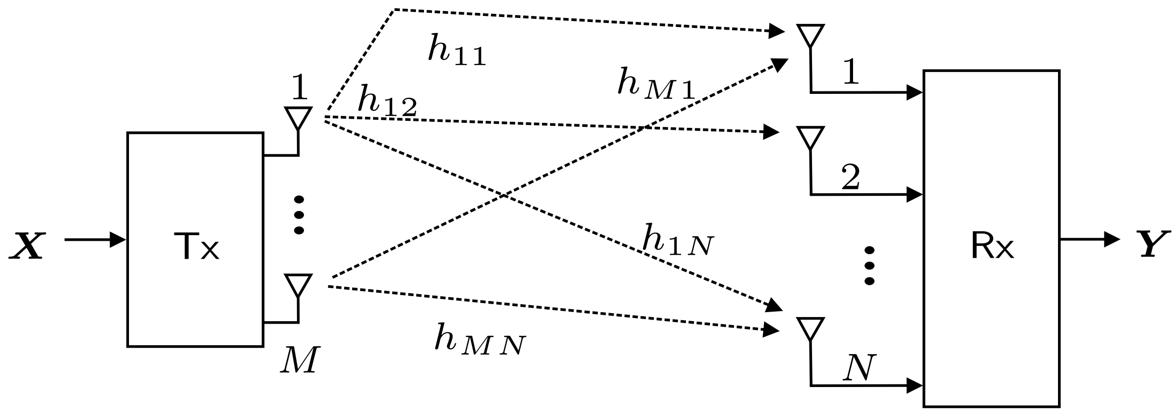


Figure 1: MIMO system

▷ $\boldsymbol{Y}, \boldsymbol{E}$: $T \times N$, $\boldsymbol{X}$: $T \times M$, $\boldsymbol{H}$: $N \times M$

▷ Codebook : $\mathcal{X} = \{\boldsymbol{X}_1, \boldsymbol{X}_2, ..., \boldsymbol{X}_K\}$ is a point in the manifold

$$\mathcal{M} = \{(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_K) : \text{tr}(\boldsymbol{X}_k^H \boldsymbol{X}_k) = 1\}$$

▷ Contribution: design codebook when $\boldsymbol{H}$ deterministic, unknown and vec $(\boldsymbol{E}) \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{\Upsilon})$ (colored noise)

Problem Formulation

▷ GLRT receiver:

$$\widehat{k} = \underset{k=1,2,\ldots,K}{\text{argmax}} \; p(\boldsymbol{y}|\boldsymbol{X}_k, \widehat{\boldsymbol{g}}_k) = \underset{k=1,2,\ldots,K}{\text{argmin}} \; ||\boldsymbol{y} - \widetilde{\boldsymbol{X}_k}\widehat{\boldsymbol{g}}_k||^2_{\boldsymbol{\Upsilon}^{-1}}$$

$\widetilde{\boldsymbol{X}_k} = \boldsymbol{I}_N \otimes \boldsymbol{X}_k$, $\widehat{\boldsymbol{g}}_k = (\widehat{\boldsymbol{X}_k}^H \widehat{\boldsymbol{X}_k})^{-1} \widehat{\boldsymbol{X}_k}^H \boldsymbol{\Upsilon}^{-\frac{1}{2}} \boldsymbol{y}$ (ML channel estimate),
$\widehat{\boldsymbol{X}_k} = \boldsymbol{\Upsilon}^{-\frac{1}{2}} \widetilde{\boldsymbol{X}_k}$, $||\boldsymbol{z}||^2_{\boldsymbol{A}} = \boldsymbol{z}^H \boldsymbol{A} \boldsymbol{z}$, $\boldsymbol{y} = \text{vec}(\boldsymbol{Y})$

▷ PEP analysis: it can be shown that (see [5]) for high SNR

$$P_{\boldsymbol{X}_i \to \boldsymbol{X}_j} = \mathcal{Q}\left(\frac{1}{\sqrt{2}}\sqrt{\boldsymbol{g}^H \boldsymbol{L}_{ij} \boldsymbol{g}}\right) \leq \mathcal{Q}\left(\frac{1}{\sqrt{2}}\,||\boldsymbol{g}||\,\sqrt{\lambda_{\min}(\boldsymbol{L}_{ij})}\right) \tag{1}$$

where $\boldsymbol{g} = \text{vec}(\boldsymbol{H}^H)$, $\boldsymbol{L}_{ij}(\mathcal{X}) = \widehat{\boldsymbol{X}_i}^H \underbrace{\left(\boldsymbol{I}_T - \widehat{\boldsymbol{X}_j}\left(\widehat{\boldsymbol{X}_j}^H \widehat{\boldsymbol{X}_j}\right)^{-1} \widehat{\boldsymbol{X}_j}^H\right)}_{\Pi_j^{\perp}} \widehat{\boldsymbol{X}_i}$

Problem Formulation

- ▷ Optimization problem: result (1) suggests the codebook merit function

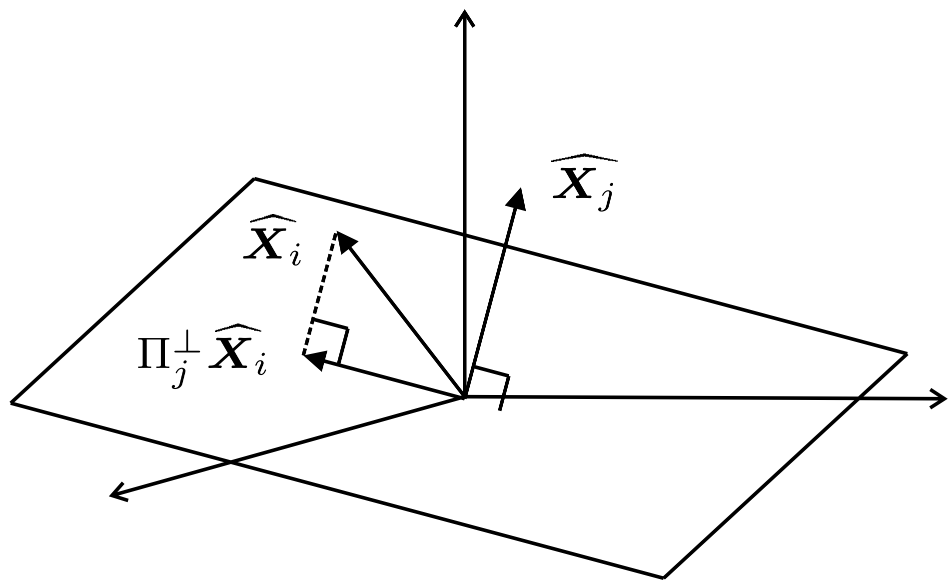
$$\mathcal{X}^* = \underset{\mathcal{X} \in \mathcal{M}}{\arg\max} \ \min\{\lambda_{\min}(\boldsymbol{L}_{ij}(\mathcal{X})) : 1 \leq i \neq j \leq K\} \quad (2)$$

- ▷ The problem in (2) is a high-dimensional, non-linear and non-smooth optimization problem!

  e.g. for $K = 256$, $T = 8$, $M = 2$: $K(K-1) = 65280$ $\boldsymbol{L}_{ij}(\mathcal{X})$ functions and $2KTM = 8192$ real variables to optimize

Codebook design : geometrical interpretation

- $\widehat{\boldsymbol{X}}_i$ should lie in the orthogonal complement of span$\{\widehat{\boldsymbol{X}_j}\}$



- $f(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_K) = f(\boldsymbol{X}_1 e^{i\theta_1}, \ldots, \boldsymbol{X}_K e^{i\theta_K})$: packing in complex projective space

Codebook Construction

▷ Two-phase methodology to tackle the optimization problem in (2)

▷ <u>Phase I</u>: solves a convex semi-definite programming (SDP) relaxation

▷ Incremental approach: Let $\mathcal{X}_{k-1}^* = \{\boldsymbol{X}_1^*, ..., \boldsymbol{X}_{k-1}^*\}$ be the codebook at the $k-1^{\underline{th}}$ stage. The new codeword is found by solving

$$\boldsymbol{X}_k^* = \underset{\text{tr}(\boldsymbol{X}_k^H \boldsymbol{X}_k) = 1}{\arg\max} \; \min_{1 \leq i \leq k-1} \{\lambda_{\text{min}}(\boldsymbol{L}_{ik}), \lambda_{\text{min}}(\boldsymbol{L}_{ki})\} \tag{3}$$

for $k = 2, ..., K$

Codebook Construction - Phase I

▷ The optimization problem (3) is equivalent to (see [5])

$$(\widehat{\boldsymbol{Y}}^*, \widetilde{\boldsymbol{X}}^*, t^*) = \arg\max \quad t \tag{4}$$

with the following constraints

$$\begin{bmatrix} \mathrm{tr}(\boldsymbol{N}_i \boldsymbol{A}_1 \widehat{\boldsymbol{Y}} \boldsymbol{B}_1) - t & \cdots & \mathrm{tr}(\boldsymbol{N}_i \boldsymbol{A}_{MN} \widehat{\boldsymbol{Y}} \boldsymbol{B}_1) \\ \vdots & & \vdots \\ \mathrm{tr}(\boldsymbol{N}_i \boldsymbol{A}_1 \widehat{\boldsymbol{Y}} \boldsymbol{B}_{MN}) & \cdots & \mathrm{tr}(\boldsymbol{N}_i \boldsymbol{A}_{MN} \widehat{\boldsymbol{Y}} \boldsymbol{B}_{MN}) - t \end{bmatrix} \succeq \boldsymbol{0},$$

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{Z}_i \\ \boldsymbol{Z}_i^H & \boldsymbol{P}_i \end{bmatrix} \succeq \boldsymbol{0} \, \forall_{1 \leq i \leq k-1}, \boldsymbol{K} \widehat{\boldsymbol{Y}} \boldsymbol{K}^H = \widetilde{\boldsymbol{X}}, \mathrm{tr}(\widetilde{\boldsymbol{X}}) = 1,$$

$$\boldsymbol{f} \widehat{\boldsymbol{Y}} \boldsymbol{f}^H = 1, \widehat{\boldsymbol{Y}} = \widehat{\boldsymbol{Y}}^H, \widehat{\boldsymbol{Y}} \succeq \boldsymbol{0}, \mathrm{rank}(\widehat{\boldsymbol{Y}}) = 1$$

and $\widetilde{\boldsymbol{X}} = \mathrm{vec}(\boldsymbol{X}_k)\mathrm{vec}^H(\boldsymbol{X}_k), b^2 = 1$, $\widehat{\boldsymbol{Y}} = \boldsymbol{z}\boldsymbol{z}^H$, $\boldsymbol{z} = \left[\mathrm{vec}^T(\widetilde{\boldsymbol{X}_k}) \; b\right]^T$, $\widetilde{\boldsymbol{X}_k} = \boldsymbol{I}_N \otimes \boldsymbol{X}_k$.

▷ The matrices $\boldsymbol{M}$, $\boldsymbol{Z}_i$ — linear in $\widehat{\boldsymbol{Y}}$

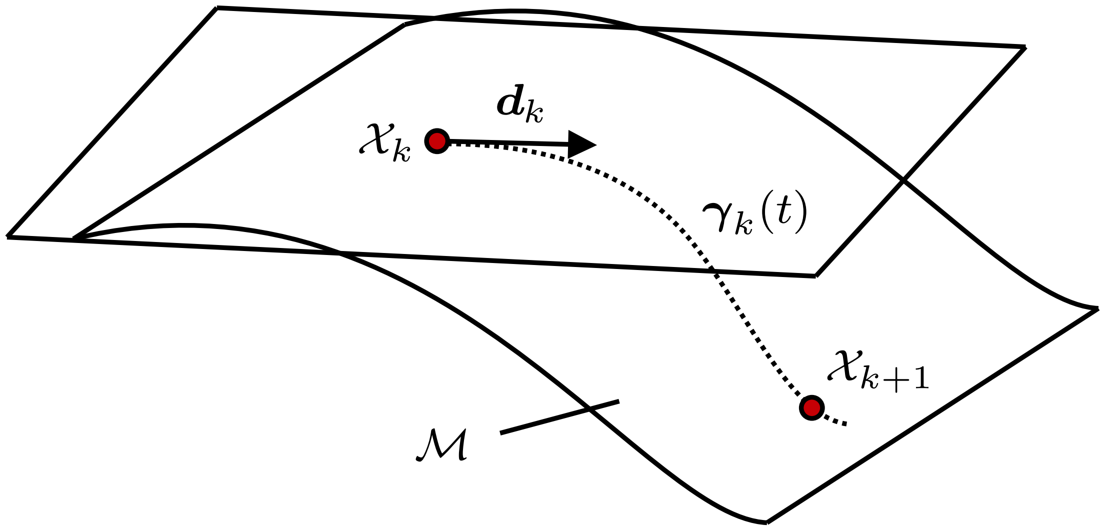
▷ The matrices $\boldsymbol{N}_i$, $\boldsymbol{P}_i$, $\boldsymbol{K}$, $\boldsymbol{f}$, $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ — constants, some depend on $\boldsymbol{\Upsilon}$

Codebook Construction - Phase 1

- ▷ Design of the codewords: high-dimensional difficult nonlinear optimization problem (rank condition in (4))
- ▷ Relaxing the rank constraint leads to an SDP [6]
- ▷ The $k^{\underline{th}}$ codeword is extracted from the output variable $\widetilde{\boldsymbol{X}}$ with a technique similar to [7]
- ▷ Initialization $\boldsymbol{X}_1^*$: randomly generated, filling columns of the matrix with eigenvectors associated to the smallest eigenvalues of the noise covariance matrix,etc.

Codebook Construction - Phase 2

▷ Phase II: optimizes a non-smooth function on a manifold



Codebook Construction - Phase 2

- ▷ Iterative algorithm, called GDA (geodesic descent algorithm)
- ▷ Identify "active" pairs (i, j) that attain minimum
- ▷ Check if there is an ascent direction $d_k \in T_{\mathcal{X}_k}\mathcal{M}$ for all active (i, j) (consists of solving LP)
- ▷ When d_k is found, perform Armijo rule along geodesic $\boldsymbol{\gamma}_k(t)$
- ▷ If no d_k is found, the algorithm stops

Computer Simulations

☐ **Example:**

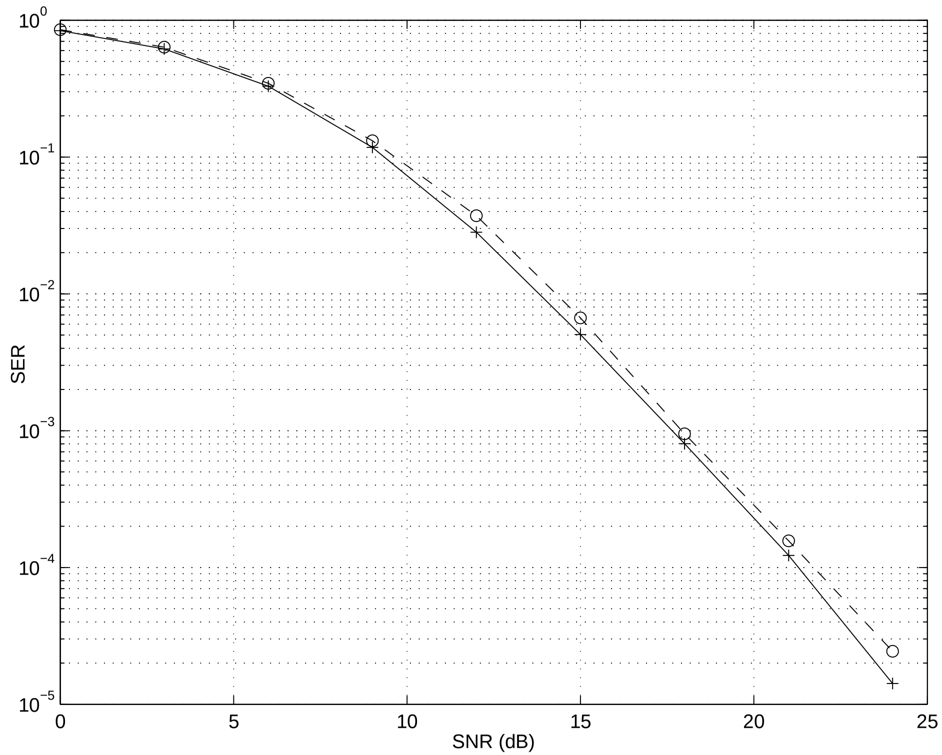


Figure 2: **Category 1 - spatio-temporally white observation noise:** T=8, M=3, N=1, K=256, $\boldsymbol{\Upsilon}= \boldsymbol{I}_{NT}$. Plus-solid curve-our codes, circle-dashed curve-unitary codes.

☐ **Example:**

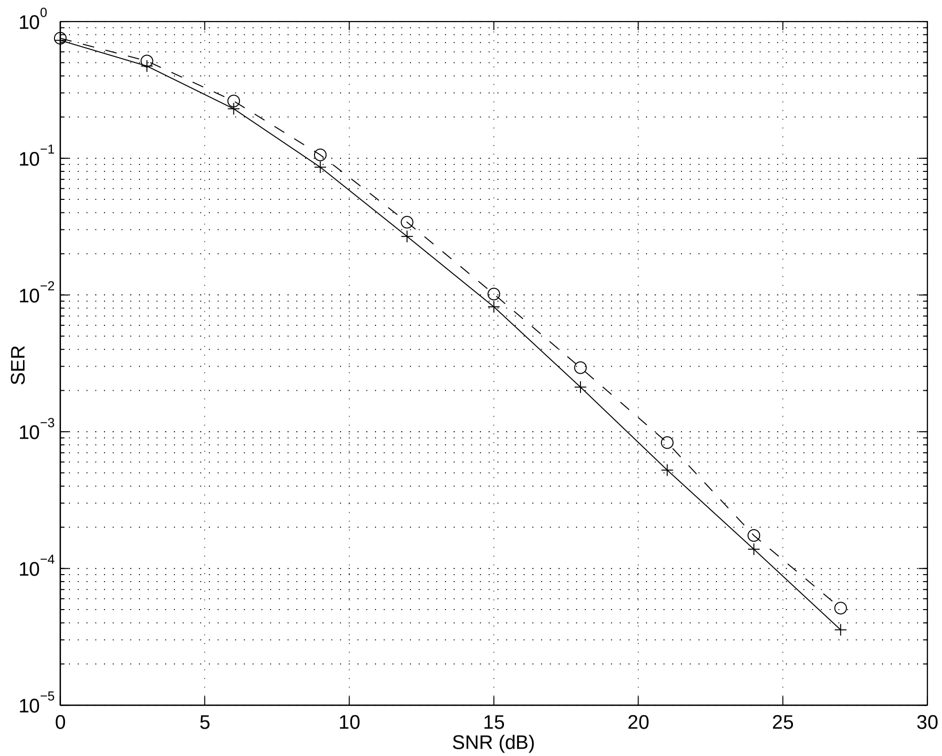


Figure 3: **Category 1 - spatio-temporally white observation noise:** T=8, M=2, N=1, K=256, $\boldsymbol{\Upsilon}= \boldsymbol{I}_{NT}$. Plus-solid curve-our codes, circle-dashed curve-unitary codes.

		PACKING RADII (DEGREES)		
T	K	MB	JAT	Rankin
4	5	75.52	75.52	75.52
4	6	70.89	70.88	71.57
4	7	69.29	69.29	69.30
4	8	67.79	67.78	67.79
4	9	66.31	66.21	66.72
4	10	65.74	65.71	65.91
4	11	64.79	64.64	65.27
4	12	64.68	64.24	64.76
4	13	64.34	64.34	64.34
4	14	63.43	63.43	63.99
4	15	63.43	63.43	63.69
4	16	63.43	63.43	63.43

Table 1: PACKING IN COMPLEX PROJECTIVE SPACE: We compare our best configurations (MB) of K points in $\mathbb{P}^{T-1}(\mathbb{C})$ against the Tropp codes (JAT) and Rankin bound. The packing radius of an ensemble is measured as the acute angle between the closest pair of lines.

		PACKING RADII (DEGREES)		
T	K	MB	JAT	Rankin
5	6	78.46	78.46	78.46
5	7	74.55	74.52	75.04
5	8	72.83	72.81	72.98
5	9	71.33	71.24	71.57
5	10	70.53	70.51	70.53
5	11	69.73	69.71	69.73
5	12	69.04	68.89	69.10
5	13	68.38	68.19	68.58
5	14	67.92	67.66	68.15
5	15	67.48	67.37	67.79
5	16	67.08	66.68	67.48
5	17	66.82	66.53	67.21
5	18	66.57	65.87	66.98
5	19	66.57	65.75	66.77

Table 2: PACKING IN COMPLEX PROJECTIVE SPACE: We compare our best configurations (MB) of K points in $\mathbb{P}^{T-1}(\mathbb{C})$ against the Tropp codes (JAT) and Rankin bound. The packing radius of an ensemble is measured as the acute angle between the closest pair of lines.

		PACKING RADII (DEGREES)	
T	K	MB	Rankin
6	7	80.41	80.41
6	8	77.06	77.40
6	9	75.52	75.52
6	10	74.20	74.21
6	11	73.22	73.22
6	12	72.45	72.45
6	13	71.82	71.83
6	14	71.31	71.32
6	15	70.87	70.89
6	16	70.53	70.53
6	17	70.10	70.21
6	18	69.73	69.94
6	19	69.40	69.70

Table 3: PACKING IN COMPLEX PROJECTIVE SPACE: We compare our best configurations (MB) of K points in $\mathbb{P}^{T-1}(\mathbb{C})$ against Rankin bound. The packing radius of an ensemble is measured as the acute angle between the closest pair of lines.

□ Example:

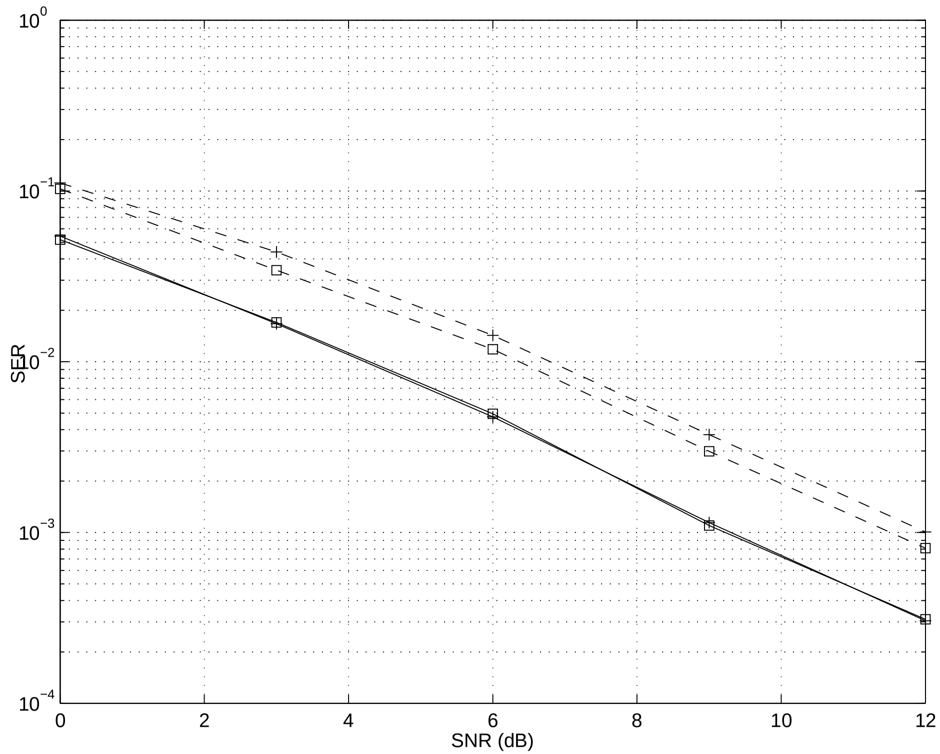


Figure 4: **Category 2 - spatially white - temporally coloured:** T=8, M=2, $N = 1$, K=67, $\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT} \otimes \Sigma(\boldsymbol{\rho})$, $\boldsymbol{\rho}$=[1; 0.85; 0.6; 0.35; 0.1; zeros(3,1)]. Solid curves-our codes, dashed curves-unitary codes, plus signed curves-GLRT receiver, square signed curves-Bayesian receiver.

☐ **Example:**

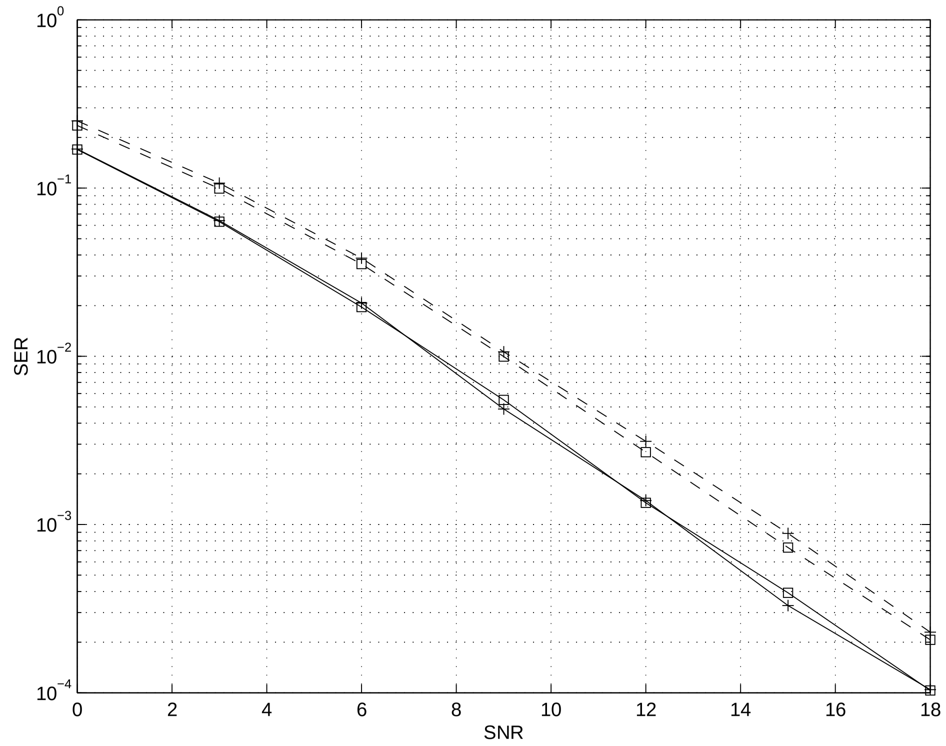


Figure 5: **Category 2 - spatially white - temporally coloured:** T=8, M=2, $N=1$, K=256, $\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT} \otimes \Sigma(\boldsymbol{\rho})$, $\boldsymbol{\rho}$=[1; 0.8; 0.5; 0.15; zeros(4,1)]. Solid curves-our codes, dashed curves-unitary codes, plus signed curves-GLRT receiver, square signed curves-Bayesian receiver.

☐ **Example:**

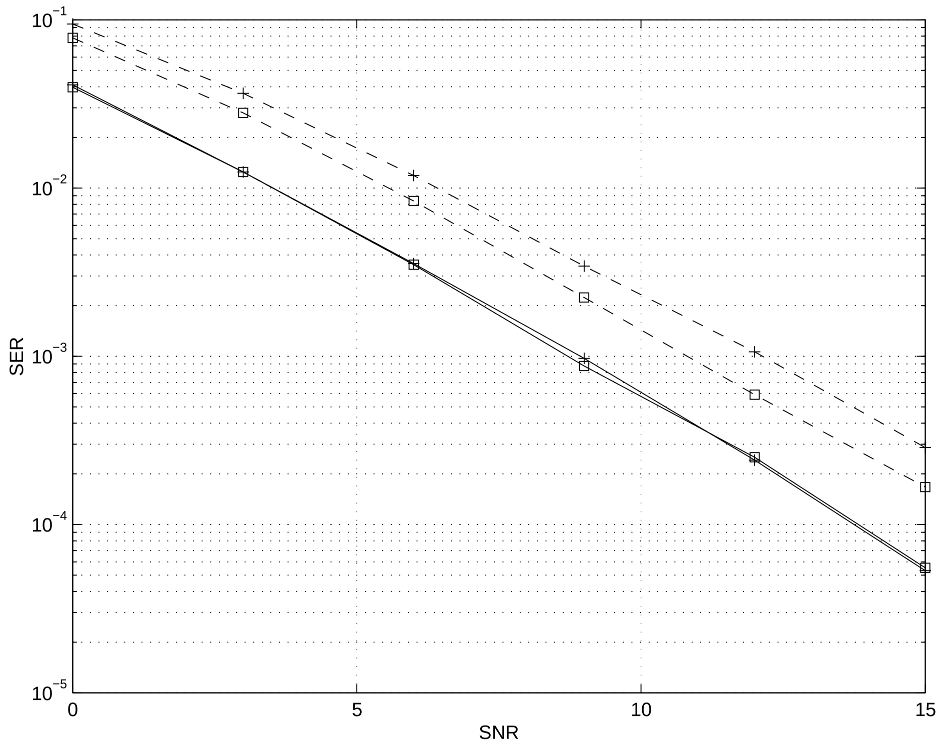


Figure 6: **Category 2 - spatially white - temporally coloured:** T=8, M=2, $N = 1$, K=32, $\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT} \otimes \Sigma(\boldsymbol{\rho})$, $\boldsymbol{\rho}$=[1; 0.8; 0.5; 0.15; zeros(4,1)]. Solid curves-our codes, dashed curves-unitary codes, plus signed curves-GLRT receiver, square signed curves-Bayesian receiver.

□ **Example:**

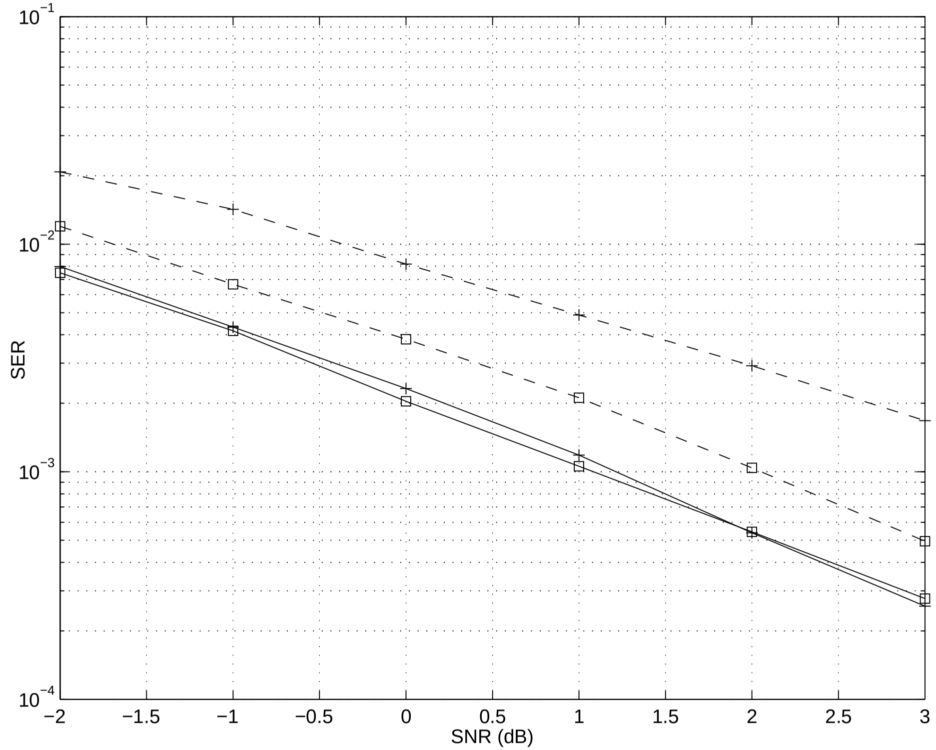


Figure 7: **Category 3 -** $\boldsymbol{\Upsilon} = \boldsymbol{\alpha}\boldsymbol{\alpha}^H \otimes \boldsymbol{\Upsilon}_s + \boldsymbol{I}_{NT} \otimes \Sigma(\boldsymbol{\rho})$**:** T=8, M=2, $N = 2$, K=32, $\boldsymbol{s}$=[1;0.7;0.4;0.15;zeros(4,1)], $\boldsymbol{\rho}$ = [1;0.8;0.5;0.15;zeros(4,1)], $\boldsymbol{\alpha}$ = [-1.146 + 1.189i;1.191- 0.038i]. Solid curves-our codes, dashed curves-unitary codes, plus signed curves-GLRT receiver, square signed curves-Bayesian receiver.

☐ **Example:**

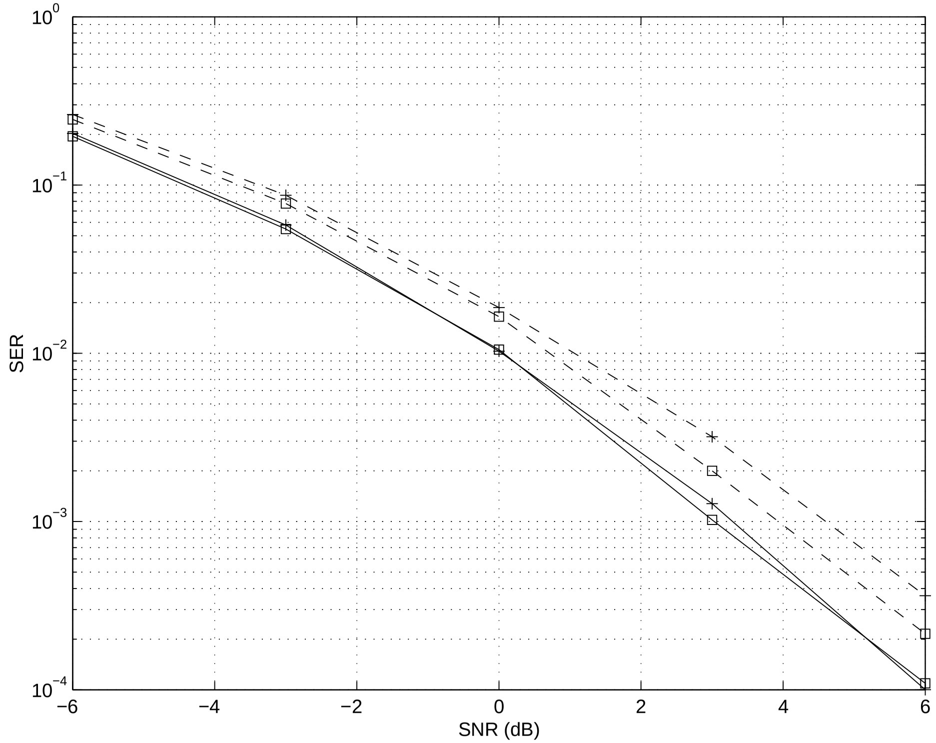


Figure 8: **Category 3 - $\boldsymbol{\Upsilon} = \boldsymbol{\alpha}\boldsymbol{\alpha}^H \otimes \boldsymbol{\Upsilon}_s + \boldsymbol{I}_{NT} \otimes \Sigma(\boldsymbol{\rho})$:** T=8, M=2, N = 2, K=67, $\boldsymbol{s}$=[1; 0.8; 0.5; 0.15; zeros(4,1)], $\boldsymbol{\rho}$ = [1; 0.7; 0.4; 0.15; zeros(4,1)], $\boldsymbol{\alpha} = [-0.453+0.007i; 0.4869+1.9728i]$. Solid curves-our codes, dashed curves-unitary codes, plus signed curves-GLRT receiver, square signed curves-Bayesian receiver.

Conclusions

- ▷ Codebook design for noncoherent setup
  - $\boldsymbol{H}$ deterministic, unknown
  - Colored noise: $\text{vec}\left(\boldsymbol{E}\right) \sim \mathcal{CN}\left(\mathbf{0}, \boldsymbol{\Upsilon}\right)$
- ▷ Results
  - outperform significantly unitary constellations for colored noise case
  - provide good packings for complex projective space ($M = 1$) (near bound performance)
  - for some cases actual Equiangular Tight Frames (ETF's)

References

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