# Non-Coherent Communication in Multiple-Antenna Systems: Receiver Design, Codebook Construction and Capacity Analysis

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# Outline

▷ **Introduction:** Motivation and Data Model

▷ **Chapter 2:** High SNR regime

– deterministic channel (PEP analysis and codebook construction)

▷ **Chapter 3:** Low SNR regime

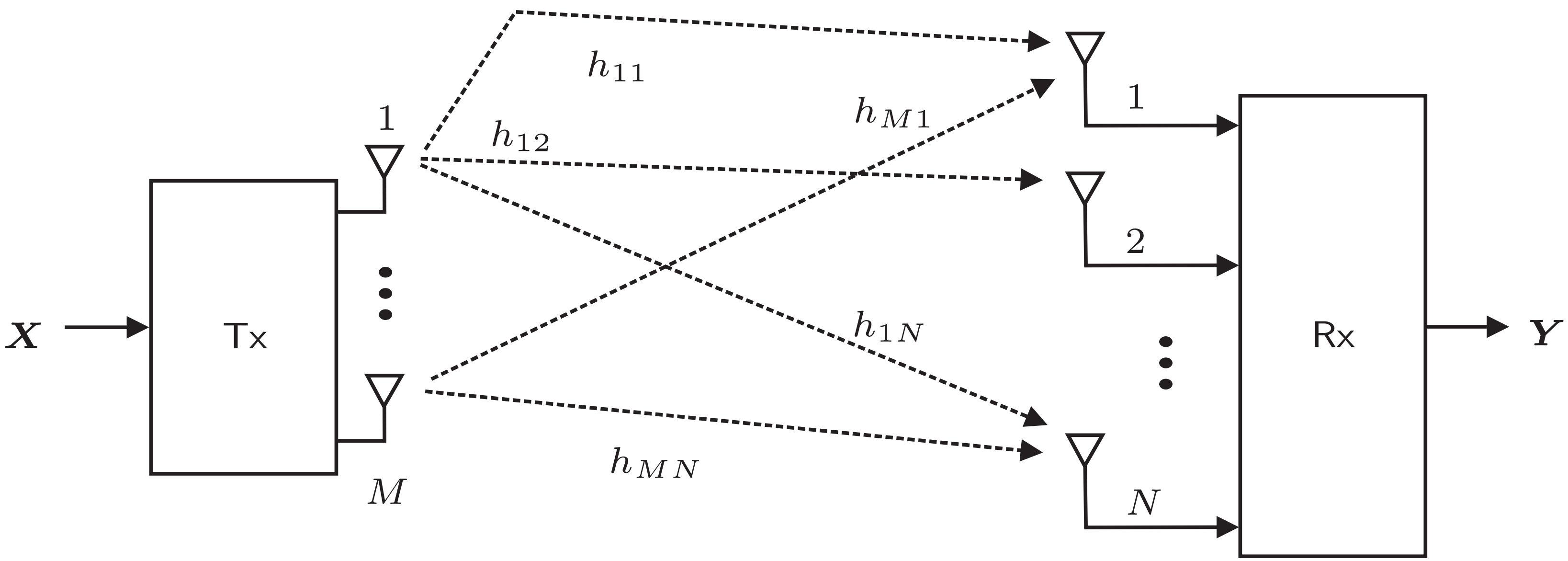
– random channel (mutual information analysis)

– deterministic channel (PEP analysis and codebook construction)

▷ **Chapter 4:** Future work

# Data Model

▷ **MIMO System:** $M$ transmit, $N$ receive antennas



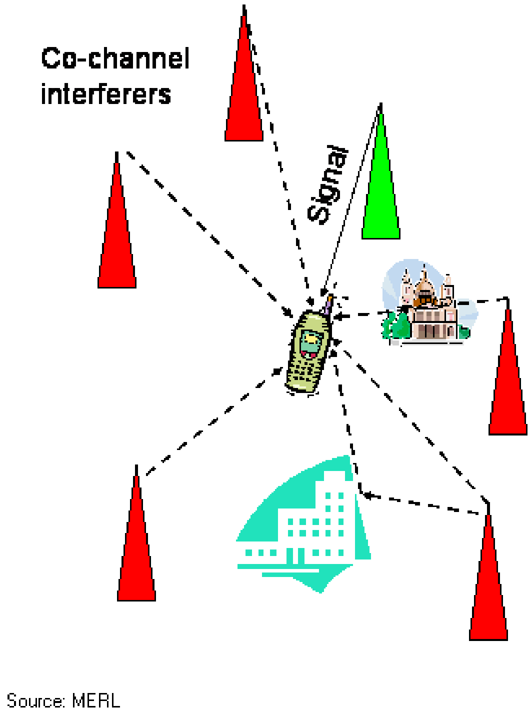
▷ **Data model:** $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{H}^H + \boldsymbol{E}$

$$\left[\quad\right] = \left[\quad\right]\left[\quad\right] + \left[\quad\right]$$

$$\underbrace{\boldsymbol{Y}}_{T\times N} \qquad \underbrace{\boldsymbol{X}}_{T\times M} \qquad \underbrace{\boldsymbol{H}^H}_{M\times N} \qquad \underbrace{\boldsymbol{E}}_{T\times N}$$

# Introduction

▷ **Motivation:** Noise is not white!!



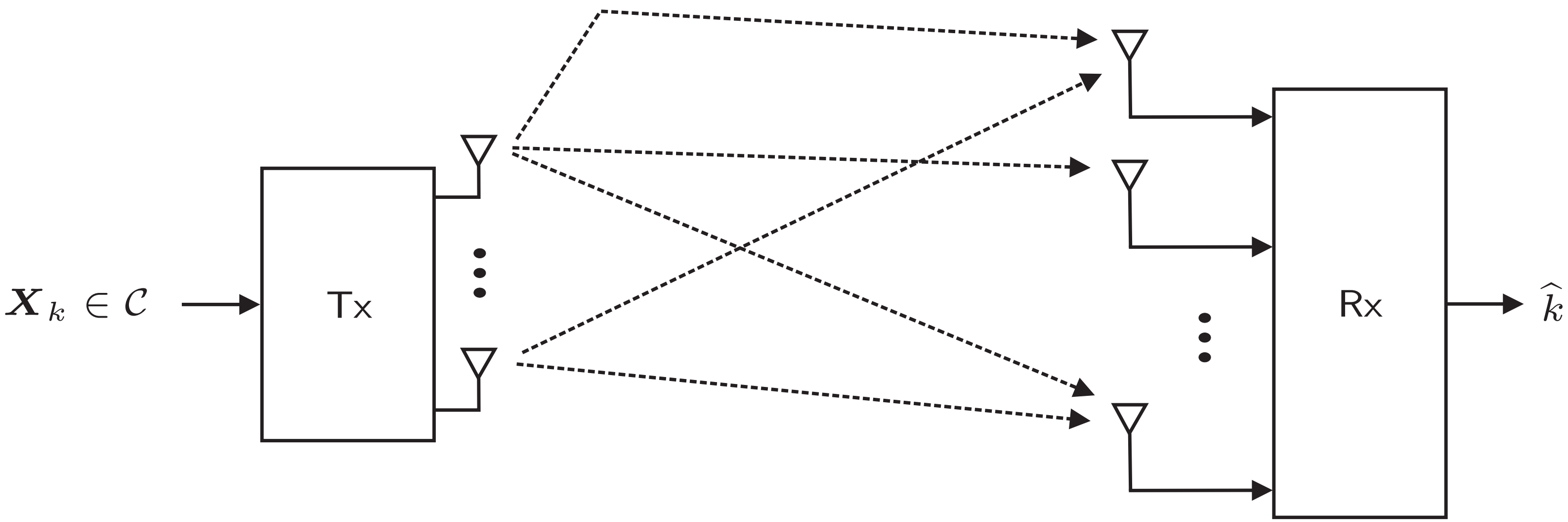

▷ "Colored" noise is more realistic!

# Chapter 2: High SNR regime

# Problem Formulation

▷ Codebook : $\mathcal{C} = \{\boldsymbol{X}_1, \boldsymbol{X}_2, ..., \boldsymbol{X}_K\}$ is a point in the manifold

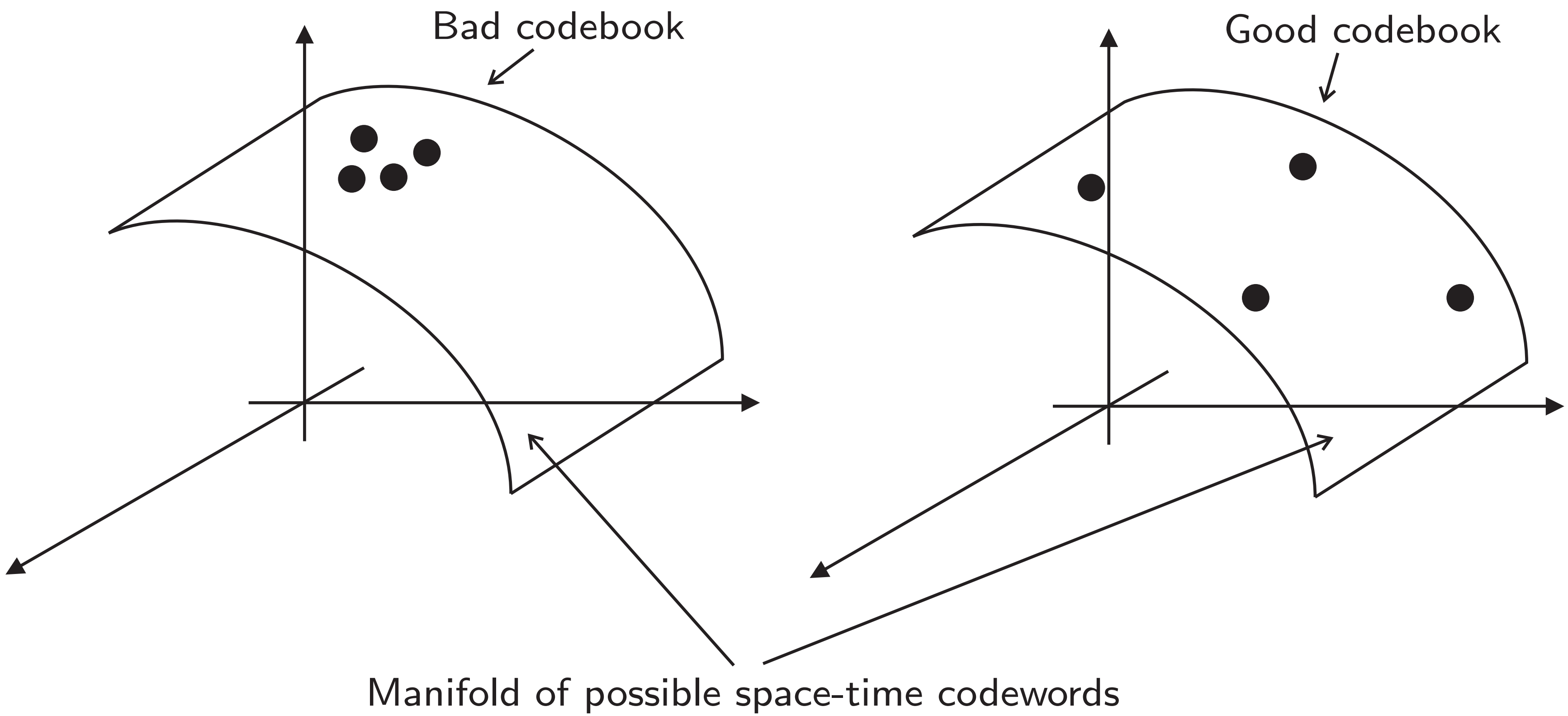
$$\mathcal{M} = \{(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_K) : \mathrm{tr}(\boldsymbol{X}_k^H \boldsymbol{X}_k) = 1\}$$




▷ Contribution: design codebook when $\boldsymbol{H}$ deterministic, unknown and vec $(\boldsymbol{E}) \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{\Upsilon})$ (colored noise)

## Problem Formulation

▷ Designing optimal codebooks = optimizing over a manifold



# Problem Formulation

▷ GLRT receiver:

$$\widehat{k} = \underset{k=1,2,\ldots,K}{\text{argmax}} \; p(\boldsymbol{y}|\boldsymbol{X}_k, \widehat{\boldsymbol{g}}_k)$$
$$= \underset{k=1,2,\ldots,K}{\text{argmin}} \; ||\boldsymbol{y} - \widetilde{\boldsymbol{X}_k}\widehat{\boldsymbol{g}}_k||^2_{\boldsymbol{\Upsilon}^{-1}}$$

$\widetilde{\boldsymbol{X}_k} = \boldsymbol{I}_N \otimes \boldsymbol{X}_k, \quad \widehat{\boldsymbol{X}_k} = \boldsymbol{\Upsilon}^{-\frac{1}{2}}\widetilde{\boldsymbol{X}_k},$

$\widehat{\boldsymbol{g}}_k = (\widehat{\boldsymbol{X}_k}^H\widehat{\boldsymbol{X}_k})^{-1}\widehat{\boldsymbol{X}_k}^H\boldsymbol{\Upsilon}^{-\frac{1}{2}}\boldsymbol{y}$ (ML channel estimate),

$\boldsymbol{y} = \text{vec}(\boldsymbol{Y})$

# Problem Formulation

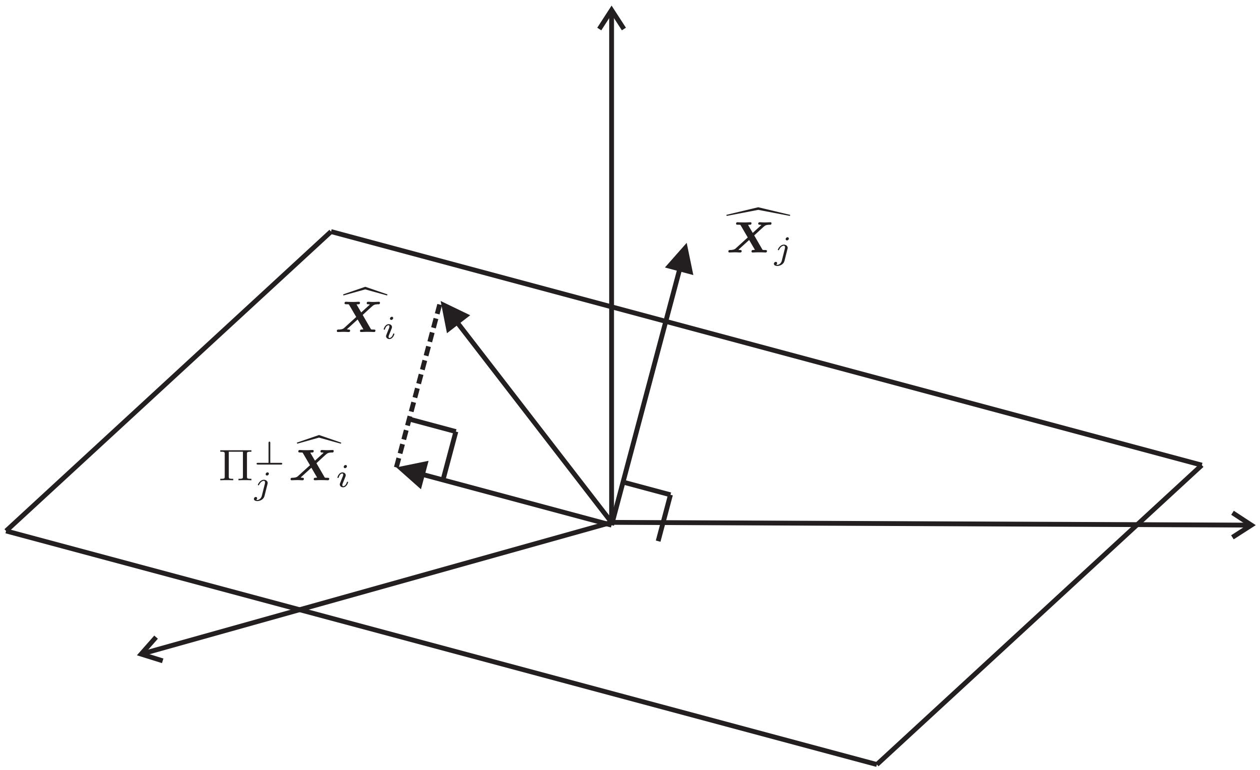
▷ PEP analysis: it can be shown that for high SNR

$$P_{\boldsymbol{X}_i \to \boldsymbol{X}_j} \quad = \quad \mathcal{Q}\left(\frac{1}{\sqrt{2}}\sqrt{\boldsymbol{g}^H\, \boldsymbol{L}_{ij}\boldsymbol{g}}\right) \leq \mathcal{Q}\left(\frac{1}{\sqrt{2}}\, ||\boldsymbol{g}||\, \sqrt{\lambda_{\min}(\boldsymbol{L}_{ij})}\right) \qquad (1)$$

where $\boldsymbol{g} = \text{vec}(\boldsymbol{H}^H)$, $\boldsymbol{L}_{ij}(\mathcal{C}) = \widehat{\boldsymbol{X}}_i^{\,H} \underbrace{\left(\boldsymbol{I}_T - \widehat{\boldsymbol{X}}_j \left(\widehat{\boldsymbol{X}}_j^{\,H} \widehat{\boldsymbol{X}}_j\right)^{-1} \widehat{\boldsymbol{X}}_j^{\,H}\right)}_{\boldsymbol{\Pi}_j^{\perp}} \widehat{\boldsymbol{X}}_i$

## Codebook design : geometrical interpretation

▷ $\widehat{\boldsymbol{X}}_i$ should lie in the orthogonal complement of span$\{\widehat{\boldsymbol{X}_j}\}$




▷ $f(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_K) = f(\boldsymbol{X}_1 e^{i\theta_1}, \ldots, \boldsymbol{X}_K e^{i\theta_K})$ : packing in complex projective space

# Problem Formulation

▷ Optimization problem: result (1) suggests the codebook merit function

$$\mathcal{C}^* = \underset{\mathcal{C} \in \mathcal{M}}{\arg\max} \quad \underbrace{\mathsf{min}\{\lambda_{\min}(\boldsymbol{L}_{ij}(\mathcal{C})) : 1 \leq i \neq j \leq K\}}_{f(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_K)} \tag{2}$$

▷ The problem in (2) is a high-dimensional, non-linear and non-smooth optimization problem!

e.g. for $K = 256$, $T = 8$, $M = 2$: $K(K-1) = 65280$ $\boldsymbol{L}_{ij}(\mathcal{C})$ functions and $2KTM = 8192$ real variables to optimize

## Codebook Construction

▷ Two-phase methodology to tackle the optimization problem in (2)

▷ Phase 1: solves a convex semi-definite programming (SDP) relaxation

▷ Incremental approach: Let $\mathcal{C}^*_{k-1} = \{\boldsymbol{X}^*_1, ..., \boldsymbol{X}^*_{k-1}\}$ be the codebook at the $k-1\underline{^{th}}$ stage. The new codeword is found by solving

$$\boldsymbol{X}^*_k = \underset{\mathrm{tr}(\boldsymbol{X}_k^H \boldsymbol{X}_k)=1}{\arg\max} \; \min_{1 \le i \le k-1} \{\lambda_{\min}(\boldsymbol{L}_{ik}),\, \lambda_{\min}(\boldsymbol{L}_{ki})\} \qquad (3)$$

for $k = 2, ..., K$

## Codebook Construction - Phase 1

▷ The optimization problem (3) is equivalent to

$$(\mathfrak{X}_k^*, \mathsf{vec}(\boldsymbol{X}_k{}^*), t^*) = \quad \arg\max \quad t \tag{4}$$

subject to

$$\begin{aligned} \mathsf{LMI}_{A_m}\,(\mathfrak{X}_k, \mathsf{vec}(\boldsymbol{X}_k), t) \succeq \mathbf{0},\ m = 1, ..., k-1 \\ \mathsf{LMI}_{B_m}\,(\mathfrak{X}_k, \mathsf{vec}(\boldsymbol{X}_k), t) \succeq \mathbf{0},\ m = 1, ..., k-1 \\ \mathsf{tr}(\mathfrak{X}_k) = 1, \ \ \mathfrak{X}_k = \mathsf{vec}(\boldsymbol{X}_k)\mathsf{vec}^H(\boldsymbol{X}_k) \end{aligned} \tag{5}$$

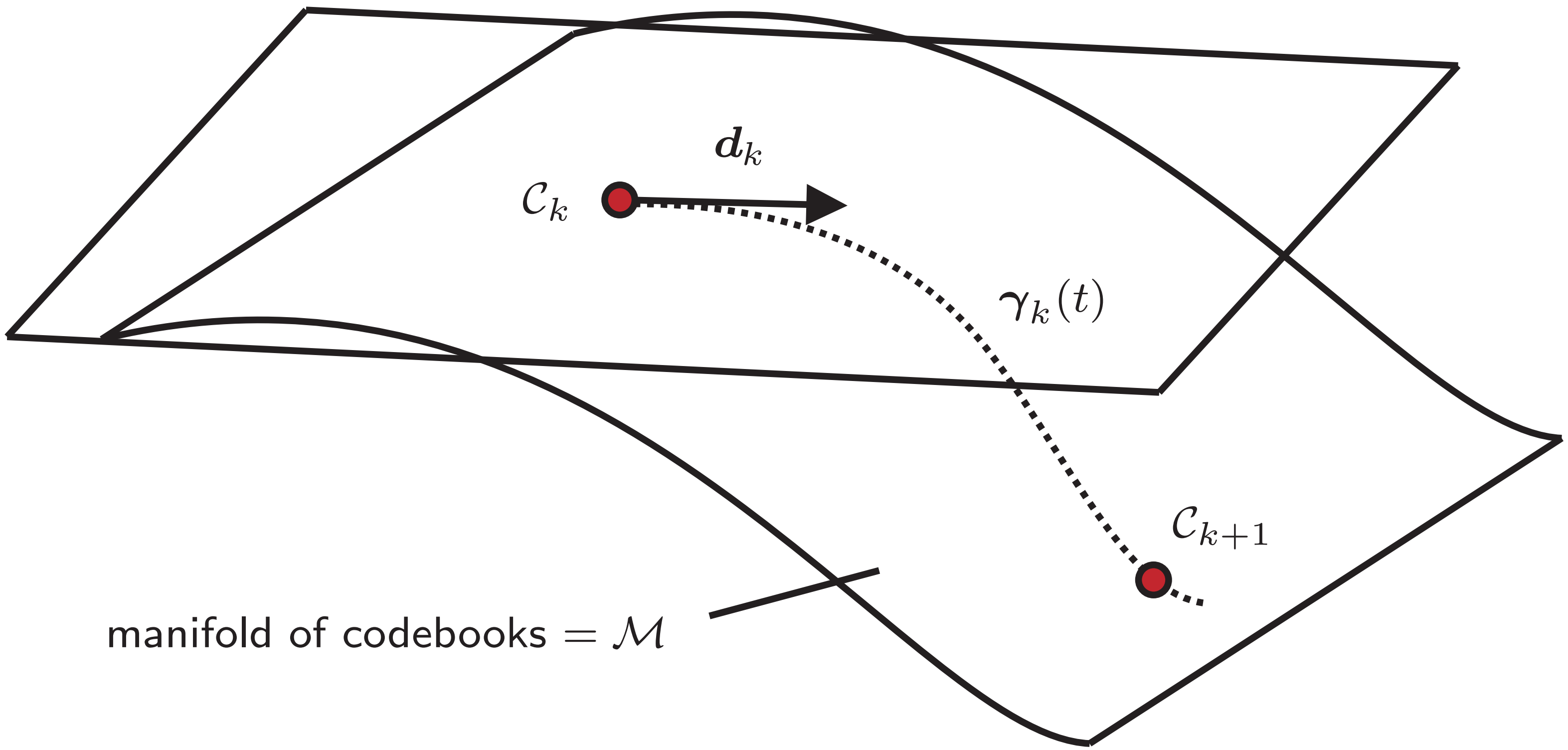
where the abbreviations $\mathsf{LMI}_{A_m}$ and $\mathsf{LMI}_{B_m}$ denote linear matrix inequalities in the variables $(\mathfrak{X}_k, \mathsf{vec}(\boldsymbol{X}_k), t)$

## Codebook Construction - Phase 1

- ▷ Design of the codewords: high-dimensional difficult nonlinear optimization problem (rank condition in (5))
- ▷ Relaxing the rank constraint leads to an SDP
- ▷ The $k\underline{^{th}}$ codeword is extracted from the output variable $\mathfrak{X}_k^*$ by randomizations
- ▷ Initialization $\boldsymbol{X}_1^*$: randomly generated, filling columns of the matrix with eigenvectors associated to the smallest eigenvalues of the noise covariance matrix,etc.

# Codebook Construction - Phase 2

▷ Phase 2: optimizes a non-smooth function on a manifold



## Codebook Construction - Phase 2

- ▷ Iterative algorithm, called GDA (geodesic descent algorithm)
- ▷ Identify "active" pairs $(i, j)$ that attain minimum
- ▷ Check if there is an ascent direction $d_k \in T_{\mathcal{C}_k}\mathcal{M}$ for all active $(i, j)$ (consists of solving LP)
- ▷ When $d_k$ is found, perform Armijo rule along geodesic $\boldsymbol{\gamma}_k(t)$
- ▷ If no $d_k$ is found, the algorithm stops

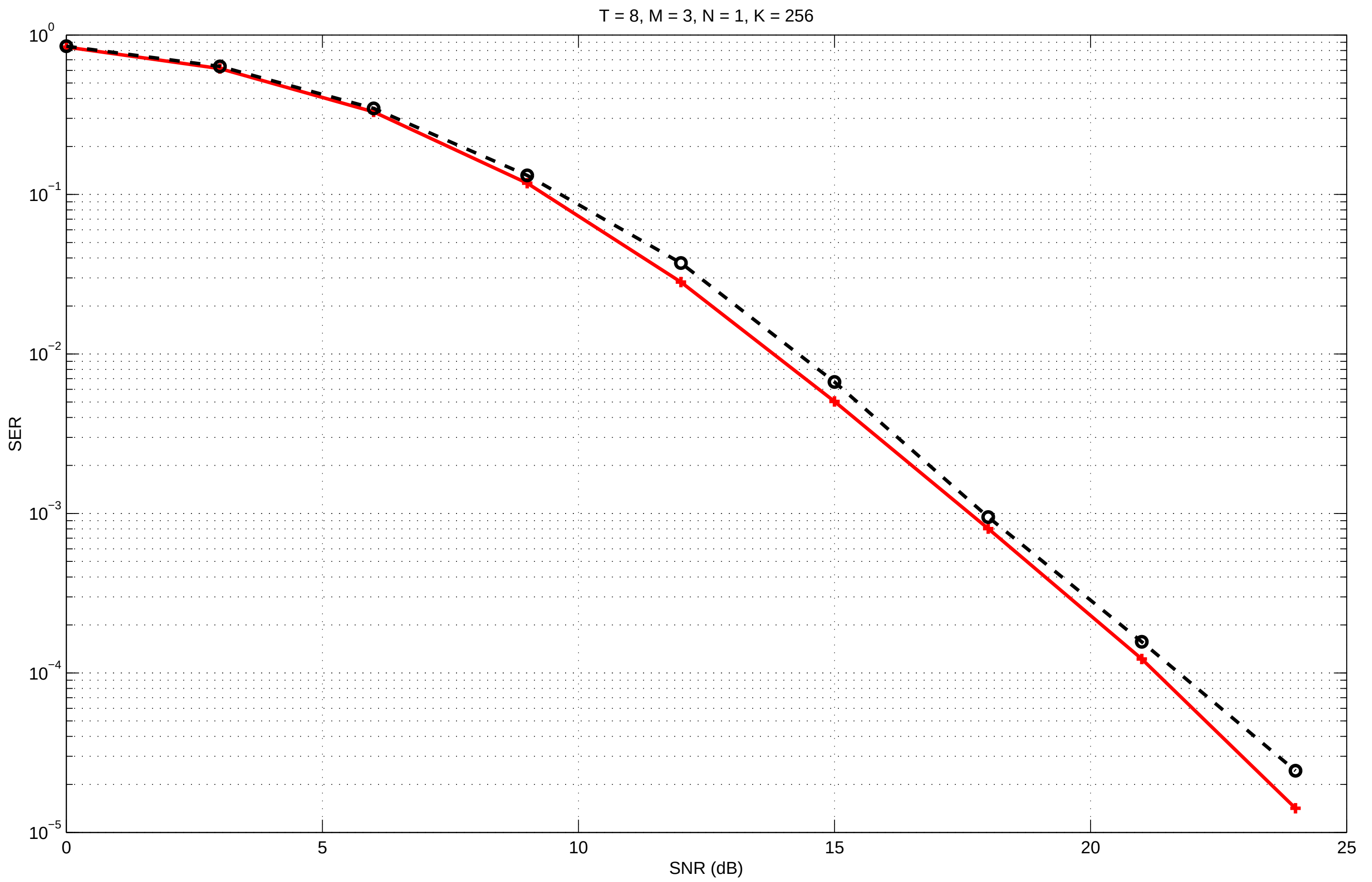
# Computer Simulations

▷ Noise correlation scenarios:

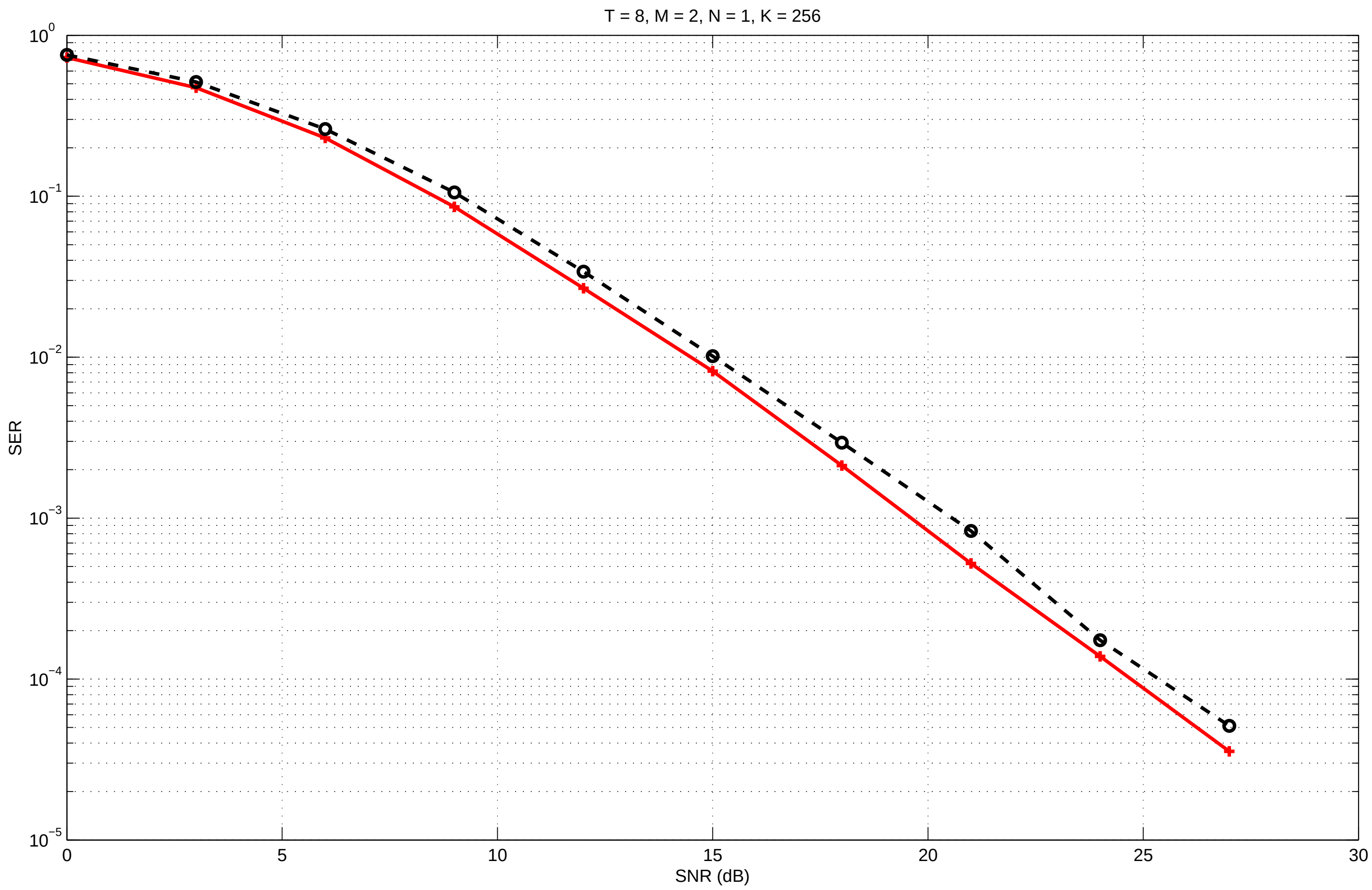
Category 1 - spatio-temporally white observation noise: $\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT}$

Category 2 - spatially white - temporally colored: $\boldsymbol{\Upsilon} = \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$

Category 3 - $\boldsymbol{E} = \boldsymbol{s}\,\boldsymbol{\alpha}^T + \boldsymbol{E}_{\text{temp}}$; $\boldsymbol{\Upsilon} = \boldsymbol{\alpha}\boldsymbol{\alpha}^H \otimes \boldsymbol{\Upsilon}_s + \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$




**Category 1 - spatio-temporally white observation noise: $\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT}$**




**Category 1 - spatio-temporally white observation noise: $\Upsilon = I_{NT}$**

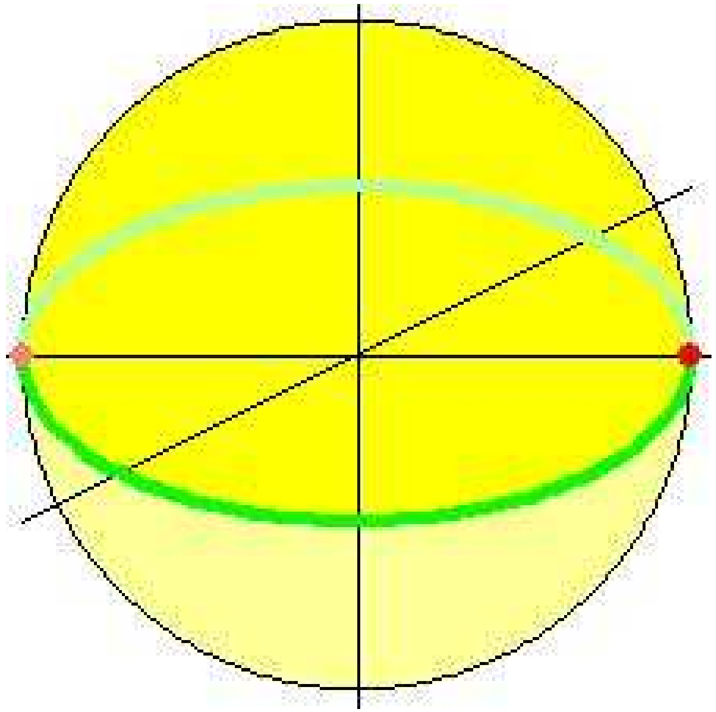
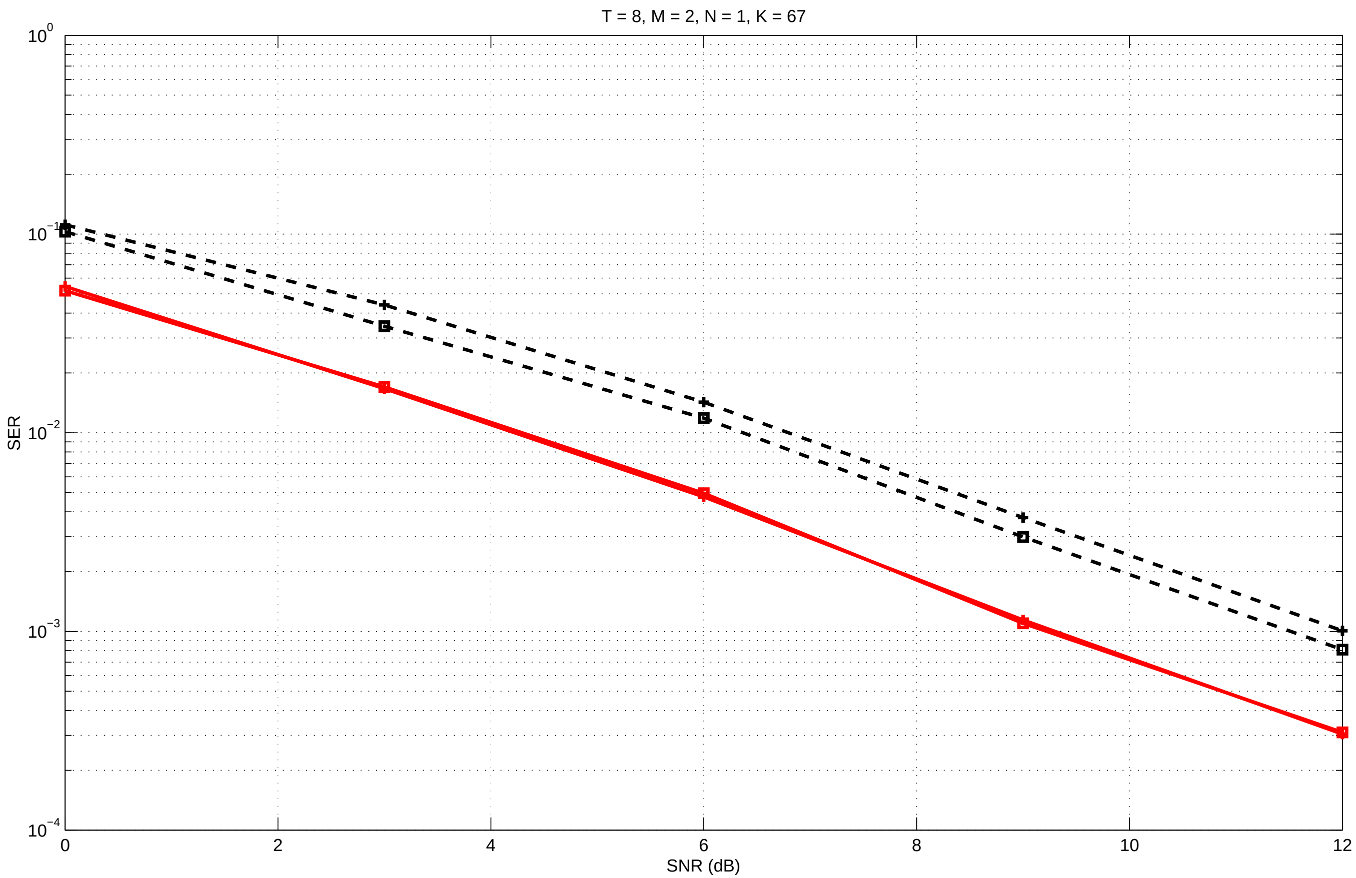


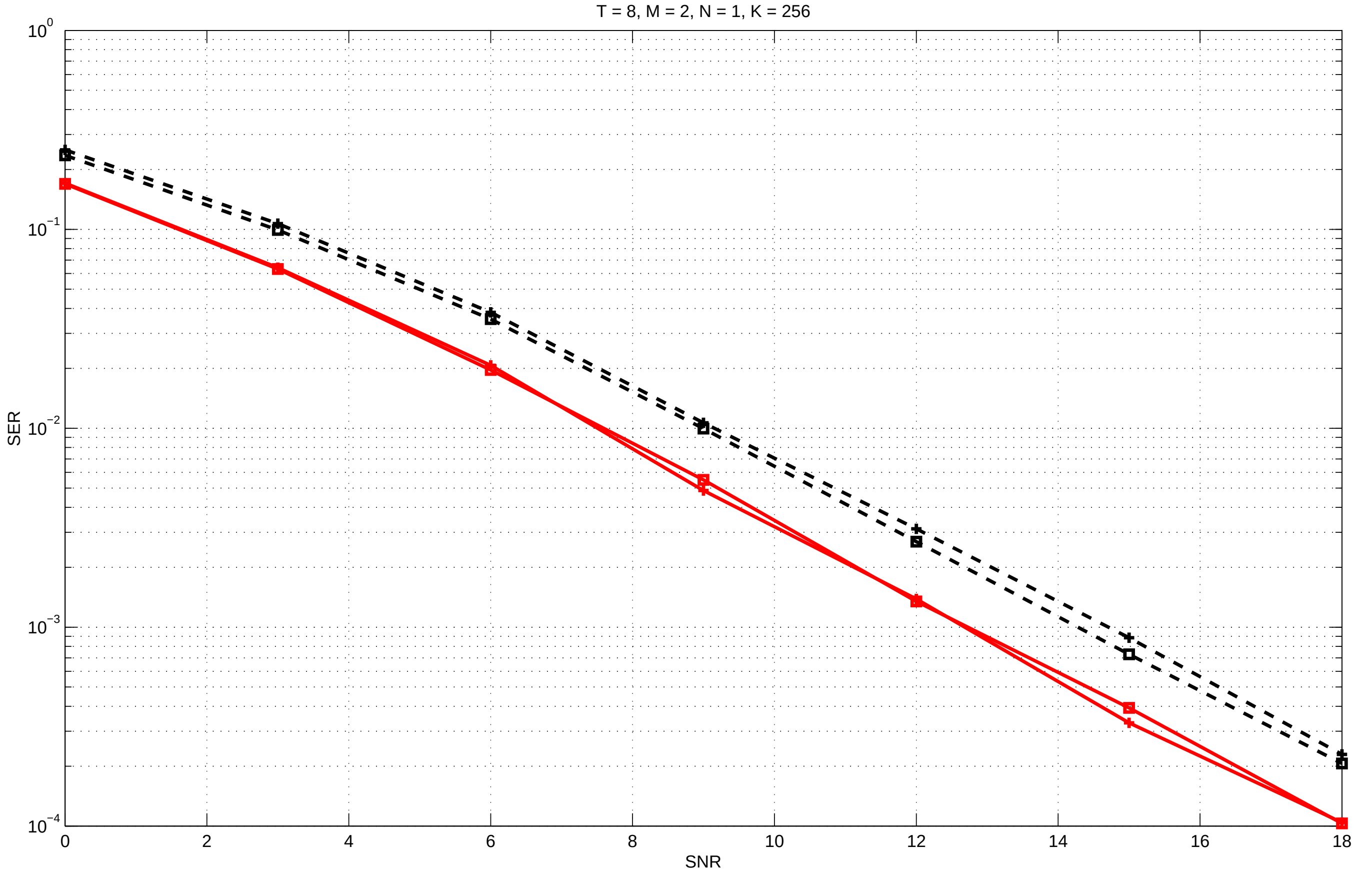
Figure 1: Real projective space $\mathbb{P}^2(\mathbb{R})$, $M = 1$, $T = 3$, $\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT}$

| | | PACKING RADII (DEGREES) | | |
|---|---|---|---|---|
| $T$ | $K$ | MB | JAT | Rankin |
| 4 | 8 | 67.79 | 67.78 | 67.79 |
| 5 | 11 | 69.73 | 69.71 | 69.73 |
| 5 | 21 | 66.42 | 65.83 | 66.42 |
| 6 | 9 | 75.52 | – | 75.52 |
| 6 | 11 | 73.22 | – | 73.22 |
| 6 | 12 | 72.45 | – | 72.45 |
| 6 | 16 | 70.53 | – | 70.53 |

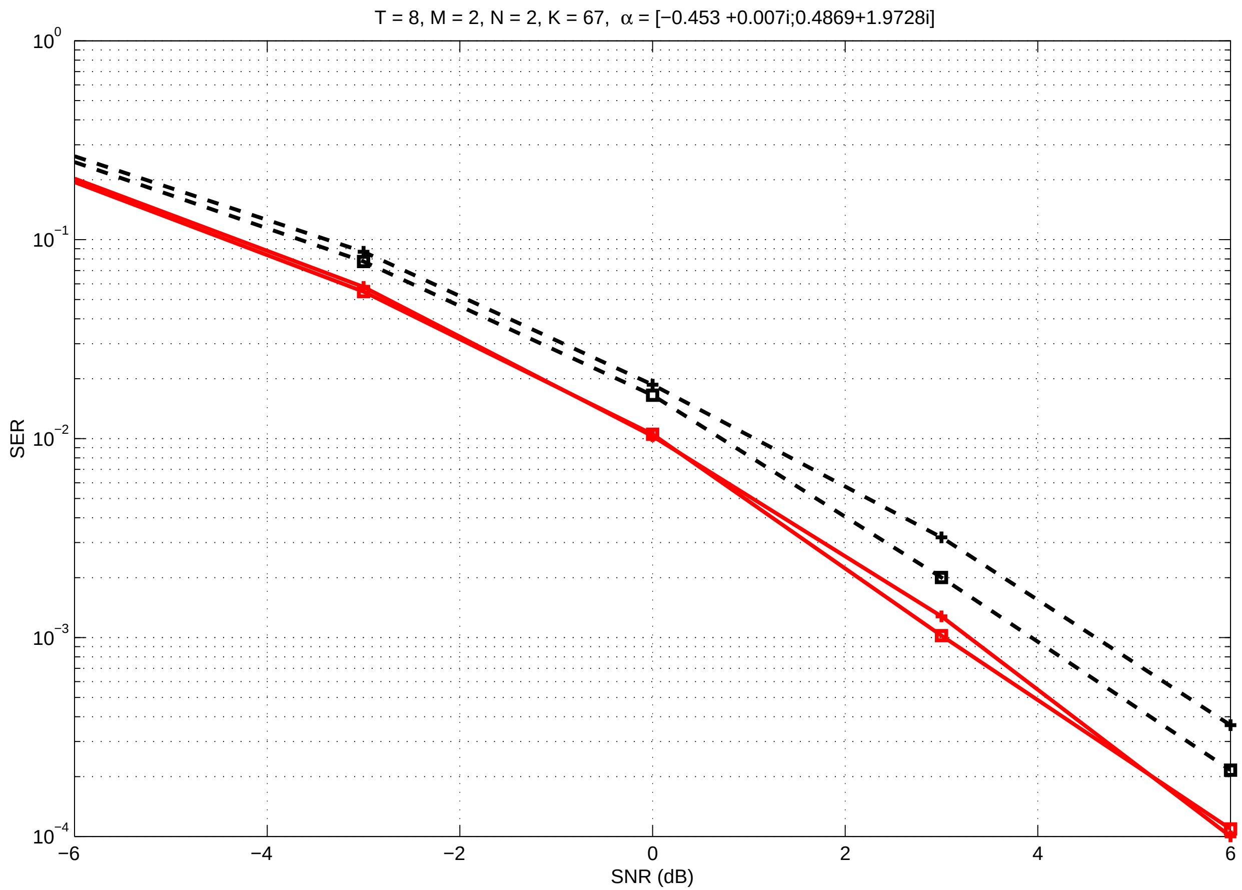
Table 1: PACKING IN COMPLEX PROJECTIVE SPACE: We compare our best configurations (MB) of $K$ points in $\mathbb{P}^{T-1}(\mathbb{C})$ against the Tropp codes (JAT) and Rankin bound.




**Category 2 - spatially white - temporally colored:** $\boldsymbol{\Upsilon} = \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$,
$\boldsymbol{\rho} = [1; 0.85; 0.6; 0.35; 0.1; \text{zeros}(3,1)]$




**Category 2 - spatially white - temporally colored:** $\boldsymbol{\Upsilon} = \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$,
$\boldsymbol{\rho} = [1; 0.8; 0.5; 0.15; \text{zeros}(4,1)]$




**Category 3** - $\boldsymbol{E} = \boldsymbol{s}\,\boldsymbol{\alpha}^T + \boldsymbol{E}_{\text{temp}}$; $\boldsymbol{\Upsilon} = \boldsymbol{\alpha}\boldsymbol{\alpha}^H \otimes \boldsymbol{\Upsilon}_s + \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$: $\boldsymbol{s}$=[ 1; 0.8; 0.5; 0.15; zeros(4,1) ], $\boldsymbol{\rho} = [1; 0.7; 0.4; 0.15; \text{zeros}(4,1)]$

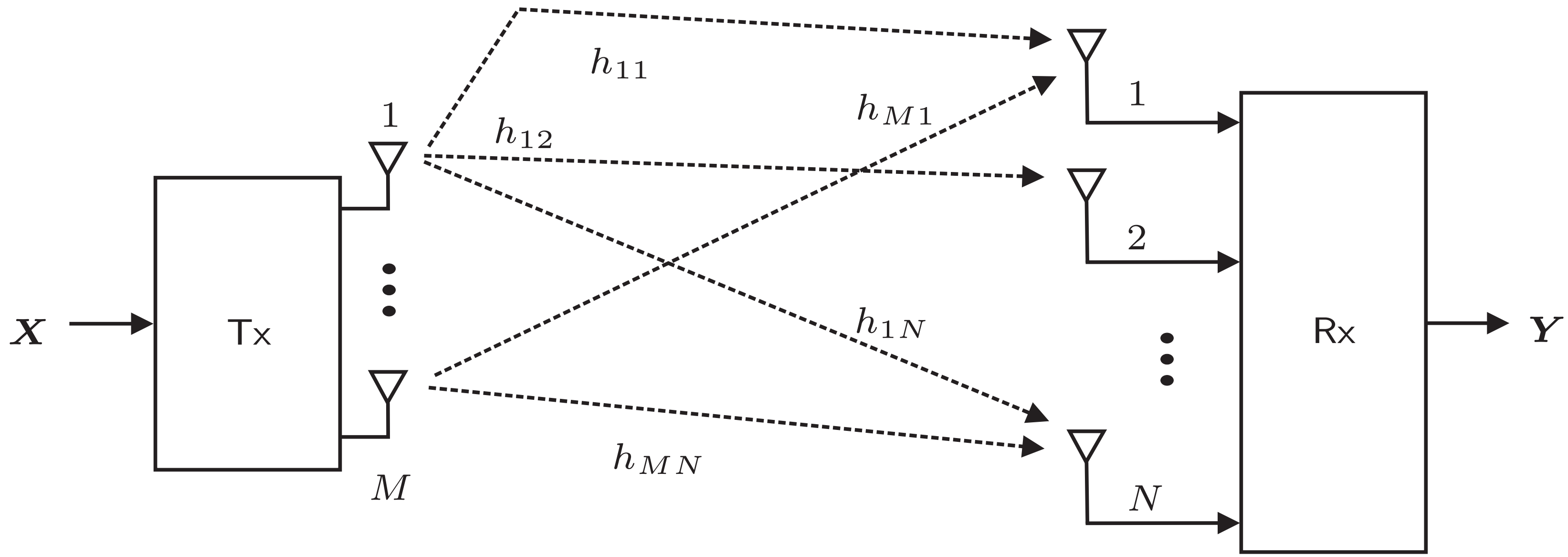
# Conclusions

- Codebook design for noncoherent setup
  - $\boldsymbol{H}$ deterministic, unknown
  - Colored noise: $\text{vec}\,(\boldsymbol{E}) \sim \mathcal{CN}\,(\mathbf{0}, \boldsymbol{\Upsilon})$
- Results
  - outperform significantly unitary constellations for colored noise case
  - small gain for white noise case
  - provide good packings for complex projective space ($M = 1$)
    (near bound performance)
  - for some cases actual Equiangular Tight Frames (ETF's)
- Publications
  - conference paper in IEEE ICASSP'2006
  - journal paper in IEEE Transactions on Signal Processing $2007$

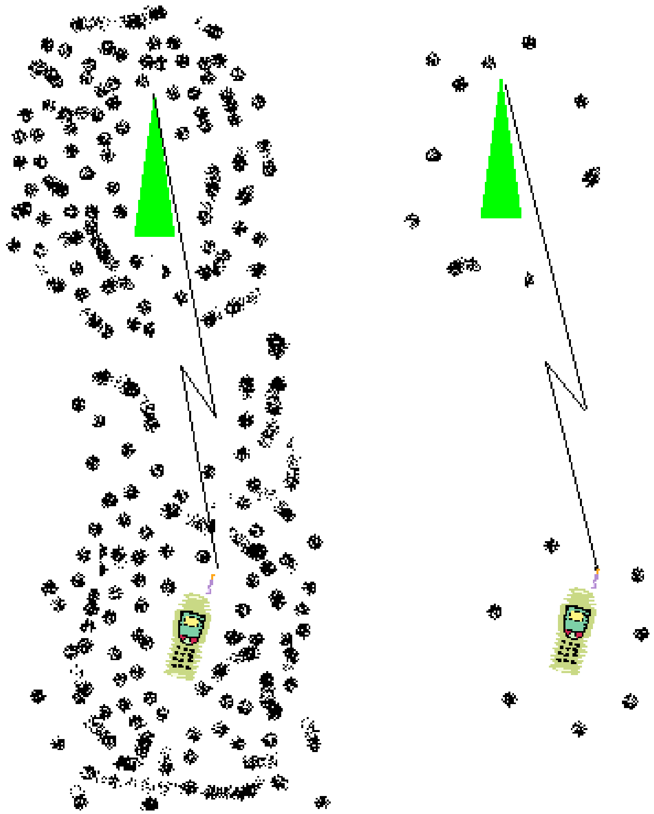
# Chapter 3: Low SNR regime – random channel

## Problem Formulation

▷ Data model: $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{H}^H + \boldsymbol{E}$




▷ Contribution: mutual information analysis for on-off and Gaussian signaling when $\boldsymbol{H}^H = \sqrt{\frac{\rho}{M}}\boldsymbol{K}_t^{\frac{1}{2}}\boldsymbol{H}_w\left(\boldsymbol{K}_r^T\right)^{\frac{1}{2}}$ and $\text{vec}\left(\boldsymbol{E}\right) \sim \mathcal{CN}\left(\mathbf{0}, \boldsymbol{\Upsilon}\right)$ (colored noise)



$\boldsymbol{K}_t = \boldsymbol{I}_M$, $\boldsymbol{K}_r = \boldsymbol{I}_N$ (left) $\boldsymbol{K}_t$,$\boldsymbol{K}_r$ rank deficient (right)

# Mutual information: on-off signaling

▷ The on-off signaling: for any $\epsilon > 1$ and assuming $\rho < 1$,

$$\boldsymbol{X} = \begin{cases} \boldsymbol{X}_{on}\rho^{-\frac{\epsilon}{2}} & \text{; w.p. } \rho^{\epsilon} \\ \boldsymbol{0} & \text{; w.p. } 1 - \rho^{\epsilon} \end{cases}$$

▷ At sufficiently low SNR

$$I(\boldsymbol{Y};\boldsymbol{X}) = \frac{\rho}{M} \operatorname{tr}\left(\boldsymbol{\Upsilon}^{-1}\left(\boldsymbol{K}_r \otimes \boldsymbol{X}_{on}\boldsymbol{K}_t\boldsymbol{X}_{on}^H\right)\right) + o(\rho), \tag{6}$$

▷ We maximize $I(\boldsymbol{Y};\boldsymbol{X})$ in (6) w.r.t $\boldsymbol{X}_{on}$, $\boldsymbol{K}_t$ and $\boldsymbol{K}_r$

# Mutual information: on-off signaling

▷ The maximum in (6) is attained by

$$\widehat{\boldsymbol{X}}_{on} = \sqrt{TM}\begin{bmatrix}\hat{\boldsymbol{x}} & \mathbf{0}_{T\times(M-1)}\end{bmatrix},\ \widehat{\boldsymbol{K}}_r = N\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}^H,\ \widehat{\boldsymbol{K}}_t(i,i) = M\delta_{i1} \tag{7}$$

where

$$(\hat{\boldsymbol{u}},\hat{\boldsymbol{x}}) = \underset{\substack{\boldsymbol{u}\in\mathbb{C}^N,\,||\boldsymbol{u}||=1\\ \boldsymbol{x}\in\mathbb{C}^T,\,||\boldsymbol{x}||=1}}{\arg\max}\ (\boldsymbol{u}\otimes\boldsymbol{x})^H\,\boldsymbol{\Upsilon}^{-1}\,(\boldsymbol{u}\otimes\boldsymbol{x}) \tag{8}$$

## Mutual information: on-off signaling

- ▷ The optimization problem in (8) always admits a solution (maximization of a continuous function over a compact set)
- ▷ For the choice in (7), the maximal mutual information (p.c.u) is equal to

$$\frac{1}{T} I(\boldsymbol{Y}; \boldsymbol{X}) = \rho\, N\, M \hat{\lambda} + o(\rho)$$

  where $\hat{\lambda} = (\hat{\boldsymbol{u}} \otimes \hat{\boldsymbol{x}})^H\, \boldsymbol{\Upsilon}^{-1}\, (\hat{\boldsymbol{u}} \otimes \hat{\boldsymbol{x}})$

- ▷ Conclusions:
  - From (7) we see that both $\boldsymbol{K}_t$ and $\boldsymbol{K}_r$ should be of rank one
  - Correlated Rayleigh fading channel is beneficial from capacity viewpoint. Gain of order $M$ with respect to uncorrelated Rayleigh fading channel
  - On-off signaling attains the known channel capacity
  - Correlation in noise is beneficial too, $\hat{\lambda} \geq 1$

## Mutual information: Gaussian modulation

▷ Gaussian modulation is a more realistic and practical case

▷ Let $\boldsymbol{x} = \text{vec}(\boldsymbol{X}) \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{P})$. At sufficiently low SNR

$$I(\boldsymbol{Y};\boldsymbol{X}) = \frac{\rho^2}{2M^2}\,\text{tr}\left(\mathsf{E}[\boldsymbol{Z}^2] - (\mathsf{E}[\boldsymbol{Z}])^2\right) + o(\rho^2) \tag{9}$$

where $\boldsymbol{Z} = \boldsymbol{\Upsilon}^{-\frac{1}{2}}\left(\boldsymbol{K}_r \otimes \boldsymbol{X}\boldsymbol{K}_t\boldsymbol{X}^H\right)\boldsymbol{\Upsilon}^{-\frac{1}{2}}$

▷ We maximize $I(\boldsymbol{Y};\boldsymbol{X})$ in (9) w.r.t $\boldsymbol{P}$, $\boldsymbol{K}_t$ and $\boldsymbol{K}_r$

## Mutual information: Gaussian modulation

▷ The maximum in (9) is attained by

$$\widehat{\boldsymbol{P}} = TM\boldsymbol{F}_1 \otimes \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^H,\ \widehat{\boldsymbol{K}}_r = N\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}^H,\ \widehat{\boldsymbol{K}}_t(i,i) = M\delta_{i1} \tag{10}$$

where

$$(\hat{\boldsymbol{u}}, \hat{\boldsymbol{x}}) = \underset{\substack{\boldsymbol{u} \in \mathbb{C}^N, ||\boldsymbol{u}|| = 1 \\ \boldsymbol{x} \in \mathbb{C}^T, ||\boldsymbol{x}|| = 1}}{\arg\max} (\boldsymbol{u} \otimes \boldsymbol{x})^H \boldsymbol{\Upsilon}^{-1} (\boldsymbol{u} \otimes \boldsymbol{x})$$

▷ The $M \times M$ matrix $\boldsymbol{F}_1$ has all the entries equal to zero except the entry (1,1) which is one

## Mutual information: Gaussian modulation

▷ For the choice in (10), the maximal mutual information (p.c.u) is equal to

$$\frac{1}{T} I(\boldsymbol{Y}; \boldsymbol{X}) = \frac{\rho^2}{2} N^2 \, T \, M^2 \, \hat{\lambda}^2 + o(\rho^2).$$

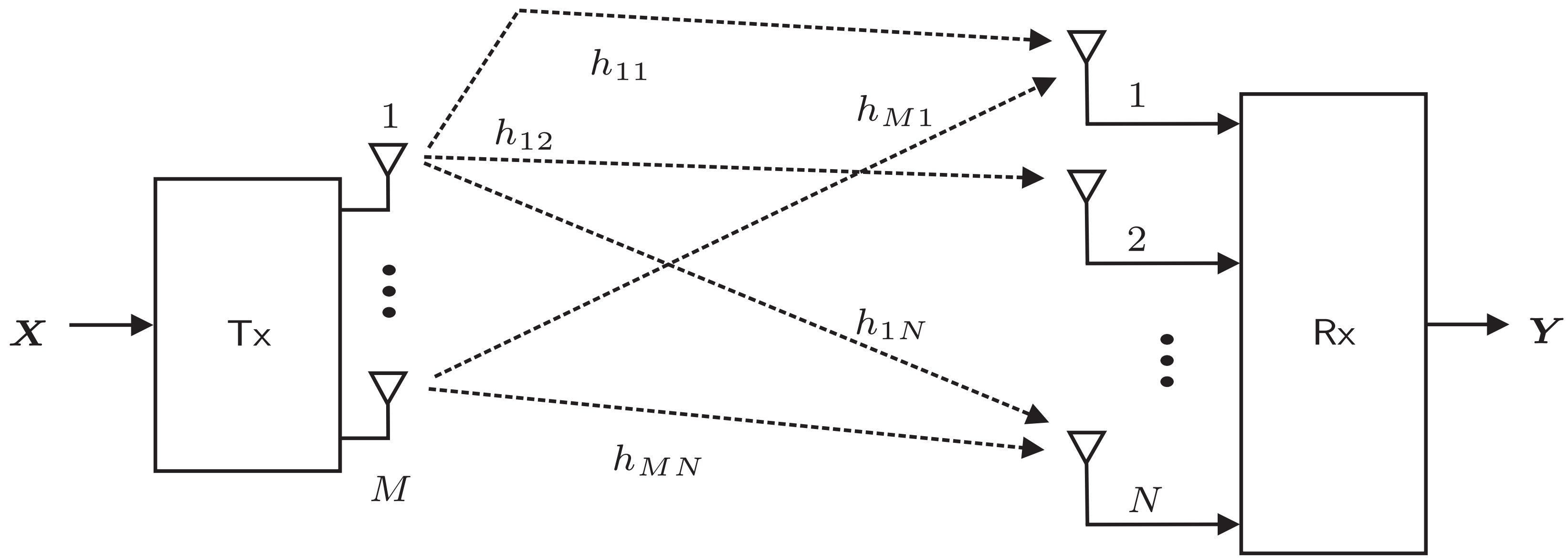
▷ Conclusions:

- From (10) we see that both $\boldsymbol{K}_t$ and $\boldsymbol{K}_r$ should be of rank one
- Correlated Rayleigh fading channel is beneficial from capacity viewpoint. Gain of order $M^2 N$ with respect to uncorrelated Rayleigh fading channel
- Correlation in noise is beneficial too, $\hat{\lambda} \geq 1$

# Chapter 3: Low SNR regime – deterministic channel

## Problem Formulation

▷ Data model: $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{H}^H + \boldsymbol{E}$




▷ Codebook : $\mathcal{C} = \{\boldsymbol{X}_1, \boldsymbol{X}_2, ..., \boldsymbol{X}_K\}$ is a point in the manifold

$$\mathcal{M} = \{(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_K) : \mathrm{tr}(\boldsymbol{X}_k^H \boldsymbol{X}_k) = 1\}$$

▷ Contribution: design codebook when $\boldsymbol{H}$ deterministic, unknown and $\mathrm{vec}\,(\boldsymbol{E}) \sim \mathcal{CN}\,(\mathbf{0}, \boldsymbol{\Upsilon})$ (colored noise)

# Problem Formulation

▷ GLRT receiver:

$$\widehat{k} = \underset{k=1,2,\ldots,K}{\text{argmax}} \; p(\boldsymbol{y}|\boldsymbol{X}_k, \widehat{\boldsymbol{g}}_k)$$
$$= \underset{k=1,2,\ldots,K}{\text{argmin}} \; ||\boldsymbol{y} - \widetilde{\boldsymbol{X}_k}\widehat{\boldsymbol{g}}_k||^2_{\boldsymbol{\Upsilon}^{-1}}$$

$\widetilde{\boldsymbol{X}_k} = \boldsymbol{I}_N \otimes \boldsymbol{X}_k, \quad \widehat{\boldsymbol{X}_k} = \boldsymbol{\Upsilon}^{-\frac{1}{2}}\widetilde{\boldsymbol{X}_k},$

$\widehat{\boldsymbol{g}}_k = (\widehat{\boldsymbol{X}_k}^H \widehat{\boldsymbol{X}_k})^{-1} \widehat{\boldsymbol{X}_k}^H \boldsymbol{\Upsilon}^{-\frac{1}{2}} \boldsymbol{y}$ (ML channel estimate),

$\boldsymbol{y} = \text{vec}\,(\boldsymbol{Y})$

# Problem Formulation

▷ PEP analysis: it can be shown that at low SNR and $T \geq 2M$

$$P_{\boldsymbol{X}_i \to \boldsymbol{X}_j} \approx \text{Prob}\left(Y > \boldsymbol{g}^H \boldsymbol{L}_{ij} \boldsymbol{g}\right), \tag{11}$$

with

$$\boldsymbol{L}_{ij} = \widehat{\boldsymbol{X}_i}^H \boldsymbol{\Pi}_j^{\perp} \widehat{\boldsymbol{X}_i}, \qquad \boldsymbol{\Pi}_j^{\perp} = \boldsymbol{I}_{TN} - \widehat{\boldsymbol{X}_j} \left(\widehat{\boldsymbol{X}_j}^H \widehat{\boldsymbol{X}_j}\right)^{-1} \widehat{\boldsymbol{X}_j}^H,$$

and

$$Y = \sum_{m=1}^{MN} \sin \alpha_m \left(|a_m|^2 - |b_m|^2\right) \text{ where } a_m, b_m \overset{iid}{\sim} \mathcal{CN}(0,1)$$

for $m = 1, \ldots, MN$. The angles $\alpha_m$ are the *principal angles* between the subspaces spanned by $\widehat{\boldsymbol{X}_i}$ and $\widehat{\boldsymbol{X}_j}$

## Problem Formulation

- PEP analysis: for $M = 1$ and $\boldsymbol{\Upsilon} = \boldsymbol{I}_{TN}$,

$$P_{\boldsymbol{x}_i \to \boldsymbol{x}_j} = P\left(\sum_{n=1}^{N} \left(|a_n|^2 - |b_n|^2\right) > ||\boldsymbol{h}||^2 \sin\alpha_{ij}\right) \tag{12}$$

  where $a_n, b_n \overset{iid}{\sim} \mathcal{CN}(0,1)$ and the angle $\alpha_{ij}$ is the acute angle between the codewords $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$

- In Chapter 2 the expression for the PEP in the high SNR regime, $M = 1$ and $\boldsymbol{\Upsilon} = \boldsymbol{I}_{TN}$ is given by

$$P_{\boldsymbol{x}_i \to \boldsymbol{x}_j} = \mathcal{Q}\left(\frac{1}{\sqrt{2}} ||\boldsymbol{h}|| \sin\alpha_{ij}\right) \tag{13}$$

  where $\mathcal{Q}(x) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt$

- Equations (12)-(13) confirm that the codewords $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ should be constructed as separate as possible

- The problem of constructing good codes corresponds to packing problem in the complex projective space

## Problem Formulation

▷ From (11), an upper bound on the PEP is readily found

$$P_{\boldsymbol{X}_i \to \boldsymbol{X}_j} \leq \text{Prob}\left(Z > ||\boldsymbol{g}||^2 \lambda_{\min}\left(\boldsymbol{L}_{ij}\right)\right), \tag{14}$$

where $Z = \sum_{m=1}^{MN} |a_m|^2$, $a_m \overset{iid}{\sim} \mathcal{CN}(0,1)$

▷ The codebook design criterion in (14) is equivalent to the one for the high SNR regime

$$\mathcal{C}^* = \underset{\mathcal{C} \in \mathcal{M}}{\arg\max} \quad \min\{\lambda_{\min}(\boldsymbol{L}_{ij}(\mathcal{C})) : 1 \leq i \neq j \leq K\}$$

## Computer Simulations: Constellations with uniform priors

▷ Noise correlation scenarios:

Category 1 - spatio-temporally white observation noise: $\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT}$

Category 2 - spatially white - temporally colored: $\boldsymbol{\Upsilon} = \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$

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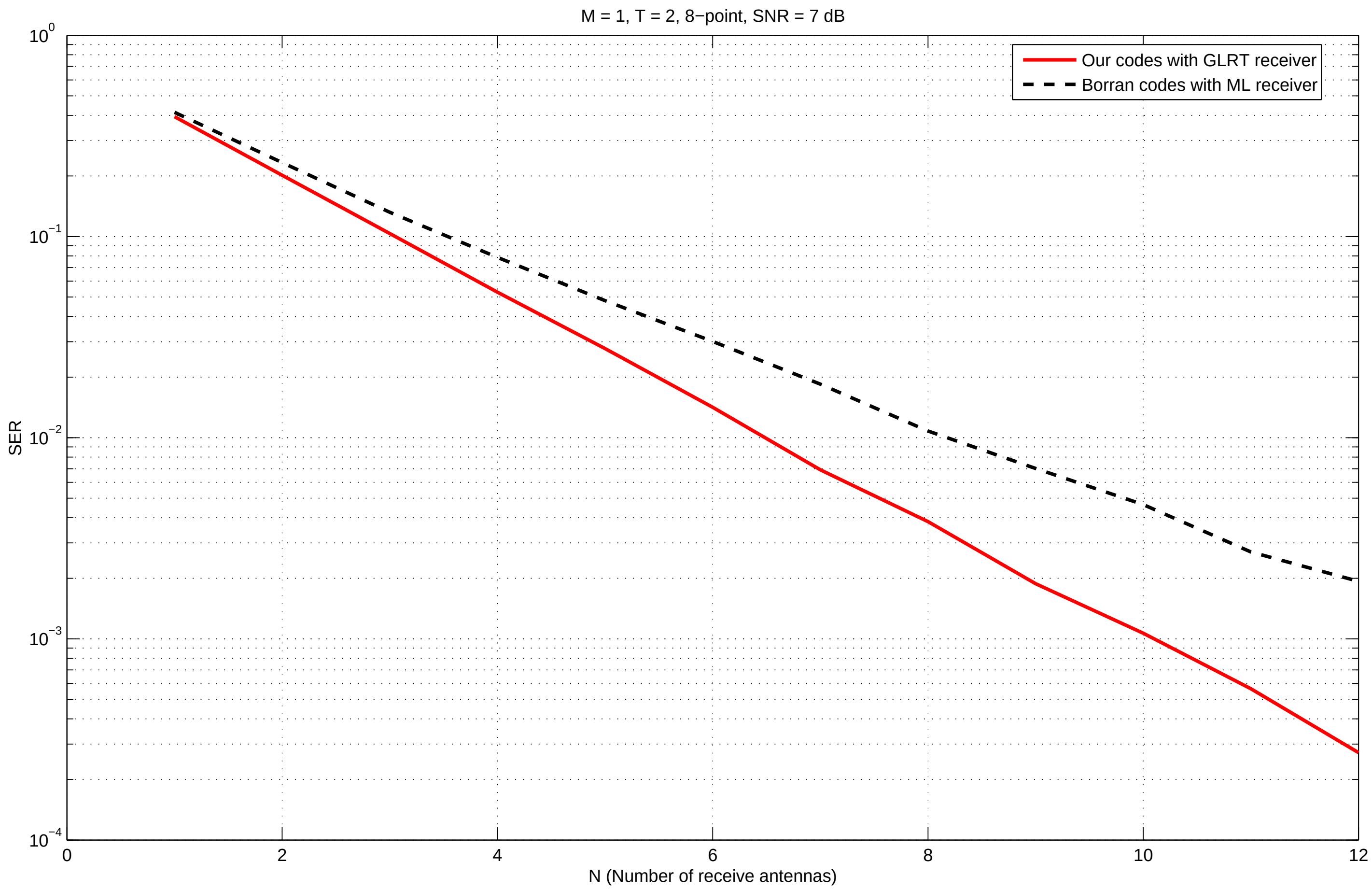
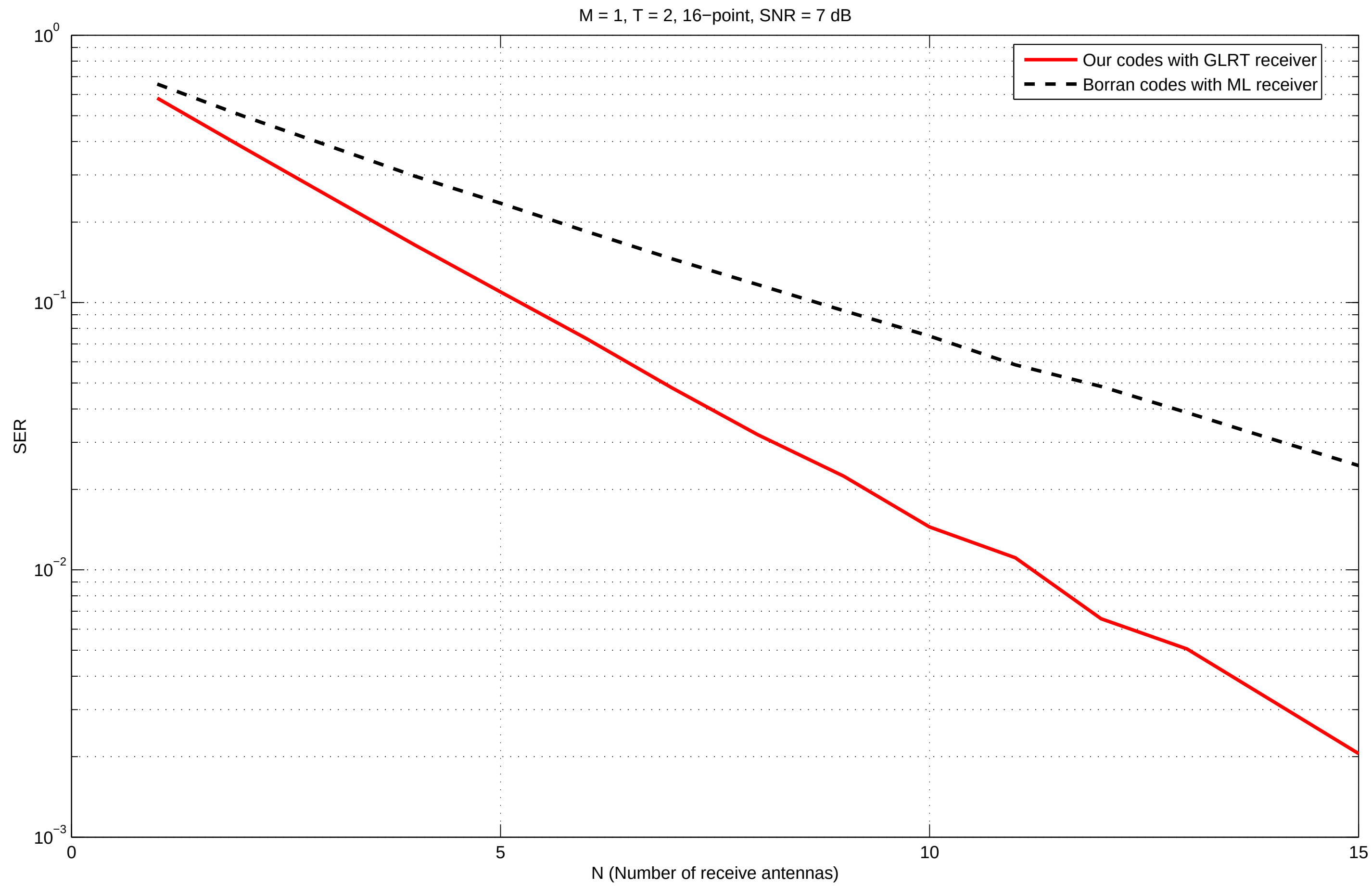



**Category 1 - spatio-temporally white observation noise: $\Upsilon = I_{NT}$**

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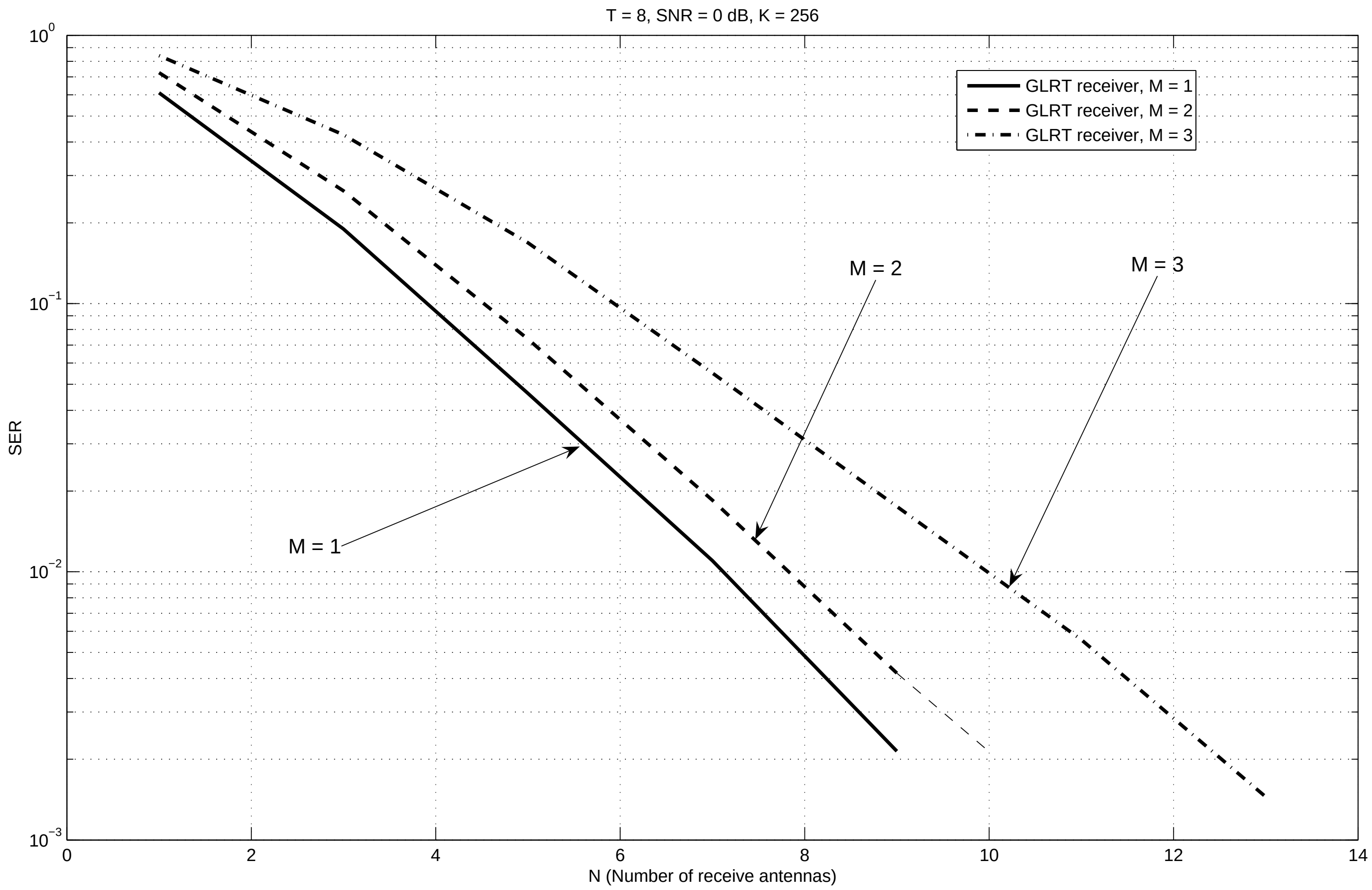


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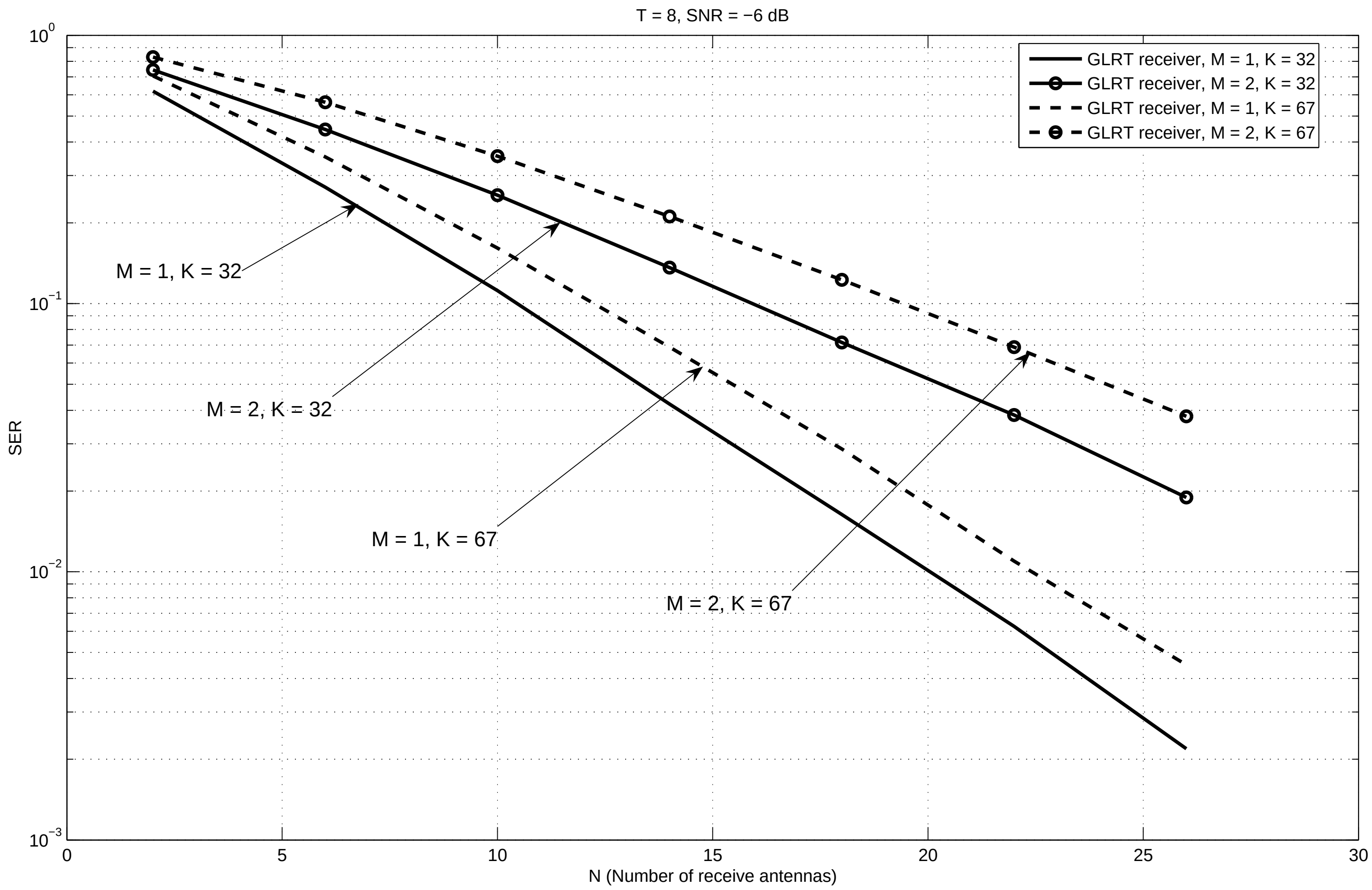
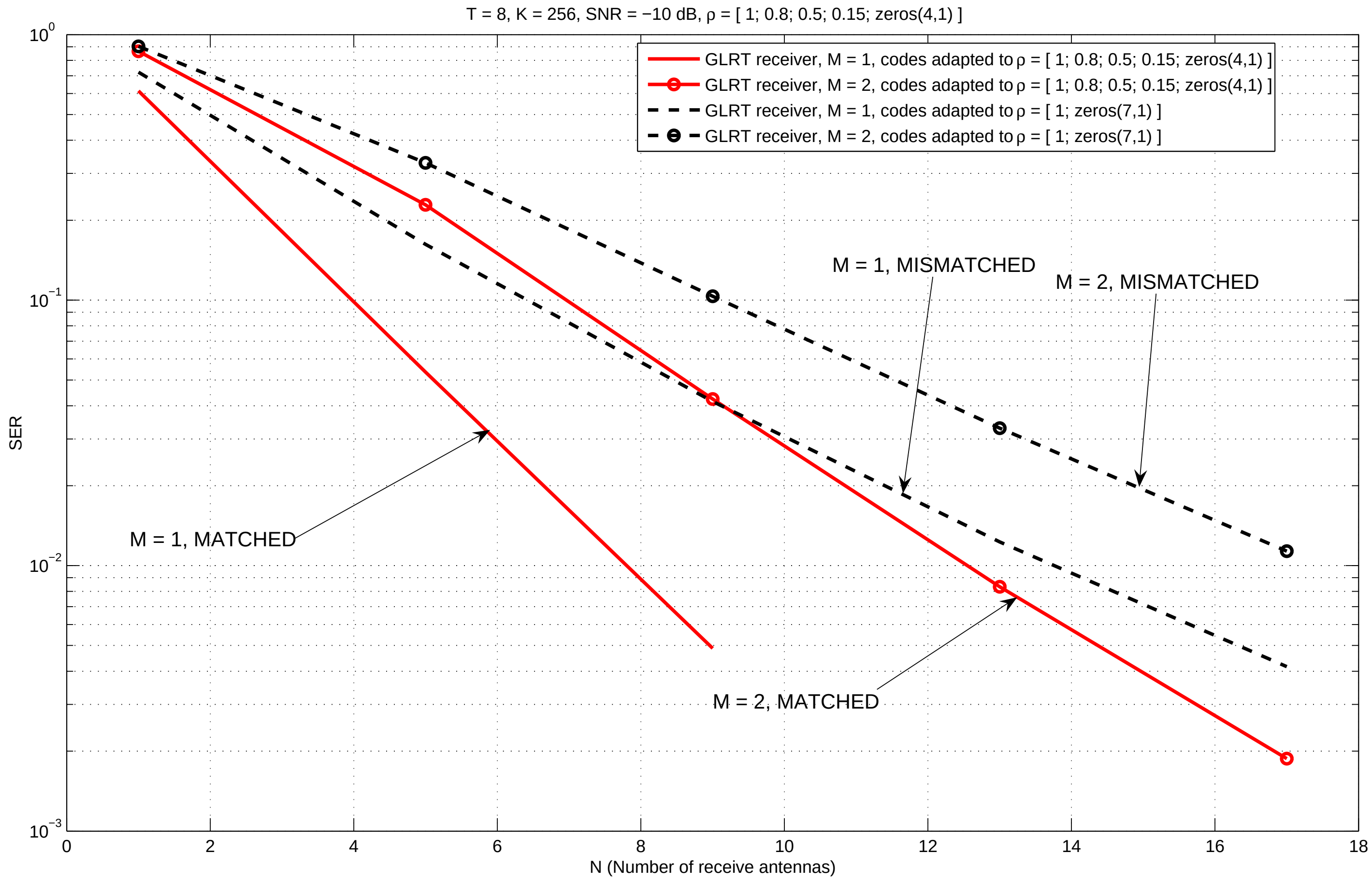
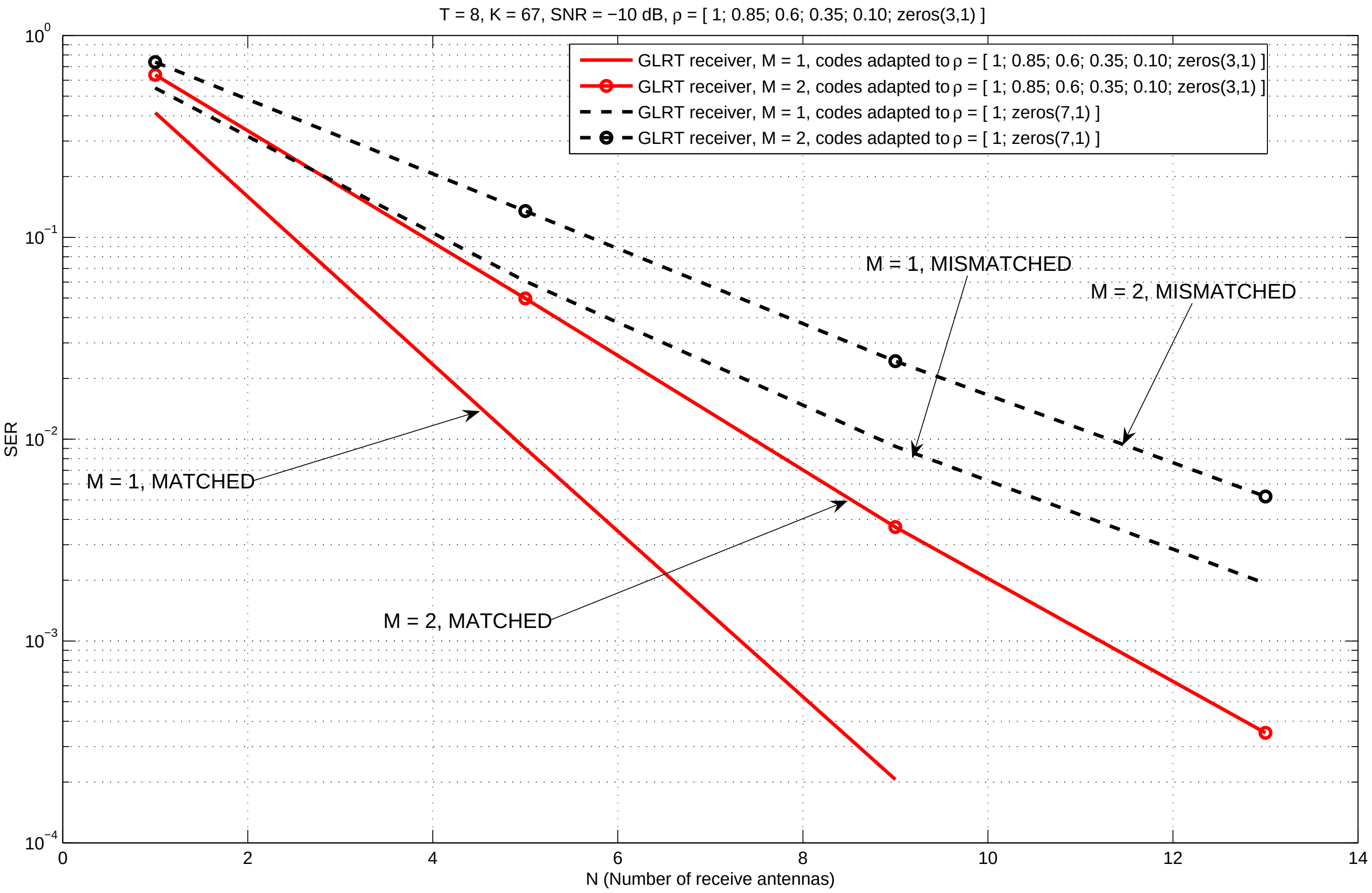


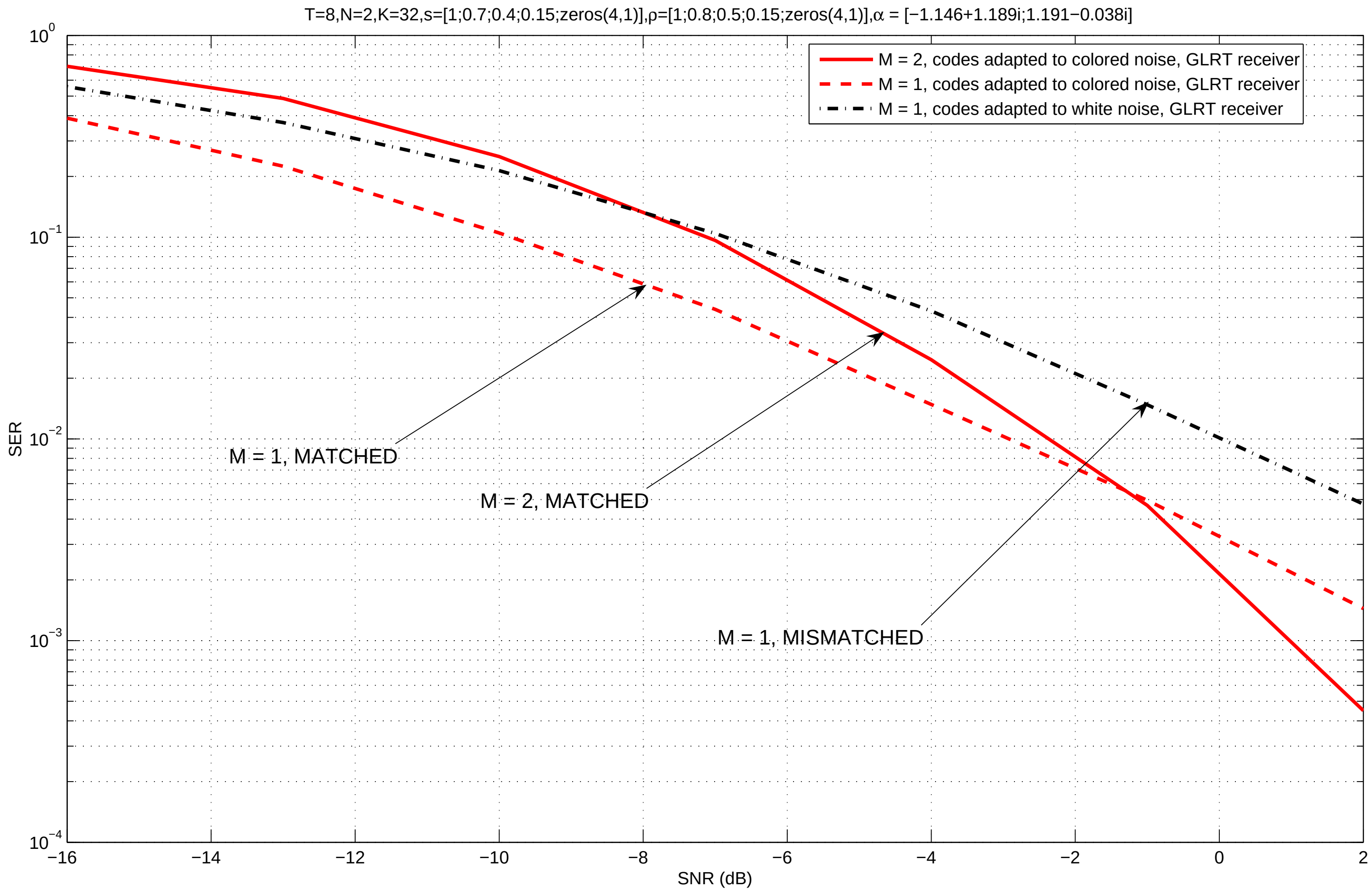

## Category 1 - spatio-temporally white observation noise: $\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT}$



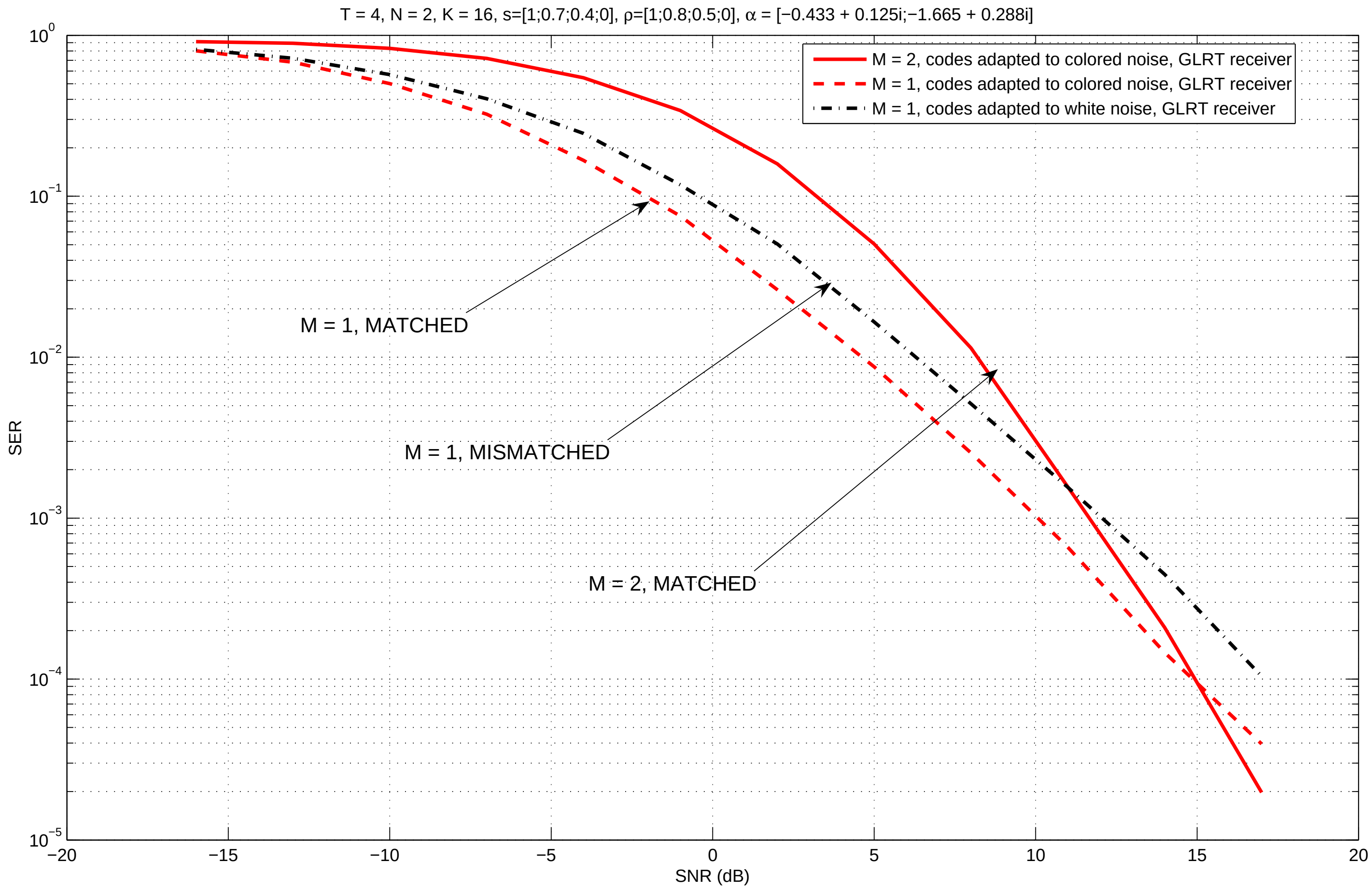

**Category 2 - spatially white - temporally colored:** $\boldsymbol{\Upsilon} = \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$




**Category 2 - spatially white - temporally colored:** $\boldsymbol{\Upsilon} = \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$




**Category 3 -** $\boldsymbol{E} = \boldsymbol{s}\,\boldsymbol{\alpha}^T + \boldsymbol{E}_{\text{temp}}$ **;** $\boldsymbol{\Upsilon} = \boldsymbol{\alpha}\boldsymbol{\alpha}^H \otimes \boldsymbol{\Upsilon}_s + \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$




**Category 3 -** $\boldsymbol{E} = \boldsymbol{s}\,\boldsymbol{\alpha}^T + \boldsymbol{E}_{\text{temp}}$ ; $\boldsymbol{\Upsilon} = \boldsymbol{\alpha}\boldsymbol{\alpha}^H \otimes \boldsymbol{\Upsilon}_s + \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$

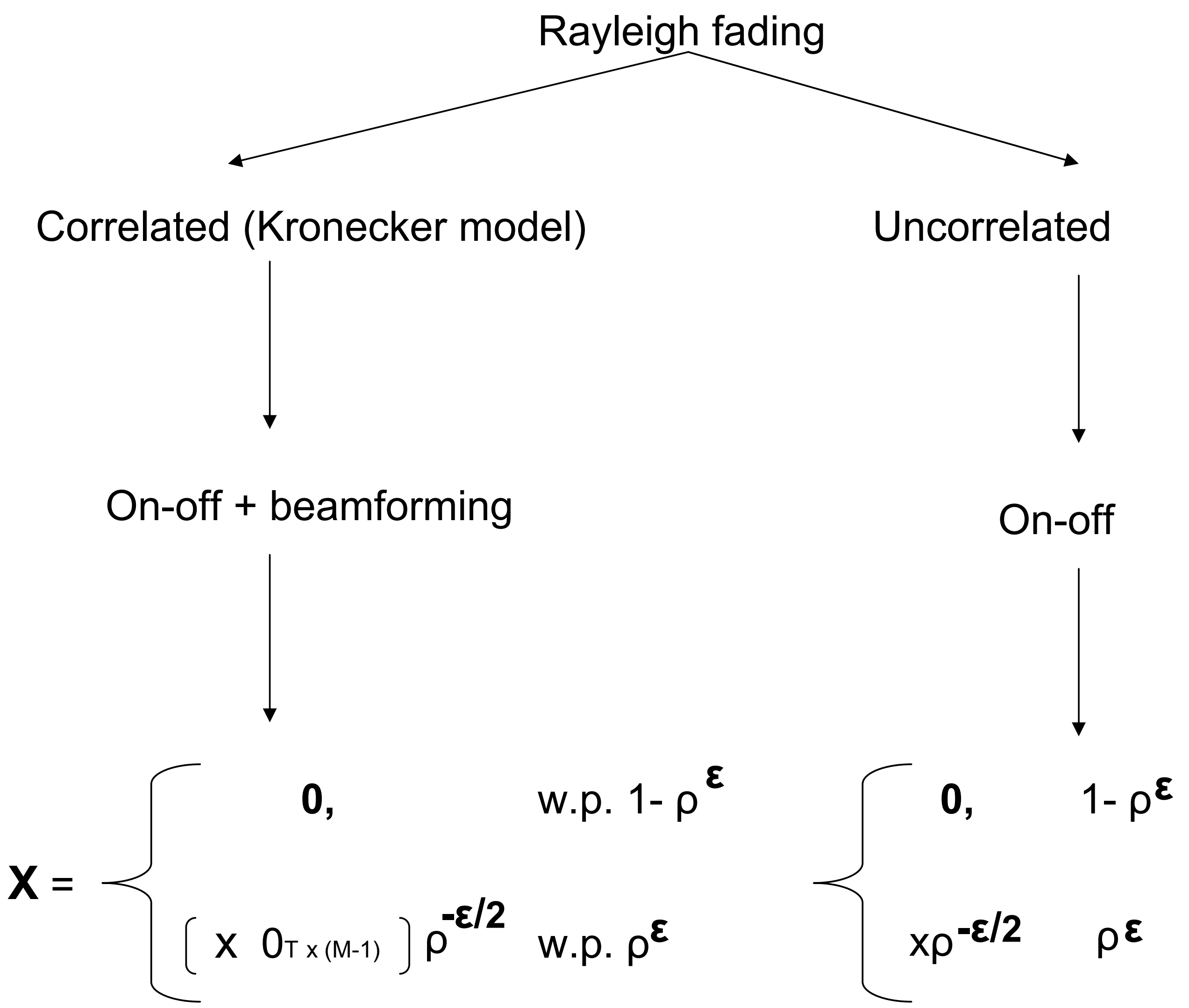
## Computer Simulations: Constellations with non-uniform priors

▷ Noise correlation scenarios:

Category 1 - spatio-temporally white observation noise: $\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT}$

Category 2 - spatially white - temporally colored: $\boldsymbol{\Upsilon} = \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$

Category 3 - $\boldsymbol{E} = \boldsymbol{s}\,\boldsymbol{\alpha}^T + \boldsymbol{E}_{\text{temp}}$; $\boldsymbol{\Upsilon} = \boldsymbol{\alpha}\boldsymbol{\alpha}^H \otimes \boldsymbol{\Upsilon}_s + \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$




Improvements over uncorrelated Rayleigh fading

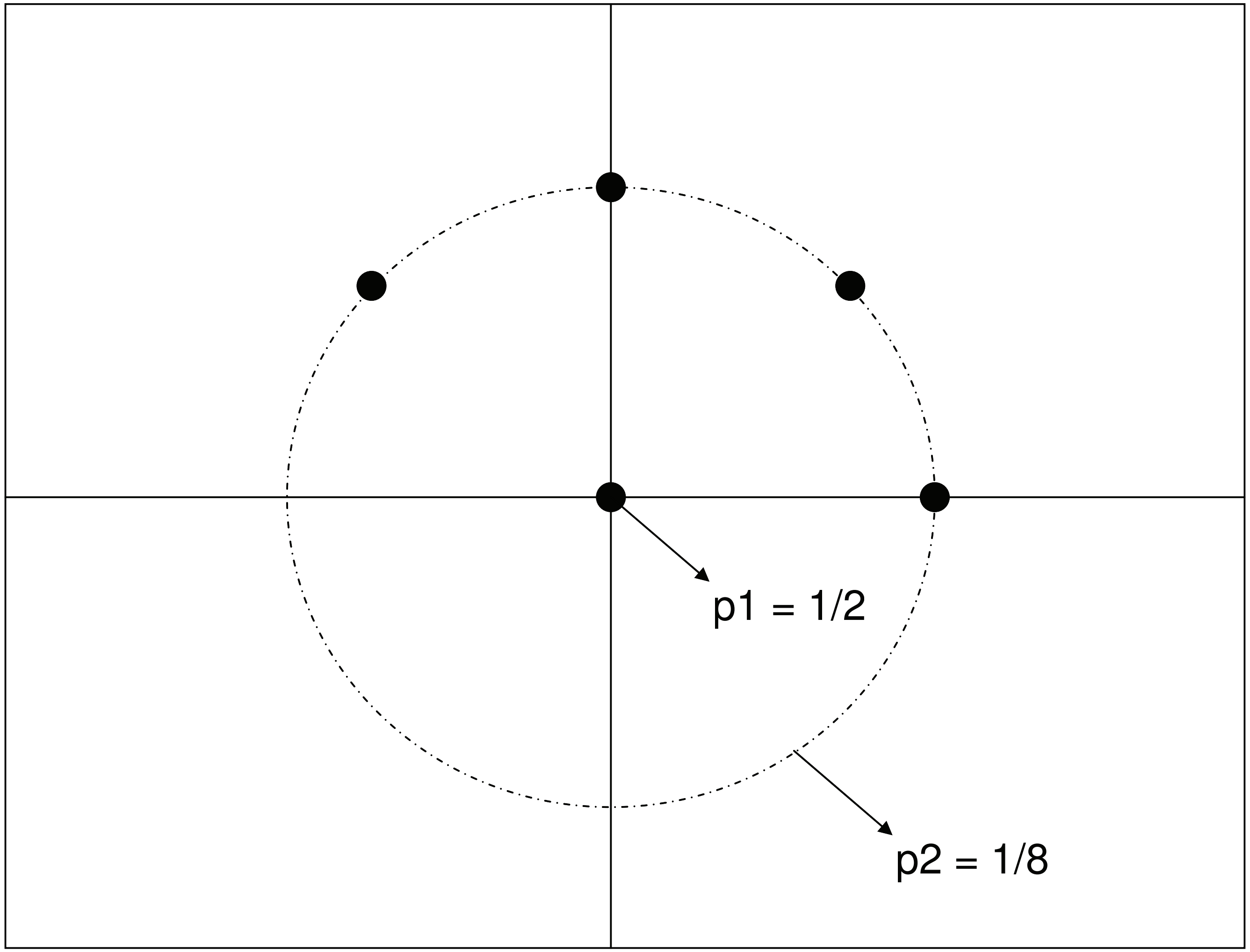
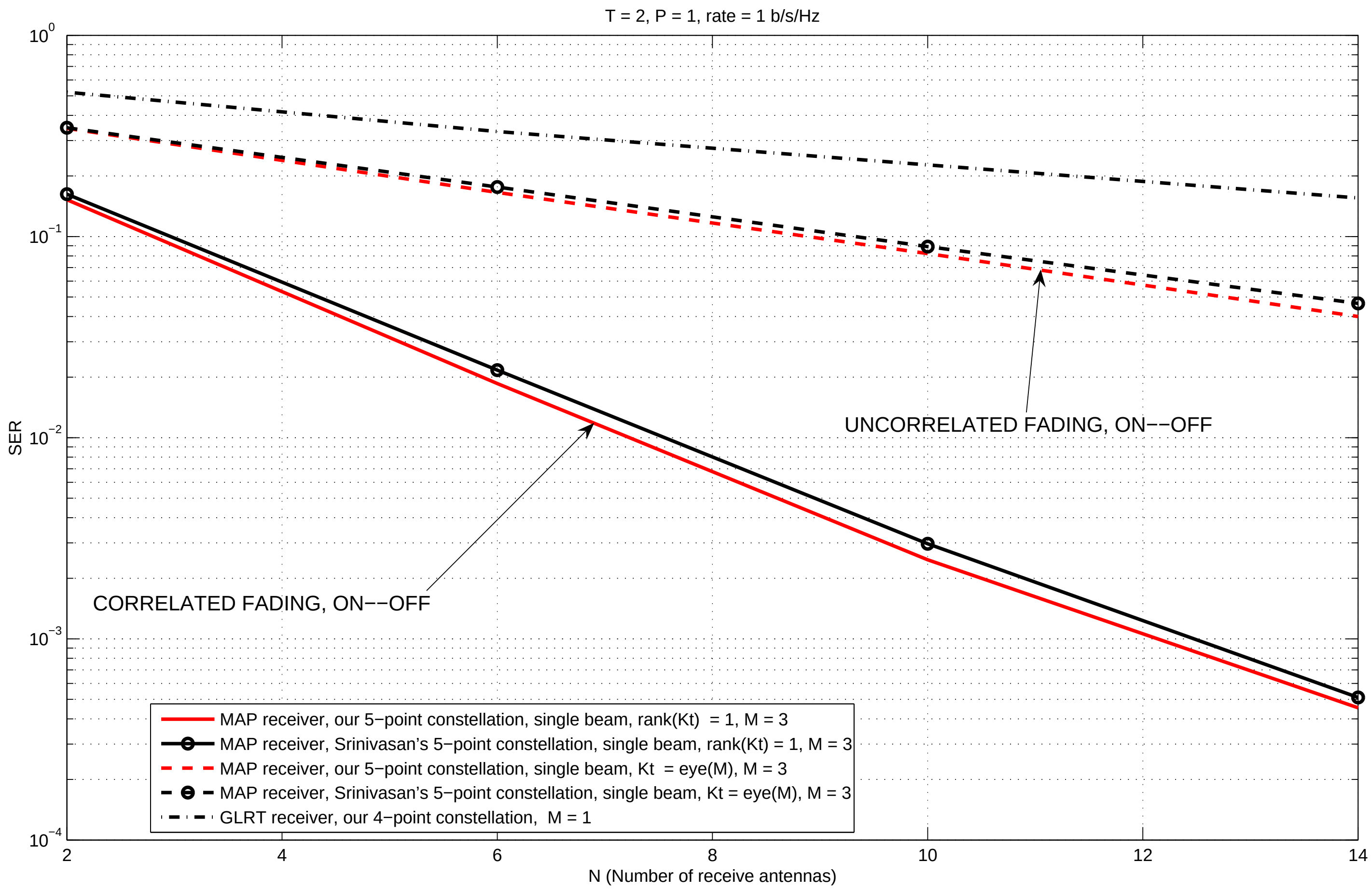
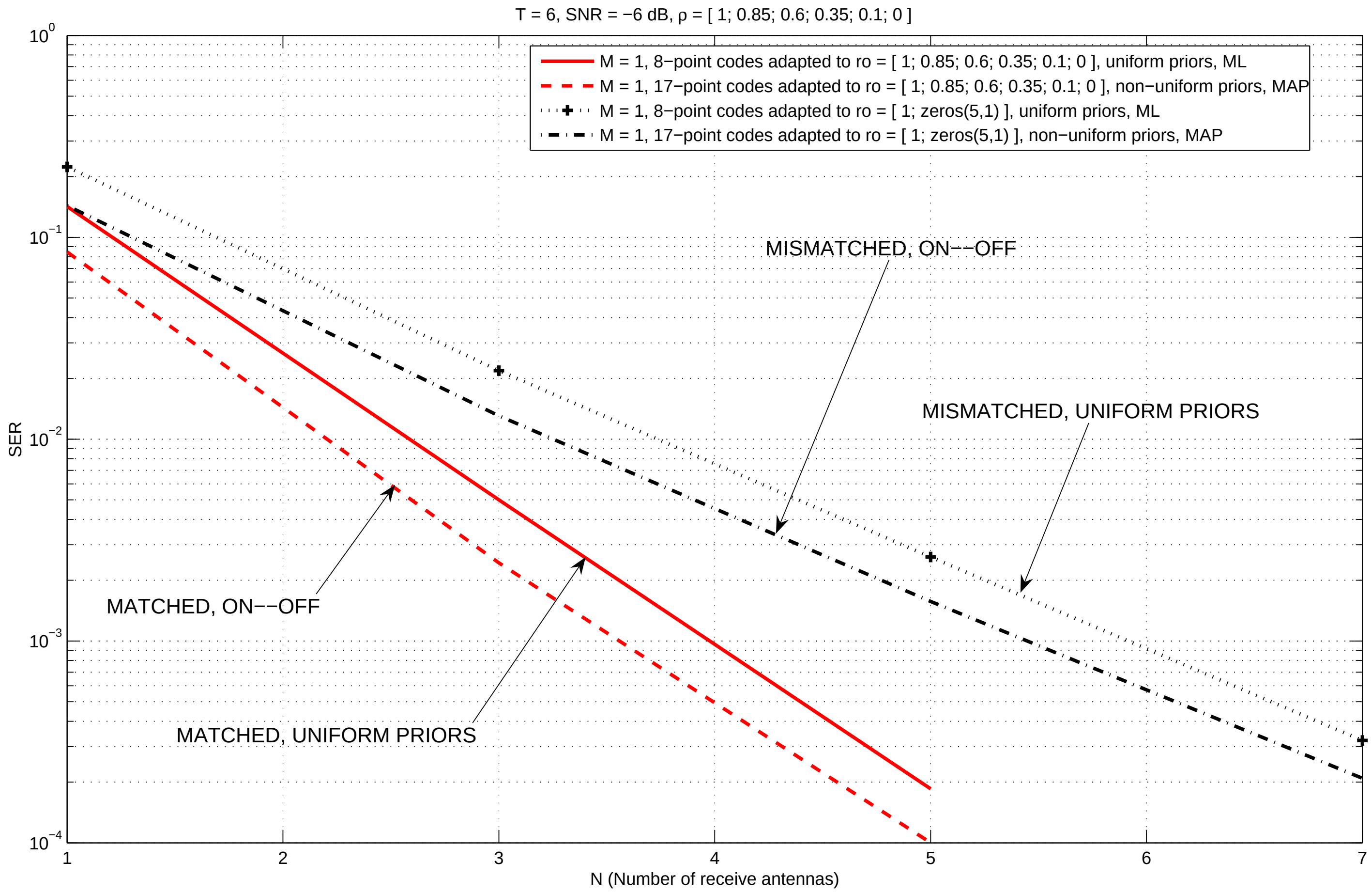


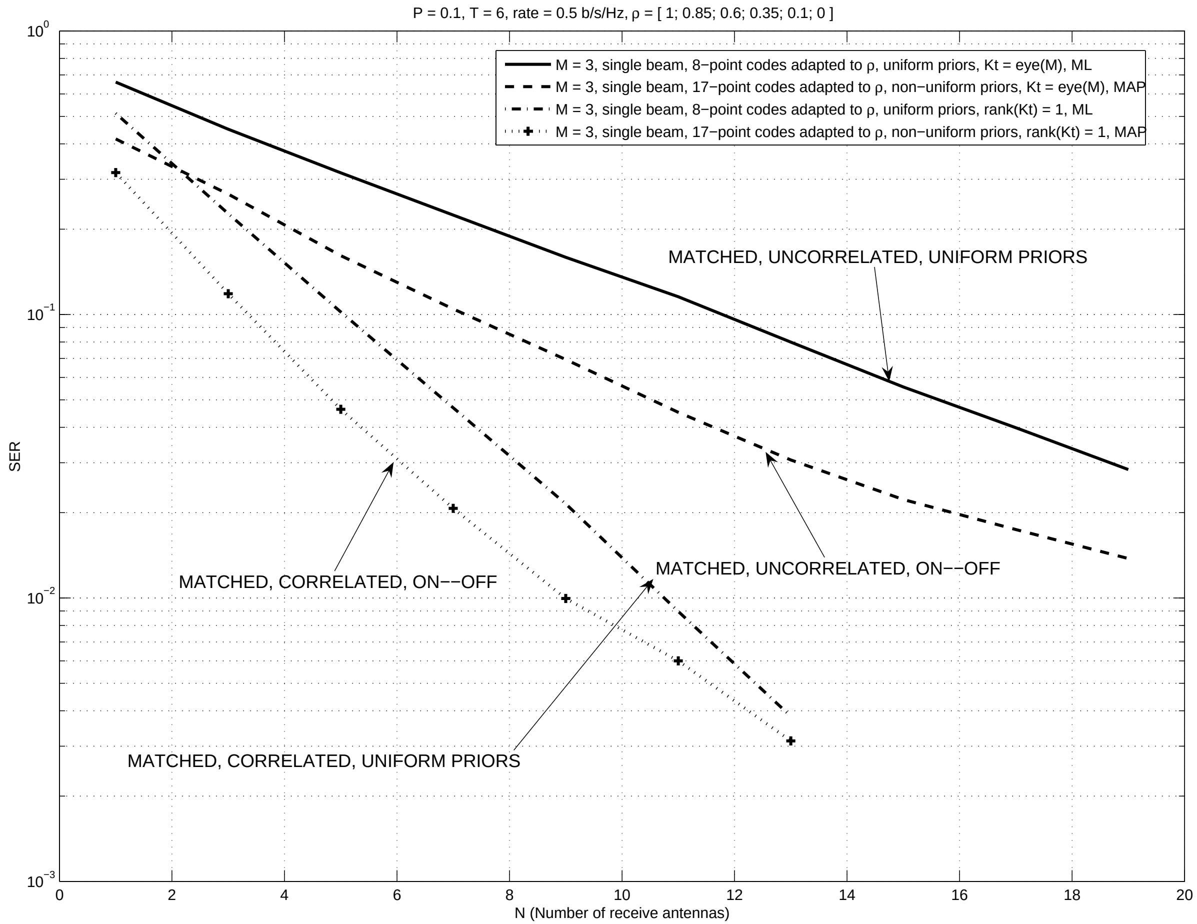

Figure 2: Non-uniform priors, $5$-point constellation, $T = 2$, real case, $\boldsymbol{x}$: codes **should** match the noise statistics



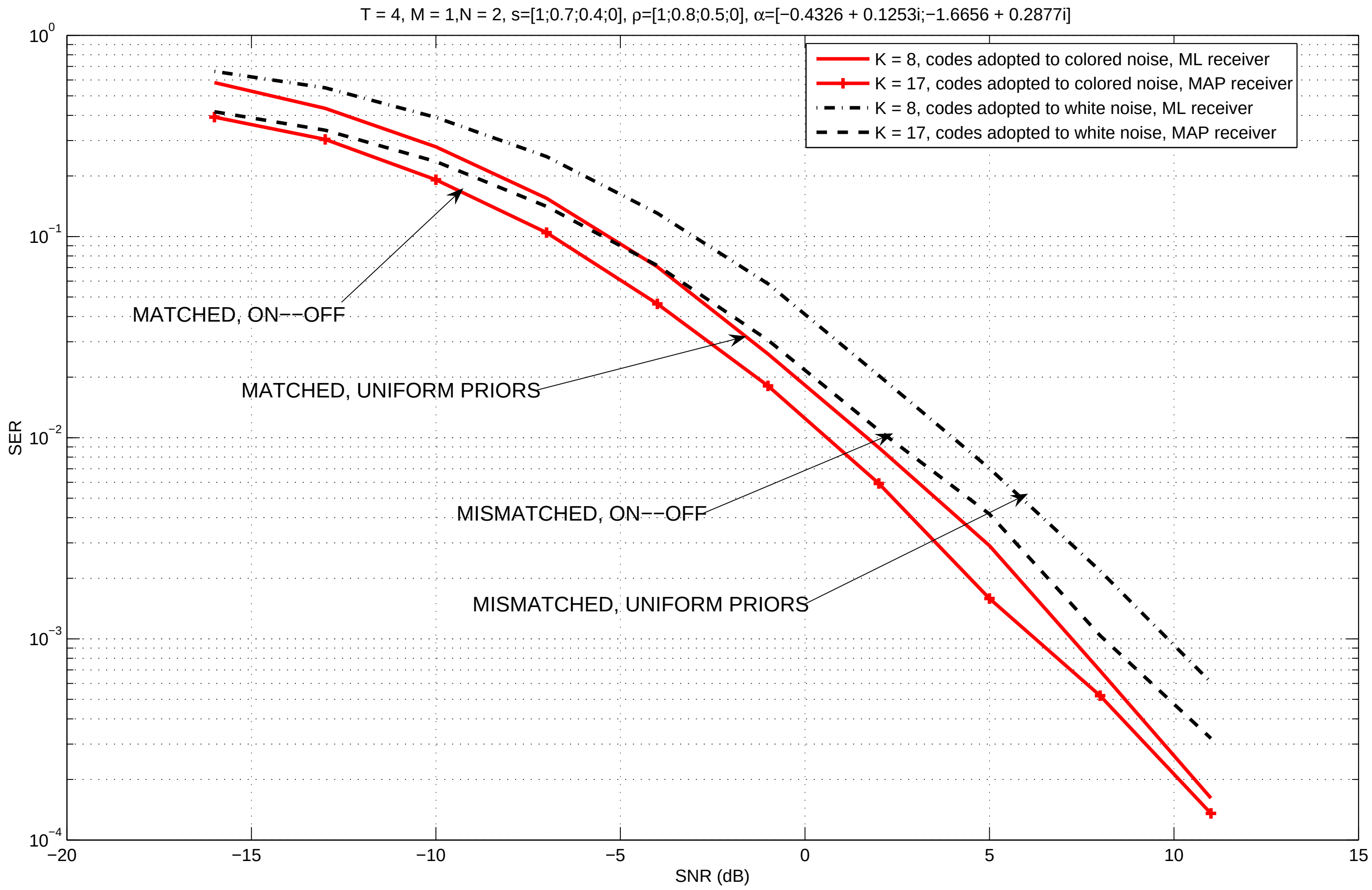

**Category 1 - spatio-temporally white observation noise: $\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT}$**




**Category 2 - spatially white - temporally colored:** $\boldsymbol{\Upsilon} = \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$




**Category 2 - spatially white - temporally colored: $\boldsymbol{\Upsilon} = \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$**




**Category 3 -** $\boldsymbol{E} = \boldsymbol{s}\,\boldsymbol{\alpha}^T + \boldsymbol{E}_{\text{temp}}$ **;** $\boldsymbol{\Upsilon} = \boldsymbol{\alpha}\boldsymbol{\alpha}^H \otimes \boldsymbol{\Upsilon}_s + \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$

## Conclusions

▷ PEP analysis and codebook design in low SNR regime when $\boldsymbol{H}$ is deterministic and unknown

▷ Results

- outperform significantly state-of-art known solutions which assume uniform prior probabilities
- also of interest for the constellations with non-uniform priors

▷ Publications

- conference paper in IEEE SPAWC'2006
- conference paper in IEEE ICASSP'2007
- journal paper in IEEE Transactions on Signal Processing 2008

## Chapter 4: Future work

- ▷ Simplified decoding
- ▷ Influence of unperfect estimate of noise covariance matrix on the error performance
- ▷ Extending the analysis in Ch. 3 to arbitrary correlation structure (done, submitted to IEEE SPAWC'2008)
- ▷ Space-frequency signaling in MIMO-OFDM systems (frequency-selective fading)
- ▷ ETF's

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THANK YOU

# Computer Simulations

☐ **Category 1 - spatio-temporally white observation noise: Constellations with equal priors**

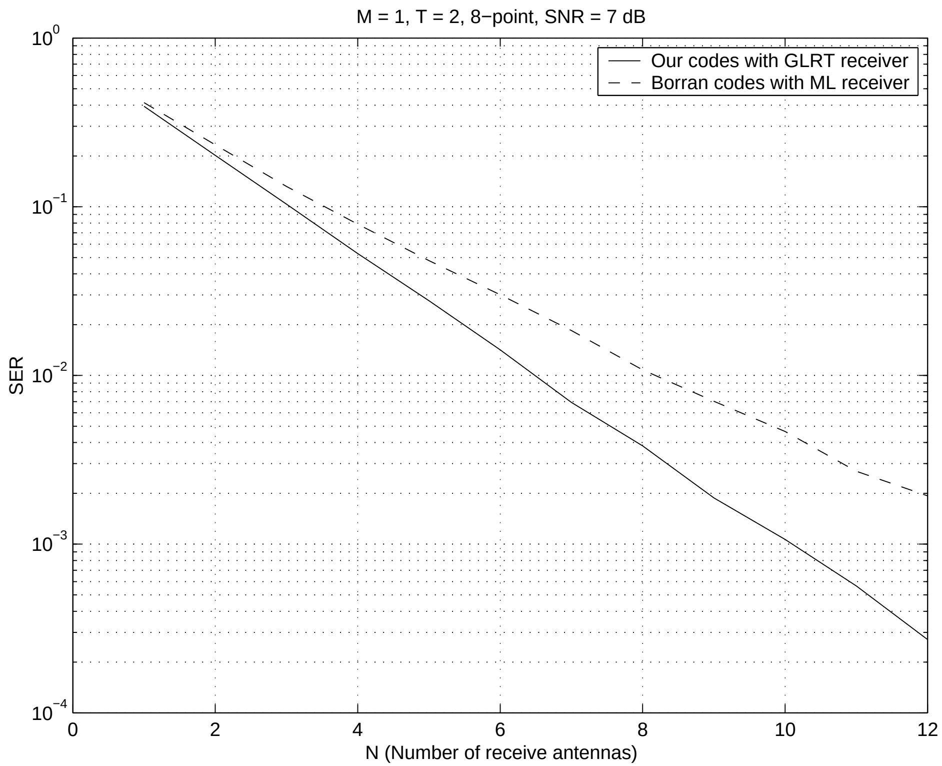


Figure 3: $M$=1, $T$=2, $K$=8, SNR = 7 dB. Solid curve:our codes with our GLRT receiver. Dashed curve:Borran codes designed for SNR = 7dB with ML receiver [1].

**☐ Category 1 - spatio-temporally white observation noise: Constellations with equal priors**

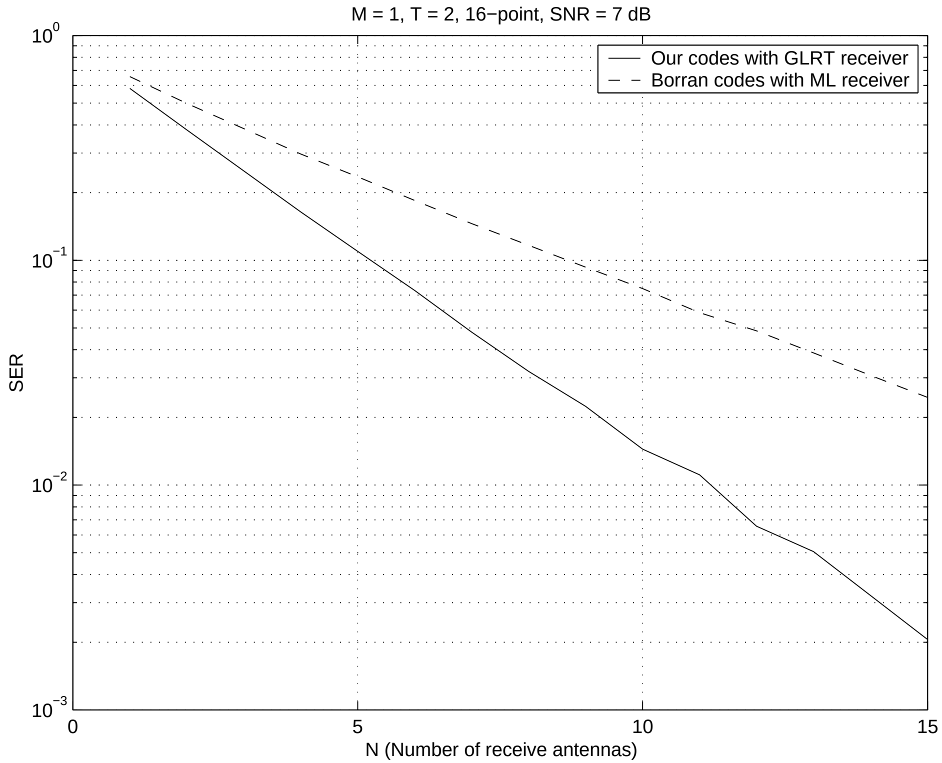


Figure 4: $M$=1, $T$=2, $K$=16, SNR = 7 dB. Solid curve:our codes with our GLRT receiver. Dashed curve:Borran codes designed for SNR = 7dB with ML receiver [1].

## □ Category 1 - spatio-temporally white observation noise: $\boldsymbol{\Upsilon} = \boldsymbol{I}_{NT}$

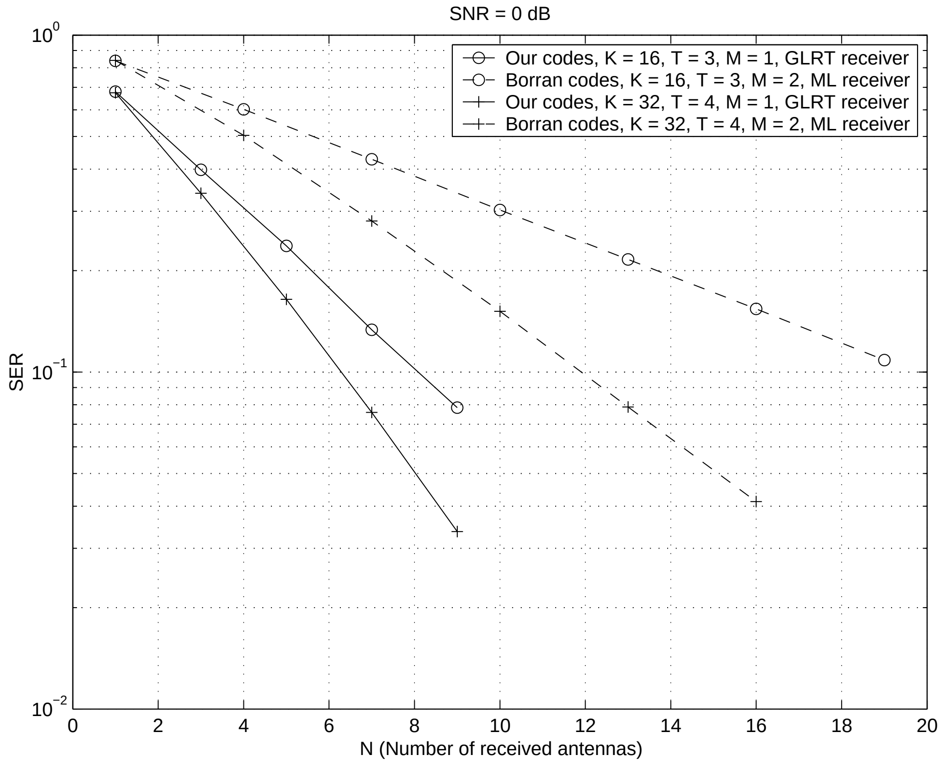


Figure 5: Solid signed curve-our codes for $K = 32$, $T = 4$, $M = 1$, dashed signed curve-Borran's codes for $K = 32$, $T = 4$, $M = 2$, solid circled curve-our codes for $K = 16$, $T = 3$, $M = 1$, dashed circled curve-Borran's codes for $K = 16$, $T = 3$, $M = 2$.

## □ Category 1 - spatio-temporally white observation noise: Constellations with unequal priors

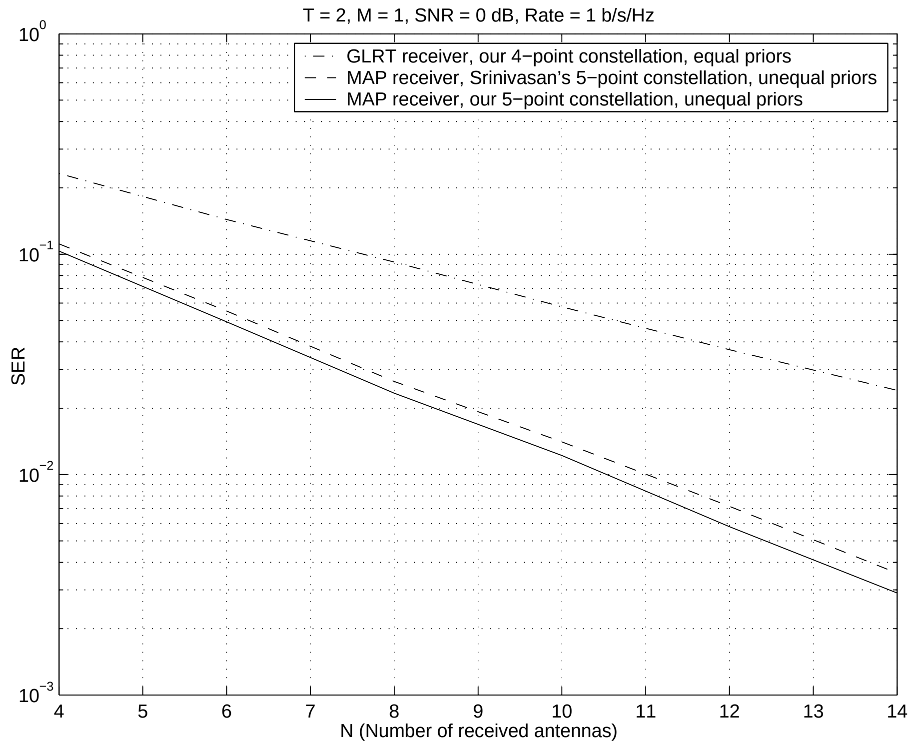


Figure 6: $T$=2, $M$=1, SNR = 0 dB, Rate = 1 b/s/Hz. Solid curve-our 5 point constellation with unequal priors, dashed curve-Srinivasan's 5 point constellation with unequal priors [2], dash-dotted curve-our 4 point constellation with equal priors. Our and Srinivasan's 5 point constellations use *maximum a-posteriori* (MAP) receiver, our 4 point constellation uses GLRT receiver.

☐ **Category 1 - spatio-temporally white observation noise: Constellations with unequal priors**

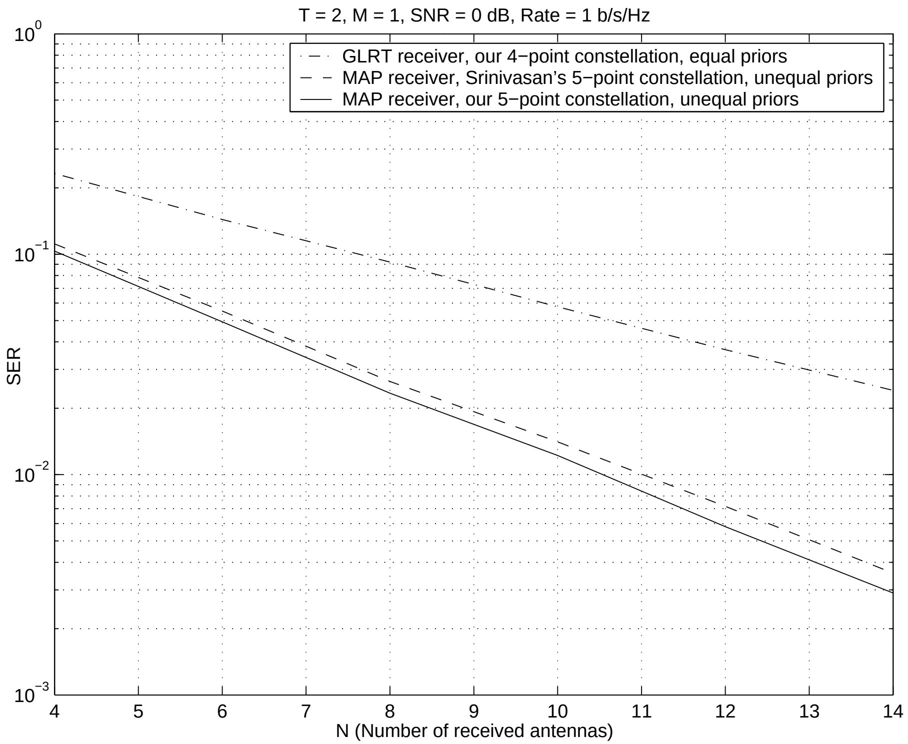


Figure 7: $T$=2, $M$=1, SNR = 0 dB, Rate = 1 b/s/Hz. Solid curve-our 5 point constellation with unequal priors, dashed curve-Srinivasan's 5 point constellation with unequal priors [2], dash-dotted curve-our 4 point constellation with equal priors. Our and Srinivasan's 5 point constellations use *maximum a-posteriori* (MAP) receiver, our 4 point constellation uses GLRT receiver.

☐ **Category 1 - spatio-temporally white observation noise: Constellations with equal priors and $M \geq 1$**

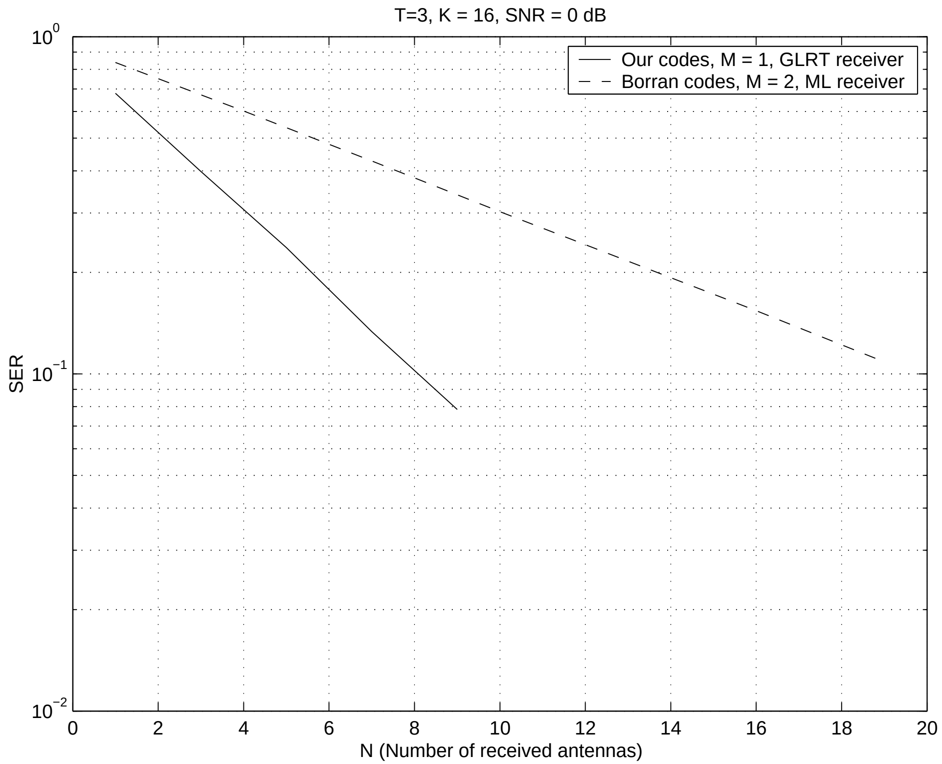


Figure 8: Solid curve-our codes for $K = 16$, $T = 3$, $M = 1$, dashed curve-Borran codes for $K = 16$, $T = 3$, $M = 2$.

☐ **Category 1 - spatio-temporally white observation noise: Constellations with equal priors and $M \geq 1$**

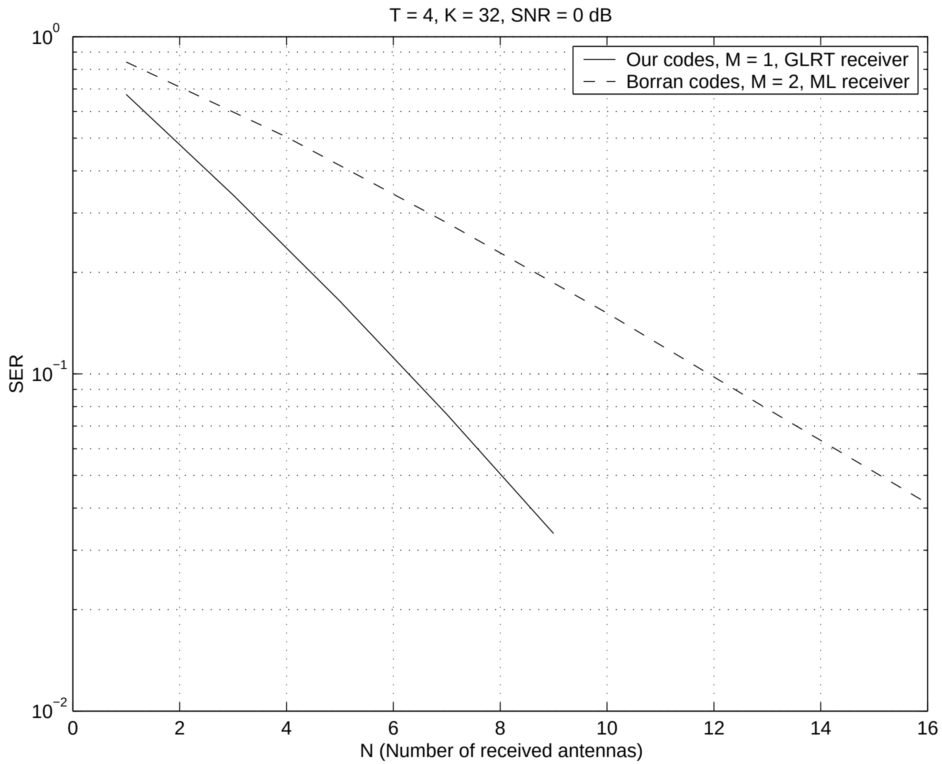


Figure 9: Solid curve-our codes for $K = 32$, $T = 4$, $M = 1$, dashed curve-Borran codes for $K = 32$, $T = 4$, $M = 2$.

## ☐ Category 1 - spatio-temporally white observation noise: Constellations with equal priors and $M \geq 1$

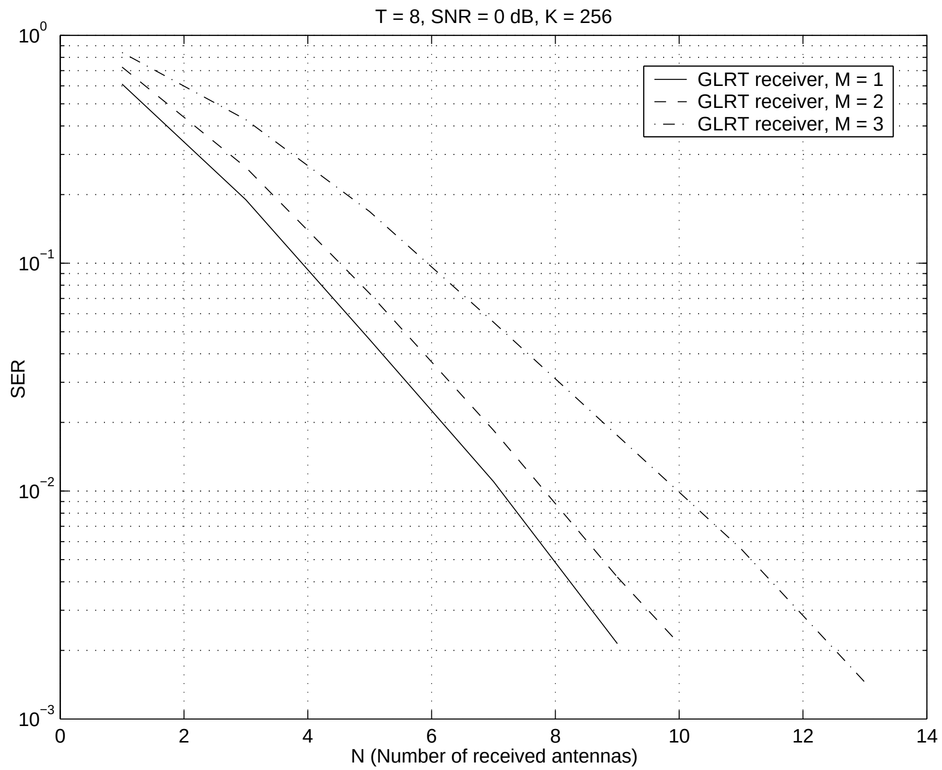


Figure 10: $T$=8, $K$=256, SNR = 0 dB. Solid curve-our codes for M = 1, dashed curve-our codes for M = 2, dash-dotted curve-our codes for M = 3. All codes use GLRT receiver.

## ☐ Category 2 - spatially white - temporally colored: $\boldsymbol{\Upsilon} = \boldsymbol{I}_N \otimes \Sigma(\boldsymbol{\rho})$

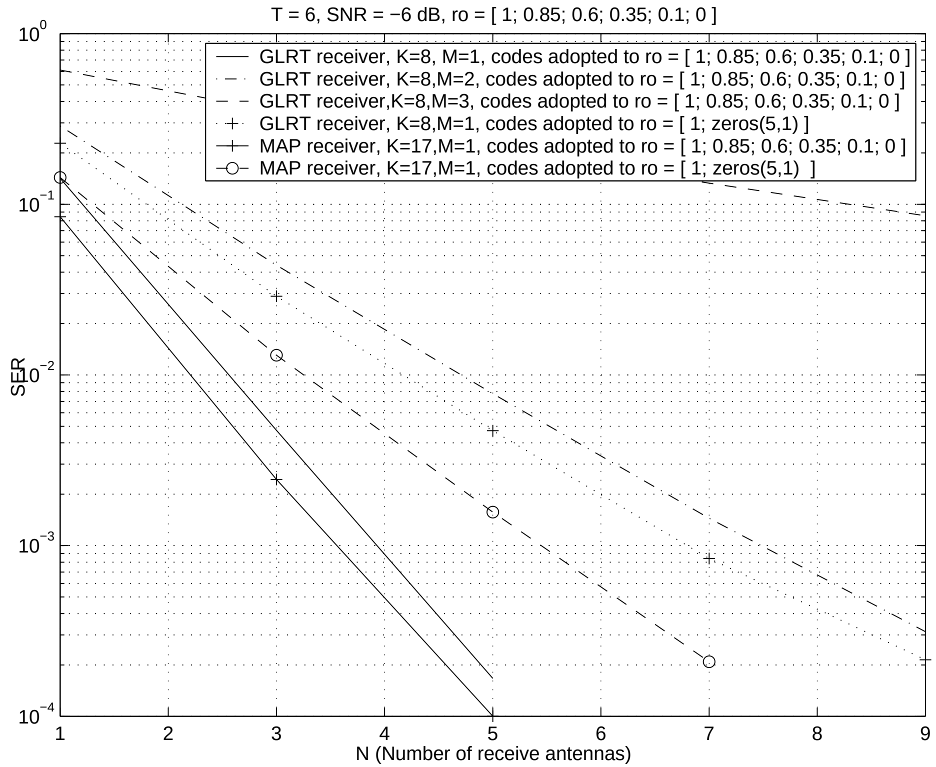


Figure 11: $T$=6, SNR=-6dB, $\boldsymbol{\rho}$=[ 1; 0.85; 0.6; 0.35; 0.1; 0 ].

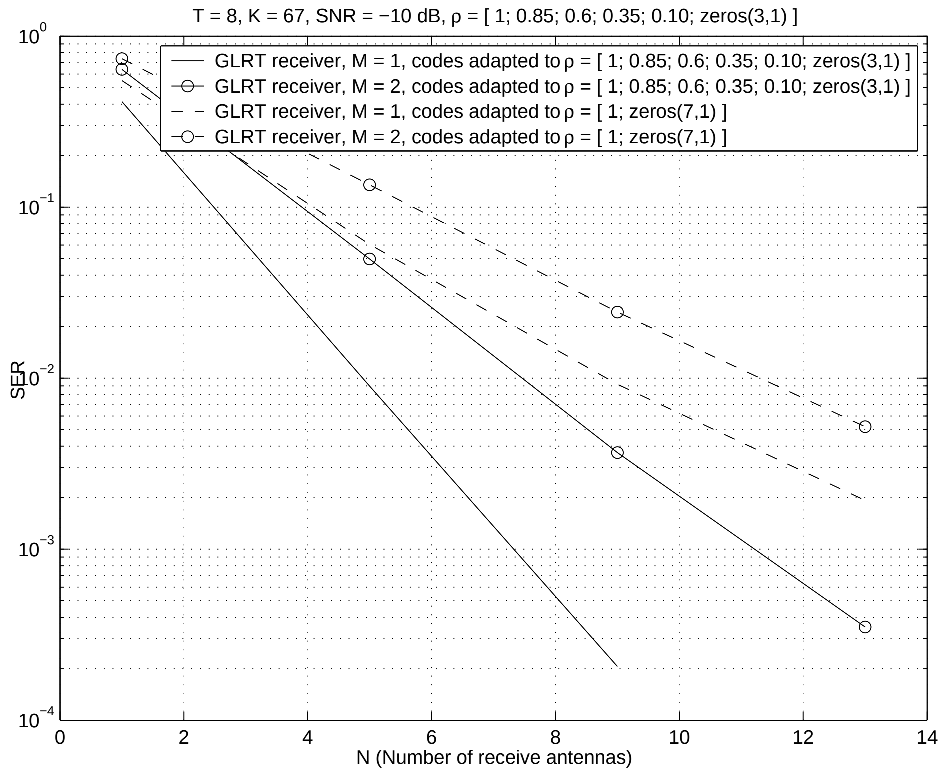


Figure 12: Category 2 - spatially white - temporally colored: $T = 8$, $K = 67$, SNR = -10 dB, $\boldsymbol{\rho}$=[ 1; 0.85; 0.6; 0.35; 0.1; zeros(3,1) ]. Solid curve-our codes for $M = 1$ adapted to $\boldsymbol{\rho}$=[ 1; 0.85; 0.6; 0.35; 0.1; zeros(3,1) ], solid-circled curve-our codes for $M = 2$ adapted to $\boldsymbol{\rho}$=[ 1; 0.85; 0.6; 0.35; 0.1; zeros(3,1) ], dashed curve-our codes for $M = 1$ adapted to $\boldsymbol{\rho}$=[ 1; zeros(7,1) ], dashed-circled curve-our codes for $M = 2$ adapted to $\boldsymbol{\rho}$=[ 1; zeros(7,1) ]. All codes use GLRT receiver.

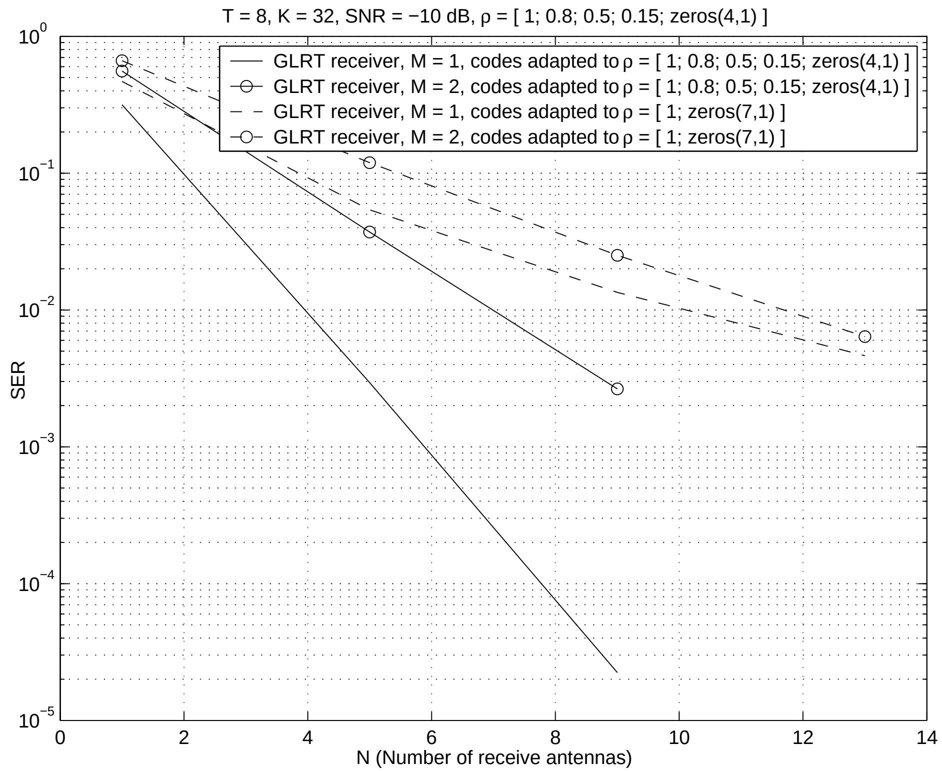


Figure 13: Category 2 - spatially white - temporally colored: $T = 8$, $K = 32$, SNR $=$ -10 dB, $\boldsymbol{\rho}$=[ 1; 0.8; 0.5; 0.15; zeros(4,1) ]. Solid curve-our codes for $M = 1$ adapted to $\boldsymbol{\rho}$=[ 1; 0.8; 0.5; 0.15; zeros(4,1) ], solid-circled curve-our codes for $M = 2$ adapted to $\boldsymbol{\rho}$=[ 1; 0.8; 0.5; 0.15; zeros(4,1) ], dashed curve-our codes for $M = 1$ adapted to $\boldsymbol{\rho}$=[ 1; zeros(7,1) ], dashed-circled curve-our codes for $M = 2$ adapted to $\boldsymbol{\rho}$=[ 1; zeros(7,1) ]. All codes use GLRT receiver.

□ **Category 3 -** $\boldsymbol{E} = \boldsymbol{s}\,\boldsymbol{\alpha}^T + \boldsymbol{E}_{\text{temp}}$

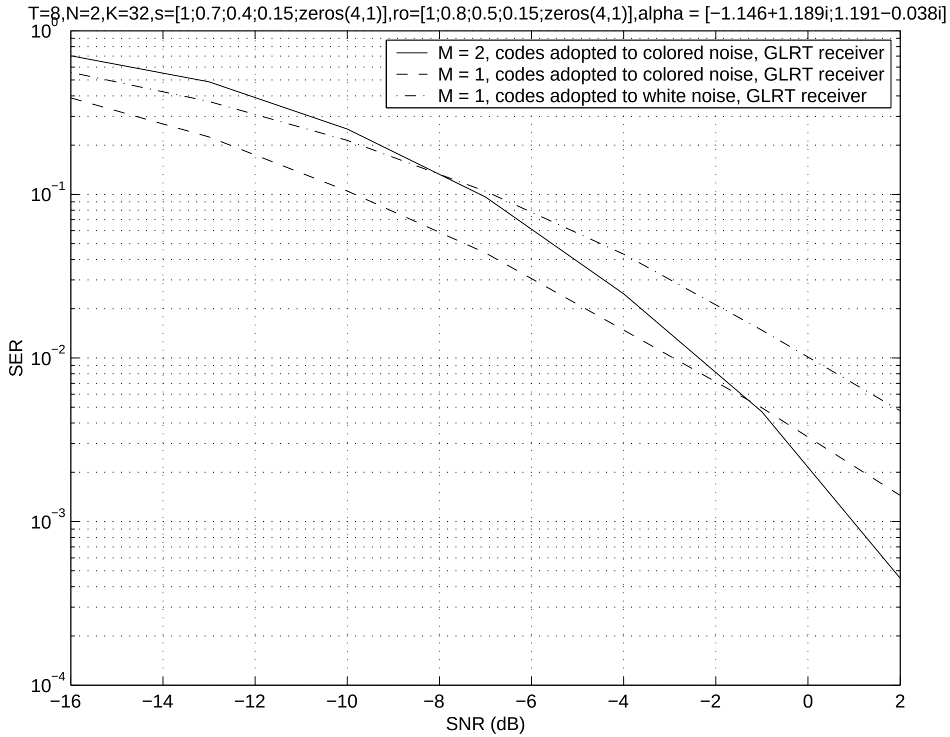


Figure 14: $T$=8, $N = 2$, $K$=32, $\boldsymbol{s}$=[1;0.7;0.4;0.15;zeros(4,1)], $\boldsymbol{\rho}$ = [1;0.8;0.5;0.15;zeros(4,1)], $\boldsymbol{\alpha}$ = [-1.146 + 1.189i;1.191- 0.038i].

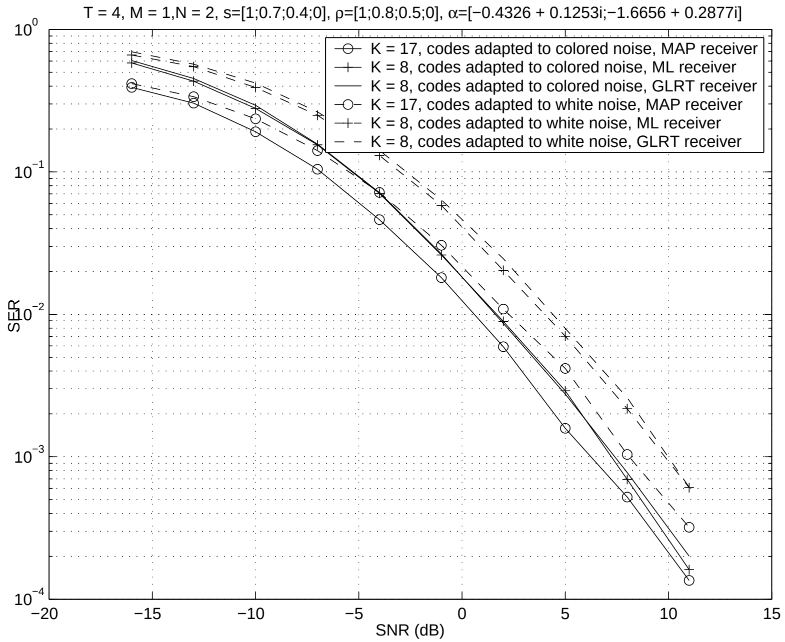


Figure 15: Category 3 - Solid-circled curve-our 17 point codes with unequal priors [2] adapted to colored noise, plus-signed solid curve-our 8 point codes with equal priors adapted to colored noise, solid curve-our 8 point codes with equal priors adapted to colored noise, dashed-circled curve-our 17 point codes with unequal priors adapted to white noise, plus-signed dashed curve-our 8 point codes with equal priors adapted to white noise, dashed curve-our 8 point codes with equal priors adapted to white noise. Circled, signed, and 8-point code curves use MAP, ML and GLRT receivers, respectively.

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