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Convex solution of a permutation problem

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ABSTRACT

In this paper, we show that a problem of finding a permuted version of k vectors from R^N such that they belong to a prescribed rank r subset, can be solved by convex optimization. We prove that under certain generic conditions, the wanted permutation matrix is unique in the convex set of doubly-stochastic matrices. In particular, this implies a solution of the classical correspondence problem of finding a permutation that transforms one collection of points in R^k into the another one. Solutions to these problems have a wide set of applications in Engineering and Computer Science.

1. Introduction

Let $x_1, \ldots, x_N \in R^k$. And let

$$D = [x_1 \quad \cdots \quad x_N]^T \in R^{N\times k}. \tag{1}$$

Let S be a subspace of R^N, with rank $S = r$, where $r \geqslant k$.

The following problem is motivated by Computer Vision – it appears frequently in face and 3D object recognition (see e.g. [1,2,6,7]):

Problem 1. Suppose that there exists a permutation matrix Π such that the columns of ΠD belong to S. Find the matrix Π.

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Also, it is related to the classical correspondence problem of finding a permutation that transforms one collection of points in R^k into the another one (see Section 4).

For large N (which is a standard case in the applications), this problem is very hard to resolve. Because of that, we shall consider its important particular case. In fact, from now on we shall consider permutation matrix Π to be of the following form

$$\Pi := \begin{bmatrix} I_{r-k+1} & 0 \\ 0 & \Pi' \end{bmatrix}, \tag{2}$$

where now $\Pi' \in R^{(N-r+k-1)\times(N-r+k-1)}$ is a permutation matrix.

We can now formulate our main problem:

Problem 2. Suppose that there exists a permutation matrix Π of the form (2), such that the columns of ΠD belong to S. Find the matrix Π.

Moreover, in Section 4, we shall discuss how the solution of Problem 2 gives an algorithm for a solution of the general case (Problem 1).

Without loss of generality, we are going to study these problems in a generic case. In particular, we have certain generic conditions on matrix D and on subspace S (precise conditions are given in Section 2.1).

Our approach to Problem 2 is to study doubly-stochastic matrices, instead of permutation matrices. Let

$$M' := \begin{bmatrix} I_{r-k+1} & 0 \\ 0 & M \end{bmatrix}, \tag{3}$$

where $M \in R^{(N-r+k-1)\times(N-r+k-1)}$ is a doubly-stochastic matrix. Now, the analogous problem to Problem 2, for doubly-stochastic matrices is the following one:

Problem 3. Suppose that there exists a doubly-stochastic matrix M' of the form (3), such that the columns of $M'D$ belong to S. Find the matrix M'.

The last problem can be resolved quickly and efficiently, see e.g. [4]. So, the goal of this paper is to use the solution of Problem 3 to resolve Problem 2. In fact, we prove the following:

Let Π be a permutation matrix of the form (2), such that the columns of ΠD belong to S.

If M' is a doubly-stochastic matrix of the form (3), such that the columns of $M'D$ belong to S, then $M' = \Pi$.

This is the main result of the paper given in Theorem 2 from Section 3. The proof relies on the Perron–Frobenius theorem applied for the special case of doubly-stochastic matrices. This theorem gives the uniqueness of the permutation matrix Π with the wanted properties in the set of all doubly-stochastic matrices.

2. Notation and auxiliary results

In this section we recall some of the properties of doubly-stochastic matrices that will be essential for the rest of the paper. These are classical results that are based on the theory of non-negative matrices and the Perron–Frobenius theorem [5], applied to the particular case of doubly-stochastic matrices.

Let M be a doubly-stochastic matrix, i.e. a non-negative square matrix whose row and column sums are all equal to 1. There exists a permutation matrix P such that PMP^T has the following lower block-triangular form:

$$PMP^T = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ M_{21} & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_{l1} & M_{l2} & \cdots & M_l \end{bmatrix},$$

where $M_1, \ldots, M_l$, are square irreducible matrices. Moreover, since M is a non-negative matrix, with row and column sums equal to 1, we have that all off-diagonal blocks are equal to zero, i.e. all blocks M_{ij} with $i > j$ are zero matrices. So, we have that for the doubly-stochastic matrix M, the matrix PMP^T has the block-diagonal form:

$$PMP^T = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_l \end{bmatrix} \tag{4}$$

with all blocks being irreducible.

The following lemma is classical and it is an application of the Perron–Frobenius theorem for irreducible doubly-stochastic matrices. It will be essential in the proof of the main result.

Lemma 1. *If M is an irreducible doubly-stochastic matrix, and w is a vector such that $Mw = w$, then all entries of w are equal, i.e. there exists $c \in \mathbb{R}$, such that*

$$w = c \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

2.1. Generic conditions

Let N, r and k be positive integers, such that $r \geqslant k$, and $N \geqslant r + 2(k-1)$. Let

$$D = \left[\begin{array}{cccc} a^1 & a^2 & \cdots & a^k \\ \hline X^1 & X^2 & \cdots & X^k \end{array}\right] \in \mathbb{R}^{N \times k}, \tag{5}$$

be a matrix where $a^i \in \mathbb{R}^{(r-k+1)\times 1}$ and $X^i \in \mathbb{R}^{(N-r+k-1)\times 1}$, $i = 1, \ldots, k$ are given vectors. We also denote

$$X = \begin{bmatrix} X^1 & X^2 & \cdots & X^k \end{bmatrix} \in \mathbb{R}^{(N-r+k-1)\times k}. \tag{6}$$

Let S be a prescribed r-dimensional vector subspace of $\mathbb{R}^N$, such that the columns of D belong to S. Then S is the span of some r linearly independent vectors:

$$S = \left\langle \left[\begin{array}{c} e^1 \\ \hline f^1 \end{array}\right], \left[\begin{array}{c} e^2 \\ \hline f^2 \end{array}\right], \ldots, \left[\begin{array}{c} e^r \\ \hline f^r \end{array}\right] \right\rangle, \tag{7}$$

where $e^i \in \mathbb{R}^{r-k+1}, f^i \in \mathbb{R}^{N-r+k-1}$, $i = 1, \ldots, r$. We denote by B the corresponding matrix:

$$B = \left[\begin{array}{cccc} e^1 & e^2 & \cdots & e^r \\ \hline f^1 & f^2 & \cdots & f^r \end{array}\right] \in \mathbb{R}^{N \times r}. \tag{8}$$

We shall assume that the matrix D and the subspace S, satisfy the following two conditions:

Condition 1. The projection of S onto the $(r-k+1)$-dimensional subspace of $\mathbb{R}^N$, formed by the first $r-k+1$ coordinates, is surjective. In other words, the matrix

$$[I_{r-k+1} \; 0]B = [e^1, \ldots, e^r],$$

is of full row-rank, i.e. has rank $r-k+1$. By reordering, without loss of generality, we can assume that the vectors $e^1, e^2, \ldots, e^{r-k+1}$ are linearly independent, i.e. that the matrix

$$Q := \begin{bmatrix} e^1 & e^2 & \cdots & e^{r-k+1} \end{bmatrix},$$

is invertible.

Moreover, let $v_1, \ldots, v_{k-1}$ be vectors defined by:

$$v_i := f^{r-i+1} - \left[f^1 \ \cdots \ f^{r-k+1}\right] Q^{-1} e^{r-i+1}, \quad i = 1, \ldots, k-1. \tag{9}$$

We furthermore require that all submatrices of the matrix $[v_1 \ \ldots \ v_{k-1}]$ formed by any $k-1$ of its rows have rank $k-1$.

Condition 2. Let T be the set of all $(N-r+k-1) \times (N-r+k-1)$ matrices R, such that all entries of R are 0's and 1's, with exactly one 1 in each row. We require that for all matrices $R \in T$ and every d such that $k \leqslant d \leqslant N-r+k-1$, we have the following: for arbitrary $1 \leqslant i_1 < i_2 < \cdots < i_d \leqslant N-r+k-1$, the submatrix of $[X \ R]$ formed by the rows $i_1, i_2, \ldots, i_d$ has the rank $k+p$, where p is the number of the nonzero columns in the submatrix of R formed by the rows $i_1, i_2, \ldots, i_d$, whenever $d \geqslant k+p$.

Since the set T has finitely many elements, it is straightforward to see that generic matrix D and subspace S satisfy the Conditions 1 and 2. In other words, the sets of matrices D and B that do not satisfy Conditions 2 and 1, respectively, are of measure zero.

3. Main result

Before proving the main result, we give a solution to the following theorem:

Theorem 1. *Let $D \in \mathbb{R}^{N\times k}$ be a matrix (5) and let S be a r-dimensional subspace of $\mathbb{R}^N$ (7) containing the columns of D, such that D and S satisfy the Conditions 1 and 2. Then we have the following:*

If $M \in \mathbb{R}^{(N-r+k-1)\times(N-r+k-1)}$ is a doubly-stochastic matrix such that all vectors $\begin{bmatrix} a^i \\ MX^i \end{bmatrix}$, $i = 1, \ldots, k$, belong to S, then M is the identity matrix.

Proof. Our first and main goal is to find a vector w which is a nonzero linear combination of the vectors $X^1, X^2, \ldots, X^k$ such that

$$Mw = w. \tag{10}$$

Having this relation, by using Lemma 1 we shall obtain strong restrictions on w.

In order to find a vector w that satisfies (10) we do the following: denote by $\Sigma^j, j = 1, \ldots, k$, the space of all vectors of the form $\begin{bmatrix} a^j \\ Z^j \end{bmatrix}$, where Z^j runs through $\mathbb{R}^{(N-r+1)\times 1}$, which belong to S, i.e.:

$$\Sigma^j = \left\{ \begin{bmatrix} a^j \\ \overline{Z^j} \end{bmatrix} \middle| Z^j \in \mathbb{R}^{N-r+k-1} \right\} \cap S.$$

Let $j \in \{1, \ldots, k\}$ be fixed. We have that $\begin{bmatrix} a^j \\ \overline{Z^j} \end{bmatrix} \in S$ if and only if there exists $\alpha_i^j \in \mathbb{R}, i = 1, \ldots, r$, such that

$$\begin{bmatrix} a^j \\ \overline{Z^j} \end{bmatrix} = \sum_{i=1}^{r} \alpha_i^j \begin{bmatrix} e^i \\ \overline{f^i} \end{bmatrix},$$

i.e. such that

$$a^j = \sum_{i=1}^{r} \alpha_i^j e^i, \tag{11}$$

$$Z^j = \sum_{i=1}^{r} \alpha_i^j f^i. \tag{12}$$

Since S satisfies the Condition 1, the matrix

$$Q = \begin{bmatrix} e^1 & e^2 & \cdots & e^{r-k+1} \end{bmatrix}$$

is invertible. Then (11) becomes

$$a^j = \begin{bmatrix} e^1 & e^2 & \cdots & e^r \end{bmatrix} \begin{bmatrix} \alpha_1^j \\ \vdots \\ \alpha_r^j \end{bmatrix} = Q \begin{bmatrix} \alpha_1^j \\ \vdots \\ \alpha_{r-k+1}^j \end{bmatrix} + \sum_{l=r-k+2}^{r} \alpha_l^j e^l,$$

and so

$$\begin{bmatrix} \alpha_1^j \\ \vdots \\ \alpha_{r-k+1}^j \end{bmatrix} = Q^{-1} a^j - \sum_{l=r-k+2}^{r} \alpha_l^j Q^{-1} e^l. \tag{13}$$

By replacing this in (12) we obtain that

$$\begin{aligned} Z^j &= \begin{bmatrix} f^1 & f^2 & \cdots & f^r \end{bmatrix} \begin{bmatrix} \alpha_1^j \\ \vdots \\ \alpha_r^j \end{bmatrix} = \begin{bmatrix} f^1 & \cdots & f^{r-k+1} \end{bmatrix} \begin{bmatrix} \alpha_1^j \\ \vdots \\ \alpha_{r-k+1}^j \end{bmatrix} + \sum_{l=r-k+2}^{r} \alpha_l^j e^l \\ &= \begin{bmatrix} f^1 & \cdots & f^{r-k+1} \end{bmatrix} Q^{-1} a^j + \sum_{l=r-k+2}^{r} \alpha_l^j \left(f^l - \begin{bmatrix} f^1 & \cdots & f^{r-k+1} \end{bmatrix} Q^{-1} e^l \right). \end{aligned}$$

In (9) we defined the vectors $v_1, \ldots, v_{k-1} \in \mathbb{R}^{N-r+k-1}$ as

$$v_i = f^{r-i+1} - \begin{bmatrix} f^1 & \cdots & f^{r-k+1} \end{bmatrix} Q^{-1} e^{r-i+1}, \quad i = 1, \ldots, k-1. \tag{14}$$

Then we have that Z^j belongs to the affine subspace spanned by $v_1, \ldots, v_{k-1}$. Finally, since $\begin{bmatrix} a^j \\ X^j \end{bmatrix} \in S$, we have:

$$\begin{bmatrix} a^j \\ Z^j \end{bmatrix} \in S \quad \Leftrightarrow \quad Z^j = X^j + \sum_{i=1}^{k-1} c_i^j v_i, \quad \text{for some } c_i^j \in \mathbb{R}. \tag{15}$$

Note that the vectors $v_i, i = 1, \ldots, k-1$ are independent on j, and hence all $\Sigma^1, \ldots, \Sigma^k$ are parallel.

Now, let M be a doubly-stochastic matrix such that $\begin{bmatrix} a^j \\ MX^j \end{bmatrix} \in S$, for all $i = 1, \ldots, k$. Then by (15) there exist real numbers $c_i^j \in \mathbb{R}, j = 1, \ldots, k, i = 1, \ldots, k-1$, such that

$$MX^j = X^j + \sum_{i=1}^{k-1} c_i^j v_i, \quad j = 1, \ldots, k. \tag{16}$$

Denote

$$\begin{aligned} V &:= \begin{bmatrix} v_1 & v_2 & \cdots & v_{k-1} \end{bmatrix} \in \mathbb{R}^{(N-r+k-1)\times(k-1)}, \\ c^j &:= \begin{bmatrix} c_1^j \\ \vdots \\ c_{k-1}^j \end{bmatrix} \in \mathbb{R}^{(k-1)\times 1}, \quad j = 1, \ldots, k, \\ C &:= \begin{bmatrix} c^1 & c^2 & \cdots & c^k \end{bmatrix} \in \mathbb{R}^{(k-1)\times k}. \end{aligned}$$

Then (16) becomes

$$MX^j = X^j + Vc^j, \quad j = 1, \ldots, k. \tag{17}$$

Moreover, since the number of rows of the matrix C is $k-1$, we have that rank $C \leqslant k-1 < k$, and so the equation

$$C \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} = 0,$$

has a nonzero solution $\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}$. Thus these β_i's satisfy:

$$\sum_{i=1}^{k} \beta_i c^i = 0. \tag{18}$$

Now let

$$w := \sum_{i=1}^{k} \beta_i X^i.$$

Then from (17) and (18) we obtain

$$Mw = w + \sum_{j=1}^{k} \beta_j V c^j = w + V \left(\sum_{j=1}^{k} \beta_j c^j \right) = w. \tag{19}$$

Thus, such defined w is an eigenvector of M, as wanted in (10).

Since M is a doubly-stochastic matrix, there exists a permutation matrix $P \in \mathbb{R}^{(N-r+k-1)\times(N-r+k-1)}$ such that

$$PMP^T = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_l \end{bmatrix} \tag{20}$$

with all diagonal blocks being irreducible. Also, we denote the size of M_i by d_i, $i = 1, \ldots, l$, and we assume that $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_l \geqslant 1$. If $d_1 = 1$ (and consequently $d_i = 1$, for all $i = 1, \ldots, l$), then $M = I$, as wanted. Otherwise, let $p := \max\{i | d_i \geqslant 2\}$, with the convention that $d_0 = +\infty$. Our aim is to prove that $p = 0$, which would finish our proof.

To that end, define vectors $x^j := PX^j$, $j = 1, \ldots, k$, and $\omega := Pw = \sum_{j=1}^{k} \beta_j x^j$, and split them accordingly with respect to (20), i.e.:

$$x^j = \begin{bmatrix} x_1^j \\ \vdots \\ x_l^j \end{bmatrix}, \quad j = 1, \ldots, k, \qquad \omega = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_l \end{bmatrix},$$

with $x_i^j, \ \omega_i \in \mathbb{R}^{d_i}$, $i = 1, \ldots, l$. Then by (19) we have that for all $i = 1, \ldots, l$:

$$M_i \omega_i = \omega_i.$$

By Lemma 1, we have that there exist $k_i \in \mathbb{R}$, $i = 1, \ldots, p$, such that

$$\omega_i = k_i \mathbf{1}_{d_i}, \quad i = 1, \ldots, p,$$

where $\mathbf{1}_{d_i} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{d_i}$. Then

$$\sum_{j=1}^{k} \beta_j x_i^j = c_i \mathbf{1}_{d_i}, \quad i = 1, \ldots, p.$$

If we denote

$$y^j := \begin{bmatrix} x_i^j \\ x_2^j \\ \vdots \\ x_p^j \end{bmatrix}, \quad j = 1, \ldots, k,$$

we obtain that

$$\sum_{j=1}^{k} \beta_j y^j - k_1 \begin{bmatrix} \mathbf{1}_{d_1} \\ \mathbf{0}_{d_2} \\ \vdots \\ \mathbf{0}_{d_p} \end{bmatrix} - k_2 \begin{bmatrix} \mathbf{0}_{d_1} \\ \mathbf{1}_{d_2} \\ \vdots \\ \mathbf{0}_{d_p} \end{bmatrix} - \ldots - k_p \begin{bmatrix} \mathbf{0}_{d_1} \\ \mathbf{0}_{d_2} \\ \vdots \\ \mathbf{1}_{d_p} \end{bmatrix} = 0,$$

i.e. since not all β_i's are equal to zero, we have that $k + p$ vectors $y^1, y^2, \ldots, y^k, \begin{bmatrix} \mathbf{1}_{d_1} \\ \mathbf{0}_{d_2} \\ \vdots \\ \mathbf{0}_{d_p} \end{bmatrix}, \begin{bmatrix} \mathbf{0}_{d_1} \\ \mathbf{1}_{d_2} \\ \vdots \\ \mathbf{0}_{d_p} \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{0}_{d_1} \\ \mathbf{0}_{d_2} \\ \vdots \\ \mathbf{1}_{d_p} \end{bmatrix}$ are linearly dependent.

However, since the matrix X satisfies the Condition 2, we have that $\sum_{i=1}^{p} d_i < k + p$, i.e.

$$\sum_{i=1}^{l} (d_i - 1) < k. \tag{21}$$

In particular we have that $p \leqslant k - 1$, and so $\sum_{i=1}^{p} d_i \leqslant 2(k-1)$. Thus PMP^T is of the form:

$$PMP^T = \begin{bmatrix} M' & 0 \\ 0 & I \end{bmatrix}, \tag{22}$$

where M' is $2(k-1) \times 2(k-1)$ matrix.

Now, if all $X^j, j = 1, \ldots, k$ satisfy $MX^j = X^j$, then by Lemma 1, the matrix X would have at least two identical rows, which contradicts the Condition 2.

So, there exists some j, such that $MX^j \neq X^j$. Without loss of generality, we can assume that $j = 1$. Then from (16) we would have that

$$PMP^T x^1 - x^1 = \sum_{i=1}^{k-1} c_i^1 P v_i.$$

However by (22), we have that the left-hand side of the above equation is a nonzero vector (and so not all c_i^1's are equal to zero), whose all entries except the first $2(k-1)$ ones are equal to zero. If we denote by $\nu_i \in \mathbb{R}^{N-r-k+1}, i = 1, \ldots, k-1$, the vectors formed by the last $N - r + k - 1 - 2(k-1) = N - r - k + 1$ components of the vector Pv_i, respectively, then the last implies that the vectors $\nu_1, \ldots, \nu_{k-1}$ are linearly dependent. However, this contradicts the second part of Condition 1 since $N - r - k + 1 \geqslant k - 1$.

Thus $M = I$, as wanted. $\square$

Finally, we can give our main result:

Theorem 2. *Let $D \in \mathbb{R}^{N\times k}$ be a matrix (5) and let S be a r-dimensional subspace of $\mathbb{R}^N$ (7) such that D and S satisfy the Conditions 1 and 2.*

Let Π be a permutation matrix of the form (2), such that the columns of ΠD belong to S.

If M' is a doubly-stochastic matrix of the form (3), such that the columns of $M'D$ belong to S, then $M' = \Pi$.

Proof. Let $D' = \Pi D = \begin{bmatrix} I_{r-k+1} & 0 \\ 0 & \Pi' \end{bmatrix} D$. Then the columns of D' belong to S, and also the columns of

$$\begin{bmatrix} I_{r-k+1} & 0 \\ 0 & M\Pi'^{-1} \end{bmatrix} D' = \begin{bmatrix} I_{r-k+1} & 0 \\ 0 & M \end{bmatrix} D,$$

belong to S. Finally, since $M\Pi'^{-1}$ is also a doubly-stochastic matrix and since D' satisfies the Condition 2, by Theorem 1 we have that $M\Pi'^{-1} = I$, i.e. $M = \Pi'$, as wanted. □

4. Applications of the main result

In all problems posed and resolved in this paper we assume the existence of a certain permutation matrix, and the problem is in finding it. Moreover, our solution also detects whether such a permutation matrix exists. Indeed, the solution of Problem 3 determines whether there exists wanted doubly-stochastic matrix, and if so, computes it. So, if it does not exist or if the obtained doubly-stochastic matrix is not a permutation matrix, we conclude that there does not exist a permutation matrix with the wanted properties.

4.1. Solution of the Problem 1

One approach to Problem 1 would be a search through all $N!$ permutation matrices.

With approach given in this paper we can reduce the original problem to the case when the wanted permutation matrix has the form (2).

In fact, if $P \in \mathbb{R}^{N\times N}$ is a permutation matrix, then there exists a permutation $\sigma^P \in \mathcal{S}_N$, such that the entries of P are $P_{i,\sigma_i^P} = 1$ for all $i = 1, \dots, N$, and zero otherwise.

Let $\mathcal{P}_{r-k+1}$ be a subset of the set of all permutation matrices P of size $N \times N$ such that $\sigma^P_{r-k+2} < \sigma^P_{r-k+3} < \cdots < \sigma^P_N$. Then, every permutation matrix from $\mathbb{R}^{N\times N}$ can be written in a unique way as a product of two permutation matrices of the form:

$$\begin{bmatrix} I_{r-k+1} & 0 \\ 0 & \Pi' \end{bmatrix} \Delta, \tag{23}$$

where $\Delta \in \mathcal{P}_{r-k+1}$.

Thus, our problem becomes to find Π' and Δ such that the columns of $\begin{bmatrix} I_{r-k+1} & 0 \\ 0 & \Pi' \end{bmatrix} \Delta D$ belong to S. So, for each $\Delta \in \mathcal{P}_{r-k+1}$, we can search for a matrix Π' by using our solution to Problem 2. Moreover, if for some Δ there exists a corresponding matrix Π', then Π' is unique.

So, computationally, we only have to check $|\mathcal{P}_{r-k+1}| = N(N-1)\cdots(N-r+k) \sim N^{r-k+1}$ combinations. Moreover, in practical applications usually $r = k$, thus giving only N possibilities for the matrix Δ through which one should search.

For more details on the implementation of the algorithm, see [4,3].

Finally, note that, since for every $\Delta \in \mathcal{P}_{r-k+1}$ there exists at most one Π' such that the matrix (23) solves the Problem 1, there are at most $|\mathcal{P}_{r-k+1}|$ permutation matrices that solve the Problem 1.

4.2. Relation with a correspondence problem

Our solution to Problem 2 implies, as a corollary, a solution to a classical correspondence problem in a generic case.

Let D be a matrix from (1), and let E be some other ordering of the rows of D, i.e.:

$$E = \begin{bmatrix} x_{\pi(1)} & \cdots & x_{\pi(N)} \end{bmatrix}^T, \tag{24}$$

for some permutation π. The correspondence problem consists of finding a permutation matrix Π_0, such that

$$\Pi_0 D = E. \tag{25}$$

By taking S_E to be a subspace spanned by the columns of E, (25) implies that the columns of $\Pi_0 D$ belong to S_E.

In previous section we have obtained the algorithm for finding a permutation matrix Π, such that the columns of ΠD belong to S_E. Also, in a generic case, the rank of S_E is equal to k, and so the number of such permutation matrices Π is at most $|\mathcal{P}_1| = N$. Moreover, by the algorithm from the previous section, we can obtain all such permutation matrices and the one that satisfies (25) will be the wanted matrix Π_0.

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