The Papoulis-Gerchberg Algorithm with Unknown Signal Bandwidth

Manuel Marques, Alexandre Neves, Jorge S. Marques, and João Sanches

Instituto Superior Técnico & Instituto de Sistemas e Robótica
1049-001 Lisboa - Portugal

manuel@isr.ist.utl.pt
alex.filipe@gmail.com
jsm@isr.ist.utl.pt
jmrs@alfa.ist.utl.pt

Abstract. The Papoulis-Gerchberg algorithm has been extensively used to solve the missing data problem in band-limited signals. The interpolation of low-pass signals with this algorithm can be done if the signal bandwidth is known. In practice, the signal bandwidth is unknown and has to be estimated by the user, preventing an automatic application of the Papoulis-Gerchberg algorithm. In this paper, we propose a method to automatically find this parameter, avoiding the need of the user intervention during the reconstruction process. Experimental results are presented to illustrate the performance of the proposed algorithm.

1 Introduction

The reconstruction of signals after a non uniform sampling of the oriiginal signal is a key problem in many areas such as communications, medical imaging, geophysics and astronomy. The reconstruction of a signal without a priori information is an ill-posed problem because the observed signals are often incomplete and only some samples are observed. For this reason, we require some information about signal for a successful reconstruction. In this work, we assume that signal is low-pass.
Several methods were proposed [4–8] to reconstruct low-pass signals from a set of non-uniform samples e.g., using Fourier Analysis and Wavelets in order to interpolate the signal. The Papoulis-Gerchberg algorithm (P-G) [1–3] is a popular technique. It amounts to alternatively applying the space and frequency information available about the signal until it converges. However, this algorithm requires the knowledge of the signal bandwidth. This parameter must be determined by the user by a manual way. This paper tries to avoid this procedure and provides an algorithm to automatically obtain the bandwidth estimate therefore making the P-G fully automatic.

2 Formulation Problem and Notation

A discrete signal with N samples can be described by a N-dimensional complex vector x. The elements of the vector are denoted by $x[0], x[1]$, $x[2], ..., x[N-1]$ and correspond to samples of the signal.

The Discrete Fourier Transform (DFT) of the signal $x \in \mathbb{C}^N$ is the vector $X \in \mathbb{C}^N$ given by

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}. \tag{1}$$

Because the DFT is a linear map in $\mathbb{C}^N$, the expression (1) can be represented in matrix form,

$$X = Fx, \tag{2}$$

where F is a $N \times N$ matrix and each element is given by

$$F_{ab} = e^{-j\frac{2\pi}{N}ab}$$

In this work, we will only consider band-limited signals. The DFT of these signals presents the following property [9]

$$X[i] = 0, i \in S \tag{3}$$

where S is a proper and fixed nonempty subset of $\{0, 1, ..., N-1\}$. The set of band-limited signals that verify (3) is a linear subspace of $\mathbb{C}^N$. The dimension of this subspace is equal to the cardinal of the complement of S and it is often denoted as the signals bandwidth. The signals are low-pass if the complement of S can be written as follows

$$\{0\} \cup \{1, 2, ..., M\} \cup \{N - M + 1, N - M + 2, ..., N - 1\}$$

In this case, the signal is characterized by the DC coefficient and by the first M harmonics. If the number of known samples L satisfies (4),

$$L \geq 2M + 1 \tag{4}$$

the signal's reconstruction without error is possible.
The signals considered in this work are low-pass ones. Therefore, they can be written as follows

$$x = Bx \tag{5}$$

where $B = F^{-1}\Gamma F$ and Γ is a $N \times N$ diagonal matrix defined by:

$$\Gamma = \text{diag}[1, \underbrace{1, ..., 1}_{M1's}, 0, ..., 0, \underbrace{1, ..., 1}_{M1's}]$$

The problem we intent to solve is to reconstruct the low-pass signal $x[n]$ knowing a subset of its samples given by

$$y = Dx \tag{6}$$

where D is diagonal matrix with binary diagonal coefficients: $d_{ii} = 1$ if the $i - th$ sample is observed and $d_{ii} = 0$ otherwise. It is important to say that we assume that M is unknown.

3 The Papoulis-Gerchberg Algorithm

Let $x[n]$ be a low-pass signal with $M+1$ harmonics. The Papoulis-Gerchberg algorithm reconstructs the signal if the condition (4) is satisfied and the M value is known. The algorithm starts with an initial signal estimate $\hat{x}_1 = y$ (6).
The iterative process consists of three steps.

1. Filter $\hat{x}_i$ with a low-pass filter, eliminating the components with frequencies higher than the frequency of the $M-th$ harmonic.

$$z_{i+1} = B_{PG}\hat{x}_i \tag{7}$$

2. Insert known samples in the estimative.

$$\hat{x}_{i+1} = Ds + (I-D)z_{i+1} \tag{8}$$

3. Verify if the process converged or not. If not, return to the first step and consider $i = i+1$.

The matrix B_{PG} performs the low-pass filtering operation and it is defined by

$$B_{PG} = F^{-1}\Gamma_{PG}F \tag{9}$$

where

$$\Gamma = \text{diag}[1, \underbrace{1, ..., 1}_{M_{PG}1's}, 0, ..., 0, \underbrace{1, ..., 1}_{M_{PG}1's}]$$

is a $N \times N$ diagonal matrix and M_{PG} is the bandwith estimate.
When the signal bandwidth is known, $M_{PG} = M$ and $B_{PG} = B$.
The algorithm convergence is proved in [1]. In this paper, the stop criterion is based on the L_1 norm between signals of two consecutive estimations.

$$\frac{1}{N}\sum_{n=1}^{N}|y_{i+1}[n] - y_i[n]| < \delta \tag{10}$$

where δ is a threshold specified by the user.

4 The Estimation of Signal Bandwidth

4.1 Motivation

As mentioned before, a perfect reconstruction of a non uniform sampled signal can be obtained by the Papoulis-Gerchberg algorithm if the number of observed samples is enough (satisfies (4)) and if the M value is known. However, in many situations, this last condition is not true and it becomes important to use an alternative way to solve the problem.
In order to motivate this problem we shown in Fig. 1(a) a discrete low-pass signal with 256 samples and bandwidth $M = 40$. The Fourier spectrum of the original signal is shown in figure 1(c). The signal was then randomly

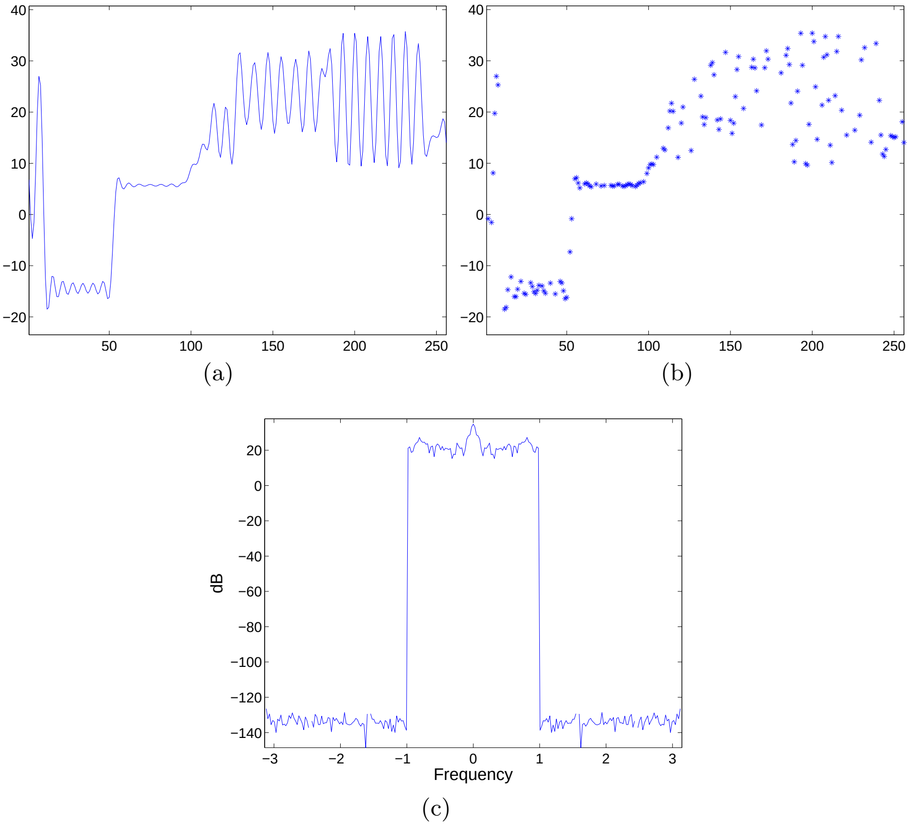


Fig. 1: Synthetic example: 60% of all samples are known.
(a) Original signal (b) Known samples (c) DFT

sampled: 60% of the samples are known and the others remain unknown (see Fig. 1(b)).

Before we describe the new method, it is relevant to discuss the importance of the signal bandwidth M. Figure 2 presents the SNR results of the estimates obtained by the P-G using M_{PG} values ranging between 1 and the largest integer satisfying (4). In this figure, we can see that a correct reconstruction (high SNR) of the original signal is obtained for a small range of M_{PG} values, close to the true value of the bandwidth M. When M_{PG} is lower or much higher than M, the P-G estimate does not converge to the original signal.

When the M_{PG} value (9) is lower than M, the reconstructed signal $\hat{x}[n]$ has a low-pass spectrum

$$\hat{X}[i] = 0, \; i \in T \tag{11}$$

where $T \supset S$. This means that we wish to find a signal that contains the known samples and less harmonics than the original. Due to this last

restriction, the original signal is not in the complement of the subspace T and the mentioned algorithm does not converge to this signal because the step 2 creates discontinuities in the time domain.
When the M_{PG} value is greater than M the reconstructed signal is low-pass with

$$\hat{X}[i] = 0,\ i \in V \tag{12}$$

where $V \subset S$. In this case, the complement of the subspace V contains the original signal x but the P-G will nor probably converge to the original signal x since there is an infinite number of signals in the subspace V satisfying (12) for the known samples.

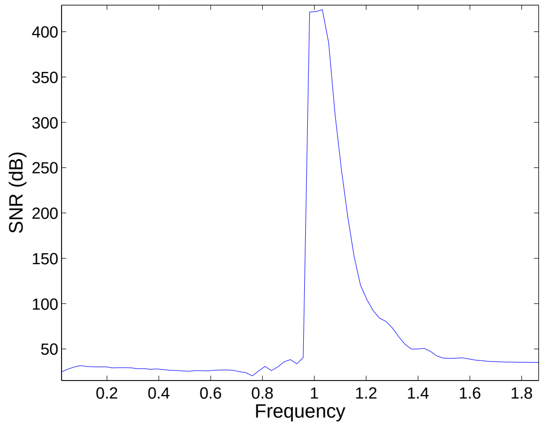


Fig. 2: Reconstruction performance SNR as function of the filter B_{PG} cut frequency

The signal to noise ratio, presented in Fig. 2, provides a valuable information about the signal bandwidth. However it cannot be used in practice because we do not know the original signal. An alternative technique must be used instead as discussed in section 4.2.

4.2 The proposed method

The main goal of the proposed method is to reconstruct the original signal from a set of non uniform samples, without knowing the signal bandwidth. The strategy followed in this paper consists of finding the lowest M that leads to a low-pass signal estimate (12) with the known samples. In other words, we wish to find a signal that contains known samples and is defined by the smallest number of Fourier coefficients $(2M + 1)$.
This is performed by applying the P-G algorithm for all M values such that inequality (4) holds. For each M_{PG} value, the percentage of energy contained in the signal's high frequencies is calculated as well as the log energy ratio

$$g[k] = log_{10} \frac{E_h(k)}{E_T(k)} \ ,\ k = 0, 1, \ldots, \lfloor L/2 \rfloor \tag{13}$$

where E_h and E_T are the high frequency and the total energies, respectively, defined as follows

$$E_T(k) = \sum_{m=0}^{N-1} |\hat{X}^k[m]|^2 \tag{14}$$

$$E_h(k) = \sum_{m=h+1}^{N-h+1} |\hat{X}^k[m]|^2 \tag{15}$$

where $\hat{X}^k$ is Fourier transform of the signal $\hat{x}_k$. This signal is obtained by the P-G algorithm with $M_{PG} = k$. The log energy ratio function provides very useful information about the signal bandwidth.
The bandwidth estimate proposed in this paper is the value of k which minimizes the difference $g[k] - g[k-1]$. It has been experimentally found that the first difference of the energy ratio $g[k]$ has its largest fall for $k = M$. This estimate is not the unique M_{PG} that allows to obtain a low pass signal but it is the first one. With this strategy, we find the smallest linear subspace (the complement of V) that contains the low pass signal verifying (6).
To illustrate the estimation of M, we show, in Fig. 3, the log energy ratio function obtained for the sampled signal presented in the Fig. 1(b).

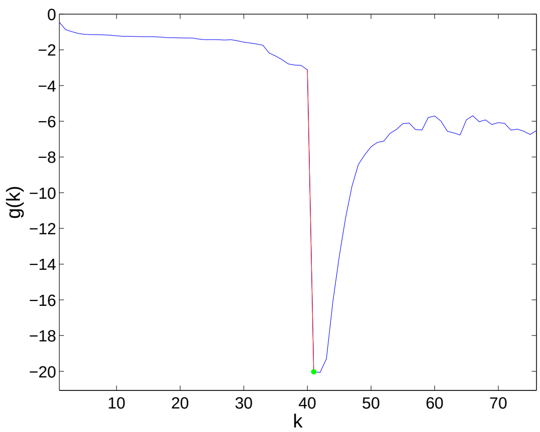


Fig. 3: The log energy ratio function for the example in figure 1(b)

The smallest difference between two consecutive values of $g[k]$ is shown with a bold line. In this case, the bandwidth estimate is equal to the true value $M = 40$.
This method can be extended to 2D low-pass signals in several ways e.g., considering a 2D signal as a set 1D low-pass signals. In the case of images we can independently apply this method to columns or rows or use a joint estimation procedure.

5 Results

To evaluate the proposed method, several experiments were performed with synthetic signals and real signals. Two experiments will be shown in this section. In the first experiment we have generated 2000 random signals with length $N = 256$ and bandwidth $M = 40$ and applied the algorithm to each of them.
The signals were generated as follows. First we split the time domain into a set of intervals with random lengths. Then, for each interval we generated one of the following signals: constant, linear or sinusoid with random parameters. Finally, we eliminated the high frequency components to guarantee that the signal is low pass with bandwidth $M = 40$.

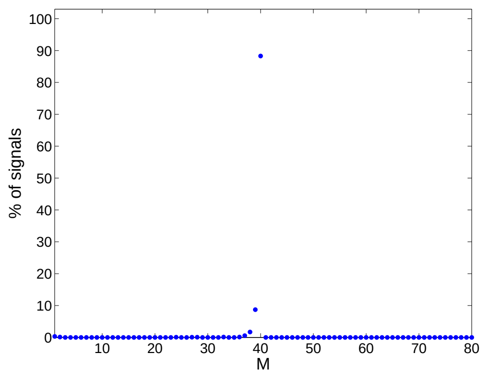
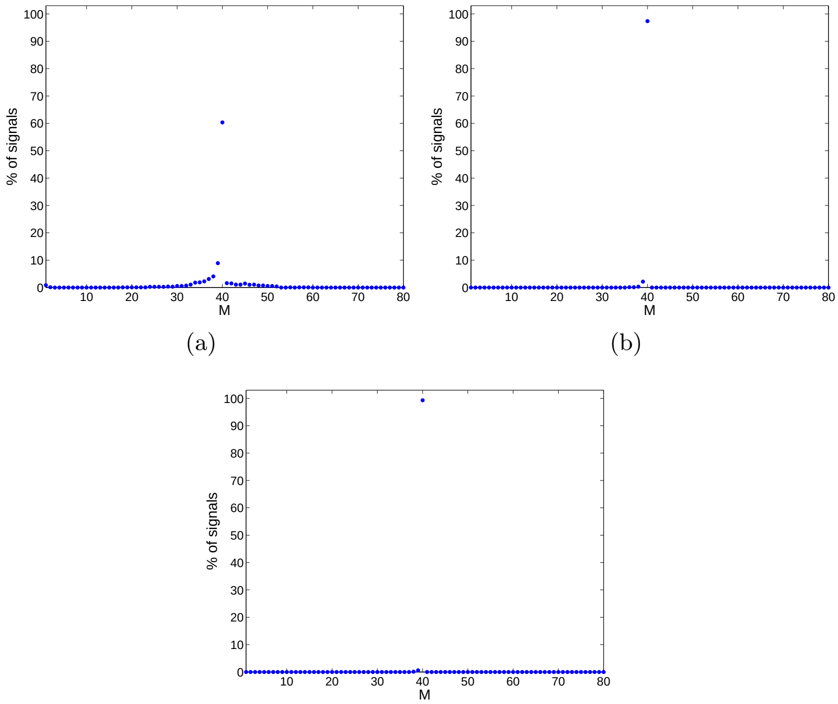


Fig. 4: Histogram of M values estimated with $\delta = 10^{-3}$ and 80% of known samples

Figures 4 and 5 display the histogram of the bandwidth estimates for several values of δ (see (10)) and percentage of unknown samples. The performance of the algorithm depends on both parameters. We observe a smooth degradation of the performance when the number of observations decreases from 80% to 40% of N.
Evaluating results presented in these figures, we can say that the percentage of known samples affects the method's performance, but even when we have a small set of known samples (40% when the lowest percentage to satisfy (4) is 31,6%) the method gives 60% of correct answers (Fig. 5(a)) and most of the errors are small.
Concerning the parameter δ, small values of δ lead to better estimates. However, the estimation method becomes slower because we have to perform L iterations.
Table 1, shows the mean and standard deviation of SNR results of the estimated signals obtained by the method for unknown M and the P-G algorithm for known M. The two methods lead to indistinguishable results when the percentage of known samples is $50\%, 60\%, 70\%, 80\%$. Only in the case 40% of known samples we observe a difference of $3dB$ between the reconstructions performed with known and unknown M.



(a) (b)

(c)

Fig. 5: Histograms of M values estimated with $\delta = 10^{-6}$: (a) 40% of known samples (b) 60% of known samples (c) 80% of known samples

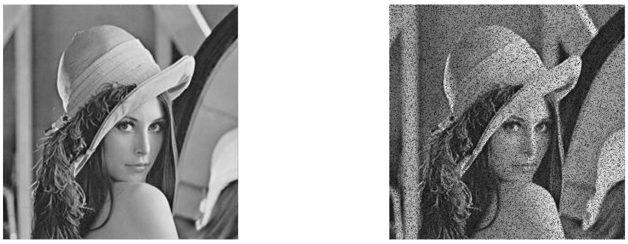
Table 1: Results of the P-G algorithm for the case of known bandwidth and unknown bandwidth

Known Samples	M known		M unknown	
(%)	Mean(dB)	STD(dB)	Mean(dB)	STD(dB)
40	24,2	15,3	21,5	16,3
50	56,9	21,7	56,0	22,9
60	82,9	14,7	82,9	14,9
70	97,3	10,3	97,3	10,3
80	108,0	7,7	108,0	7,8

The proposed method was used to reconstruct real signal and images from an incomplete set of samples. To illustrate the method's performance in the case of images we applied the proposed algorithm to the *Lena* image with 15% of unknown samples (see Fig. 6(a)).

We created an enlarged version of the *Lena* image with 512×512 pixels and bandwidth $M = 128$. This was done by padding with zeros an initial version of the *Lena* image with 256×256 and zeroing the high fre-

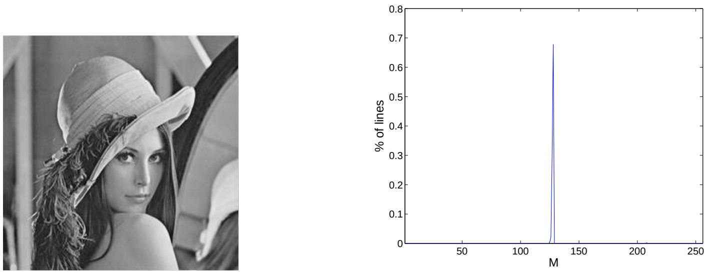
quency coefficients of the Fourier spectrum. We then applied the proposed algorithm to each row independently.
In this case, the Fig. 6(a) has 512×512 pixels and $M = 128$.



(a) (b)

Fig. 6: Non uniform sampling: (a) Original image (b) Sampled image with 85% of known samples

Figure 7(b) shows the histogram of the M estimates for the image's rows. The Fig. 7(a) displays the reconstructed image obtained with the proposed method which is indistinguishable from the original. A SNR=30 dB was obtained.



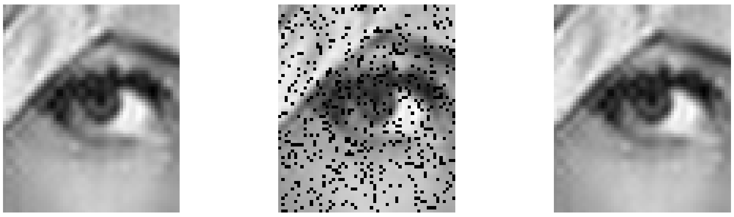
(a) (b)

Fig. 7: Method's results: (a) Estimated image (b) Histogram of M values

For a better evaluation of the method's performance, we show a detail of the eye. The original image (Fig. 8(a)) and the reconstructed image (Fig. 8(c)) are almost undistinguishable.

6 Conclusions

This paper presents a method to estimate the signal bandwidth for non-uniform sampled signal. This allows an automatic reconstruction of non-uniform sampled signal using the Papoulis-Gerchberg algorithm, when the



(a) (b) (c)

Fig. 8: Eye Reconstruction: (a) Original image (b) Sampled image (c) Estimated image

bandwidth is unknown. Very good experimental results are obtained with synthetic and real signals.

References

1. Papoulis, A.: A new algorithm in spectral analysis and band-limited extrapolation. IEEE Transactions on Circuits Syst. **19** (1975) 735–742
2. Papoulis, A.: A new method in image restoration. JSTAC, Paper VI-3 (1973)
3. Gerchberg, R. W.: Super-resolution through error energy reduction. Optica Acta **21-9** (1974) 709–720
4. Wingham, D. J.: The reconstruction of a band-limited function and its Fourier transform from a finite number of samples at arbitrary locations by singular value decomposition. IEEE Transactions on Circuit Teory **40** (1992) 559–570
5. Yen, J. L.: On nonuniform sampling of bandwidth-limited signals. IRE Transactions on Circuit Teory **CT-3** (1956) 251–259
6. Thomas, J. O. and Yao, K.: On some stability and interpolatory properties of nonuniform sampling expansions. IEEE Transactions on Circuit Teory **14** (1967) 404–408
7. Benedetto, J. and Chui, C. K. (ed): Irregular sampling and frames. In: Wavelets: A Tutorial in Teory and Applications, Academic Press (1992) 445–507
8. Feichtinger, H. G., Grchenig, K. and Strohmer, T: Efficient numerical methods in non-uniform sampling theory. Numerische Mathematik **69** (1995) 423–440
9. Ferreira, P. J. and Marvasti, F. A. (ed.): Iterative and Noninterative Recovery of Missing Samples for 1-D Band-Limited Signals. In: Sampling Theory and Practice, Plenum Publishing Corporation, 2001 235–282