Nonlinear Optimization

Part II
Conditions for optimality and duality theory

Instituto Superior Técnico – Carnegie Mellon University

jxavier@isr.ist.utl.pt

Duality

Primal problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h_i(x) = 0 \quad i = 1, \dots, p \\ & g_j(x) \leq 0 \quad j = 1, \dots, m \\ & x \in X \end{array}$$

- $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the cost function
- x is the optimization variable

Lagrangian:

$$L : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\} \qquad L(x;\lambda,\mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

Lagrange dual function:

$$L : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R} \cup \{-\infty\} \qquad L(\lambda,\mu) = \inf \{L(x;\lambda,\mu) \,:\, x \in X\}$$

Dual function is concave (usually, no explicit formula)

Lagrange dual problem:

$$\begin{array}{ll} \text{maximize} & L(\lambda, \mu) \\ \text{subject to} & \mu \geq 0 \end{array}$$

Dual problem is convex

Toy example:

Primal

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & x \geq 0 \\ & 1_n^\top x = 1 \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & c = \mu + \lambda 1_n \\ & \mu \geq 0 \end{array}$$

Optimal value of the two problems is $\min\{c_1, c_2, \ldots, c_n\}$

Theorem (Weak duality) If x is primal feasible and (λ, μ) is dual feasible then

$$L(\lambda, \mu) \leq f(x).$$

Thus,

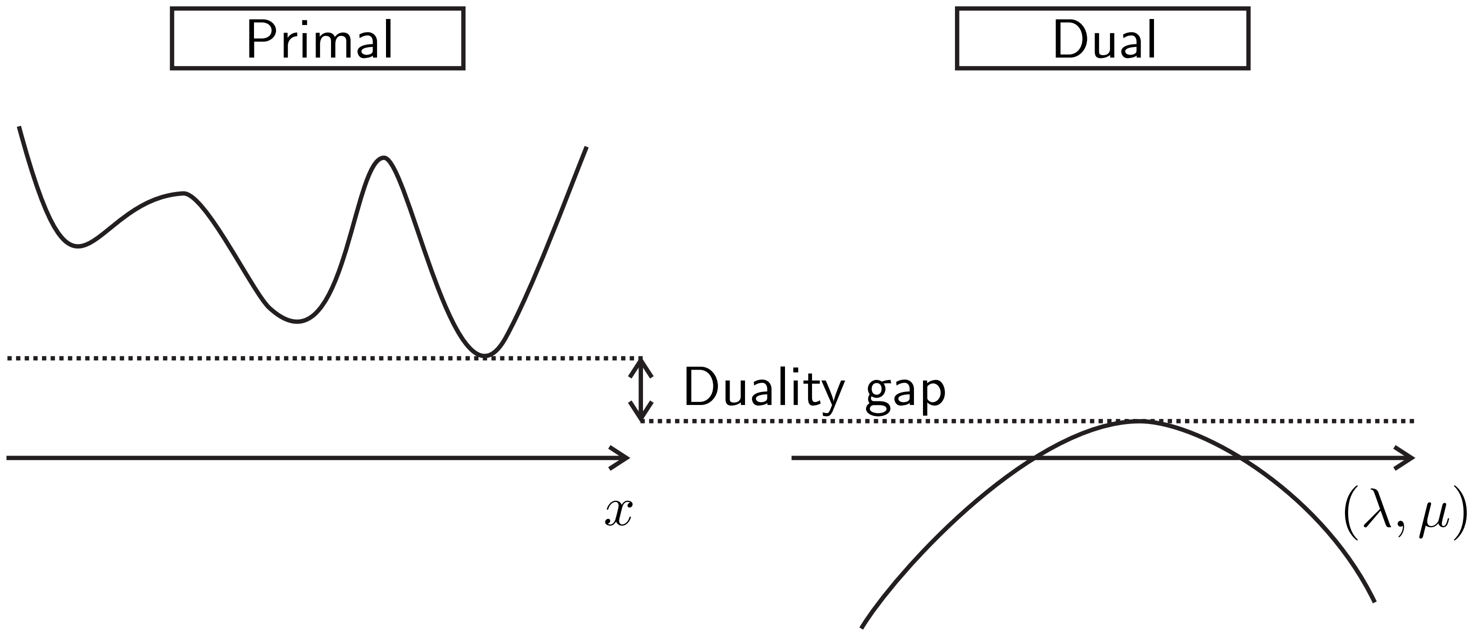
$$d^\star \leq p^\star$$

where

$$p^\star := \inf \{f(x) \,:\, h(x) = 0, g(x) \leq 0, x \in X\}$$

and

$$d^\star := \sup \{L(\lambda, \mu) \,:\, \mu \geq 0\}$$



The duality gap is defined as $p^\star - d^\star$ and belongs to $[0, +\infty]$

Corollary

- Primal is unbounded implies dual is infeasible

  (Proof: $p^\star = -\infty \Rightarrow d^\star = -\infty$)

- Dual is unbounded implies primal is infeasible

  (Proof: $d^\star = +\infty \Rightarrow p^\star = +\infty$)

Theorem (Strong duality) Let the primal be

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X \\ & Ax \le b \end{array}$$

Assume:

- X is convex
- $f : X \to \mathbb{R}$ is convex
- (Slater point) there exists $x_0 \in \operatorname{ri} X$ such that $Ax_0 \le b$

Then, strong duality holds ($p^\star = d^\star$) and the dual problem is solvable when $p^\star = d^\star$ are finite

When there is no Slater point, anything can happen

Example: no Slater point, no duality gap, dual is solvable

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & x \in X = [-1, 1] \\ & x \geq 1 \end{array}$$

Example: no Slater point, no duality gap, dual is not solvable

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & (x, y) \in X = \{(x, y) \, : \, x^2 \leq y\} \\ & y \leq 0 \end{array}$$

Example (support vector machines):

Primal

$$\begin{array}{ll} \text{minimize} & \|s\|^2 \\ \text{subject to} & X^\top s \geq (r+1)1_K \\ & Y^\top s \leq (r-1)1_L \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -\frac{1}{4}\|X\mu - Y\nu\|^2 + 1_K^\top \mu + 1_L^\top \nu \\ \text{subject to} & 1_K^\top \mu = 1_L^\top \nu \\ & \mu \geq 0, \nu \geq 0 \end{array}$$

Dual shows that $p^\star (= d^\star)$ only depends on X, Y through their inner-products

$$\begin{bmatrix} X^\top X & X^\top Y \\ Y^\top X & Y^\top Y \end{bmatrix}$$

The dual can be simplified:

$$\begin{array}{ll} \text{minimize} & \|X\mu - Y\nu\|^2 \\ \text{subject to} & 1_K^\top \mu = 1_L^\top \nu = 1 \\ & \mu \geq 0, \nu \geq 0 \end{array}$$

Geometrical interpretation: computing $p^\star$ is equivalent to evaluating

$$\mathrm{dist}\,(\mathrm{co}\,X, \mathrm{co}\,Y)$$

Example: Portfolio optimization

- T euros to invest among n assets
- r_i is the random rate of return of ith asset
- diversity constraint: no more than 80% of the investment should be concentrated in any $k \leq n$ stocks

Goal: maximize expected return subject to the constraints

Optimization problem:

$$
\begin{array}{ll}
\text{maximize} & r^\top x \\
\text{subject to} & x \geq 0,\ 1_n^\top x = T \\
& x_{[1]} + x_{[2]} + \cdots + x_{[k]} \leq 0.8T
\end{array}
$$

Optimization variable is $x \in \mathbb{R}^n$

Equivalent problem:

$$
\begin{array}{ll}
\text{maximize} & r^\top x \\
\text{subject to} & x \geq 0,\ 1_n^\top x = T \\
& 1_n^\top \nu - k\lambda \leq 0.8T \\
& x \leq \nu - \lambda 1_n \\
& \nu \geq 0
\end{array}
$$

Optimization variable is $(x, \nu, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$

Theorem (Strong duality) Let the primal be

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X \\ & Ax \leq b \\ & g(x) = (g_1(x), \ldots, g_m(x)) \leq 0 \end{array}$$

Assume:

- $X = C \cap P$ where C is convex and P is a polyhedron
- $f,\, g_1,\, \ldots,\, g_m \,:\, C \to \mathbb{R}$ are convex
- (Slater point) there exists $x_0 \in (\mathrm{ri}\, C) \cap P$ such that $Ax_0 \leq b$ and

$$g(x_0) = (g_1(x_0), \ldots, g_m(x_0)) < 0$$

Then, strong duality holds ($p^\star = d^\star$) and the dual problem is solvable when $p^\star = d^\star$ are finite

Duality gap depends on how the problem is represented

Example:

$$\begin{array}{ll} \text{minimize} & e^{-y} \\ \text{subject to} & x \geq 0 \\ & y = 0 \end{array}$$

No duality gap: $p^\star = d^\star = 1$

"Same" example:

$$\begin{array}{ll} \text{minimize} & e^{-y} \\ \text{subject to} & \sqrt{x^2 + y^2} \leq x \end{array}$$

There is a duality gap: $p^\star = 1$ and $d^\star = 0$

Example: nonconvex problem with zero duality gap ($A \in \mathsf{S}^n$)

Primal

$$\begin{array}{ll} \text{minimize} & x^\top A x \\ \text{subject to} & x^\top x = 1 \end{array}$$

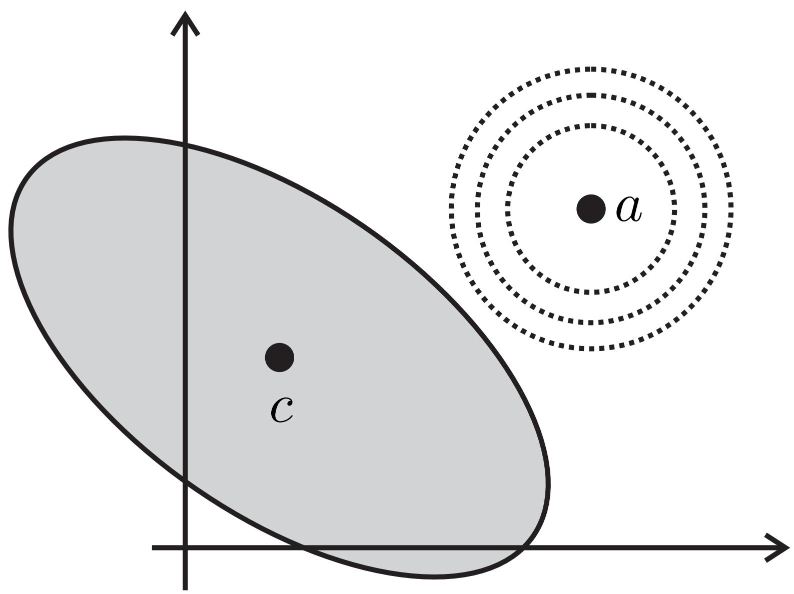
Dual

$$\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & A - \lambda I_n \succeq 0 \end{array}$$

$p^\star = d^\star = \lambda_{\min}(A)$

Example: projection onto an ellipsoid

$$\begin{array}{ll} \text{minimize} & \|x-a\|^2 \\ \text{subject to} & (x-c)^\top A(x-c) \le 1 \end{array}$$



By the strong duality theorem $p^\star = d^\star$

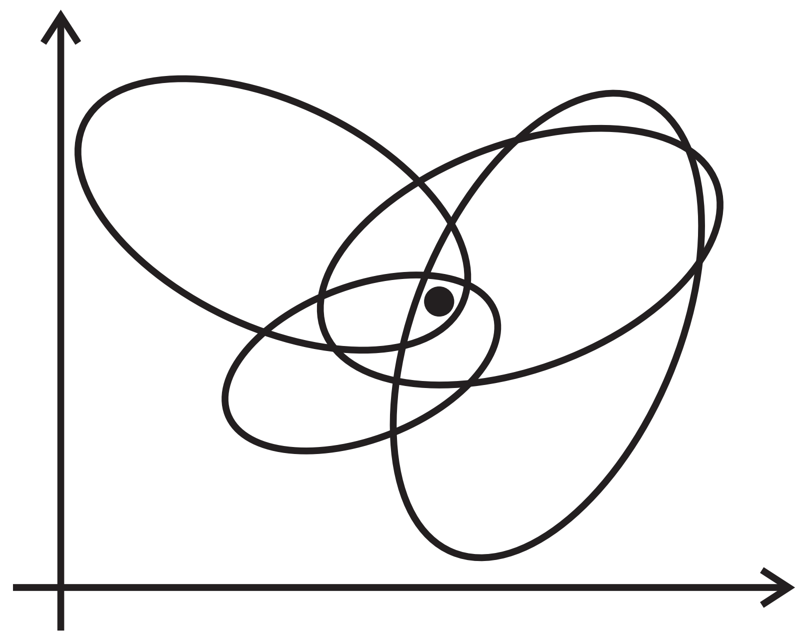
The dual program is

$$\begin{array}{ll} \text{maximize} & -\frac{1}{2}\left(c-a\right)^\top \left(I_n + \mu A\right)^{-1}\left(c-a\right) - \mu + \left\|c-a\right\|^2 \\ \text{subject to} & \mu \geq 0 \end{array}$$

Application of duality: certificates of infeasibility

m ellipsoids $E_i = \{x \in \mathbb{R}^n \,:\, x^\top A_i x + 2b_i^\top x + c_i \leq 0\}$ are given

Do their interiors intersect ?



Equivalently: is this system of strict quadratic inequalities solvable ?

$$\begin{cases} x^\top A_1 x + 2b_1^\top x + c_1 & < & 0 \\ x^\top A_2 x + 2b_2^\top x + c_2 & < & 0 \\ & \vdots & \\ x^\top A_m x + 2b_m^\top x + c_m & < & 0 \end{cases}$$

System is unsolvable iff $p^\star \geq 0$ where

$$p^\star = \inf\{s \,:\, g(x) \leq s1_m\}$$

where $g = (g_1, g_2, \dots, g_m)$, $g_i(x) = x^\top A_i x + 2b_i^\top x + c_i$

Through the strong duality theorem:

$$p^\star \geq 0 \quad \Leftrightarrow \quad \exists_{\mu \geq 0, \mu \neq 0} : c(\mu) - b(\mu)^\top A(\mu)^{-1} b(\mu) \geq 0$$

where

$$A(\mu) := \sum_{i=1}^{m} \mu_i A_i \quad b(\mu) := \sum_{i=1}^{m} \mu_i b_i \quad c(\mu) := \sum_{i=1}^{m} \mu_i c_i$$

Using Schur's complement, the answer boils down to feasibility of a LMI:

$$\mu \geq 0, \quad 1^\top \mu > 0, \quad \begin{bmatrix} A(\mu) & b(\mu) \\ b(\mu)^\top & c(\mu) \end{bmatrix} \succeq 0$$

A feasible μ for the above LMI **certifies** that $\bigcap_{i=1}^{m} \operatorname{int} E_i = \emptyset$

Example: approximating sum of ellipsoids

- Discrete-time linear dynamical system:

$$\begin{cases} x[n+1] = Ax[n] + Bu[n] & (A : m \times m \text{ and } B \text{ are known}) \\ x[0] = 0 & (\text{system initially at rest}) \\ \|u[t]\| \leq 1 & (\text{inputs } u[t] \text{ are unknown}) \end{cases}$$

- What is the set $S_N \subset \mathbb{R}^m$ of reachable states at $n = N$?

Since

$$x[N] = A_1 u[1] + A_2 u[2] + \cdots + A_N u[N] \qquad (A_n := A^{N-n} B)$$

the set S_N is a sum of ellipsoids

S_N can be written as

$$S_N = \mathsf{E}(W_1) + \mathsf{E}(W_2) + \cdots + \mathsf{E}(W_N)$$

for appropriate matrices W_n, where $\mathsf{E}(W) := \{x \in \mathbb{R}^m \,:\, x^\top W x \leq 1\}$

Approximate S_N by one outer ellipsoid $\mathsf{E}(W)$:

$$\begin{array}{ll} \text{minimize} & \mathsf{vol}\,(\mathsf{E}(W)) \\ \text{subject to} & S_N \subset \mathsf{E}(W) \end{array}$$

$\mathsf{vol}(\mathsf{E}(W)) \propto \det\left(W^{-1}\right)$ is the volume of the ellipsoid $\mathsf{E}(W)$

How to code the constraint ?

$S_N = \mathsf{E}(W_1) + \cdots + \mathsf{E}(W_N) \subset \mathsf{E}(W)$ if and only if

$$\sup\{(x_1 + \cdots + x_N)^\top W (x_1 + \cdots + x_N) \,:\, x_i^\top W_i x_i \leq 1\} \leq 1$$

By weak duality, a sufficient condition for $S_N \subset E(W)$ to hold is:

$$\exists_{\mu=(\mu_1,\mu_2,\ldots,\mu_N)} \,:\, \mu \geq 0,\ 1^\top \mu \leq 1,\ B(\mu, W) \preceq 0$$

where

$$B(\mu, W) := \begin{bmatrix} W - \mu_1 W_1 & W & \cdots & W \\ W & W - \mu_2 W_2 & \cdots & W \\ \vdots & \ddots & \ddots & \vdots \\ W & \cdots & W & W - \mu_N W_N \end{bmatrix}$$

Convex approximation of the original problem:

$$\begin{array}{ll} \text{minimize} & -\log\det(W) \\ \text{subject to} & \mu \geq 0 \\ & 1^\top \mu \leq 1 \\ & B(\mu, W) \preceq 0 \end{array}$$

The variable to optimize is $(\mu, W) \in \mathbb{R}_+^N \times \mathsf{S}_{++}^N$

Primal problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h_i(x) = 0 \quad\quad i = 1, \ldots, p \\ & g(x) \preceq_K 0 \\ & x \in X \end{array}$$

- $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the cost function
- $g : \mathbb{R}^n \to \mathbb{R}^m$
- K is a convex cone in $\mathbb{R}^m$
- x is the optimization variable

Lagrangian:

$$L : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\} \qquad L(x; \lambda, \mu) = f(x) + \langle \lambda, h(x) \rangle + \langle \mu, g(x) \rangle$$

Lagrange dual function:

$$L : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R} \cup \{-\infty\} \qquad L(\lambda, \mu) = \inf \{ L(x; \lambda, \mu) \, : \, x \in X \}$$

Lagrange dual problem:

$$\begin{array}{ll} \text{maximize} & L(\lambda, \mu) \\ \text{subject to} & \mu \succeq_{K^*} 0 \end{array}$$

Dual problem is convex

Theorem (Weak duality) If x is primal feasible and (λ, μ) is dual feasible then

$$L(\lambda, \mu) \leq f(x).$$

Thus,

$$d^\star \leq p^\star$$

where

$$p^\star := \inf \{ f(x) \,:\, h(x) = 0, g(x) \leq 0, x \in X \}$$

and

$$d^\star := \sup \{ L(\lambda, \mu) \,:\, \mu \succeq_{K^*} 0 \}$$

Corollary

- Primal is unbounded implies dual is infeasible
- Dual is unbounded implies primal is infeasible

Theorem (Strong duality) Let the primal be

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X \\ & A(x) \leq b \\ & g(x) \preceq_K 0 \end{array}$$

Assume:

- $X = C \cap P$ where C is convex and P is a polyhedron
- $f : C \to \mathbb{R}$ is convex, $A(\cdot)$ is a linear map and g is K-convex
- (Slater point) there exists $x_0 \in (\mathrm{ri}\, C) \cap P$ such that $A(x_0) \leq b$ and

$$g(x_0) \prec_K 0$$

Then, strong duality holds ($p^\star = d^\star$) and the dual problem is solvable when $p^\star = d^\star$ are finite

Example: "MAXCUT"-like optimization problem

$$\begin{array}{ll} \text{maximize} & x^\top A x \\ \text{subject to} & x_i^2 = 1 \quad i = 1, \ldots, n \end{array}$$

Equivalent reformulation:

$$\begin{array}{lll} \text{maximize} & \mathrm{tr}(AX) & \\ \text{subject to} & X_{ii} = 1 & i = 1, \ldots, n \\ & X \succeq 0 & \\ & \mathrm{rank}\, X = 1 & \end{array}$$

Convex relaxation:

$$\begin{array}{ll} \text{maximize} & \mathrm{tr}(AX) \\ \text{subject to} & X_{ii} = 1 \quad i = 1, \ldots, n \\ & X \succeq 0 \end{array}$$

Dual of "MAXCUT"-like problem is

$$
\begin{array}{ll}
\text{minimize} & \mathbf{1}^\top \lambda \\
\text{subject to} & A - \mathsf{Diag}(\lambda) \preceq 0
\end{array}
$$

Strong duality holds: dualizing again (bi-dual) gives

$$
\begin{array}{ll}
\text{maximize} & \mathsf{tr}(AX) \\
\text{subject to} & X_{ii} = 1 \quad i = 1, \ldots, n \\
& X \succeq 0
\end{array}
$$

Karush-Kuhn Tucker (KKT) conditions

Convex primal problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h_i(x) = 0 \quad i = 1, \ldots, p \\ & g_j(x) \leq 0 \quad j = 1, \ldots, m \\ & x \in X \end{array}$$

Dual problem:

$$\begin{array}{ll} \text{maximize} & L(\lambda, \mu) \\ \text{subject to} & \mu \geq 0 \end{array}$$

Associated KKT system is:

$$\begin{cases} x \in \arg\min \{L(y; \lambda, \mu) \,:\, y \in X\} & (\text{“stationarity”}) \\ h(x) = 0,\ g(x) \leq 0 & (\text{primal feasibility}) \\ \mu \geq 0 & (\text{dual feasibility}) \\ g(x)^\top \mu = 0 & (\text{complementary slackness}) \end{cases}$$

System of conditions posed on $(x, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m$

Theorem

$$\left\{\begin{array}{l} x^\star \text{ is a primal solution} \\ (\lambda^\star, \mu^\star) \text{ is a dual solution} \\ \text{Strong duality (SD) holds} \end{array}\right. \quad \Leftrightarrow \quad (x^\star, \lambda^\star, \mu^\star) \text{ solves the KKT system}$$

Assumption: primal & dual solvability + SD

Solving the primal through the dual:

- solve dual and get $(\lambda^\star, \mu^\star)$
- set of primal solutions is

  $$X^\star = X^\star(\lambda^\star, \mu^\star) \cap \{x \,:\, h(x) = 0,\ g(x) \leq 0\} \cap \{x \,:\, g(x)^\top \mu^\star = 0\}$$

  where

  $$X^\star(\lambda^\star, \mu^\star) := \arg\min \{L(x; \lambda^\star, \mu^\star) \,:\, x \in X\}$$

Example:

- Primal

$$\begin{aligned} \text{minimize} \quad & -x \\ \text{subject to} \quad & x \in X = [-1, 1] \\ & x \le 0 \end{aligned}$$

$X^\star = \{0\}$

- Dual

$$\begin{aligned} \text{maximize} \quad & -|\mu - 1| \\ \text{subject to} \quad & \mu \ge 0 \end{aligned}$$

Dual solution is $\mu^\star = 1$ and $X^\star(\mu^\star) = [-1, 1]$

- There holds:

$$X^\star = X^\star(\mu^\star) \cap \{x \,:\, x \le 0\} \cap \{x \,:\, x\mu^\star = 0\}$$

Special case:

- f is strictly convex
- X is convex
- $h(x) = Ax - b$
- $g = (g_1, g_2, \ldots, g_m)$ is a convex map

Then, $X^\star(\lambda^\star, \mu^\star)$ is a singleton, i.e.,

$$X^\star = X^\star(\lambda^\star, \mu^\star)$$

Example:

- Primal

$$\begin{array}{ll} \text{minimize} & x^2 \\ \text{subject to} & x \in X = [-1, 1] \\ & x \leq 0 \end{array}$$

$X^\star = \{0\}$

- Dual problem

$$\begin{array}{ll} \text{maximize} & L(\mu) \\ \text{subject to} & \mu \geq 0 \end{array}$$

where

$$L(\mu) = \begin{cases} -\frac{\mu^2}{4} & , 0 \leq \mu \leq 2 \\ 1 - \mu & , \mu \geq 2 \end{cases}$$

Dual solution is $\mu^\star = 0$ and $X^\star(\mu^\star) = X^\star$

Example: separable problems

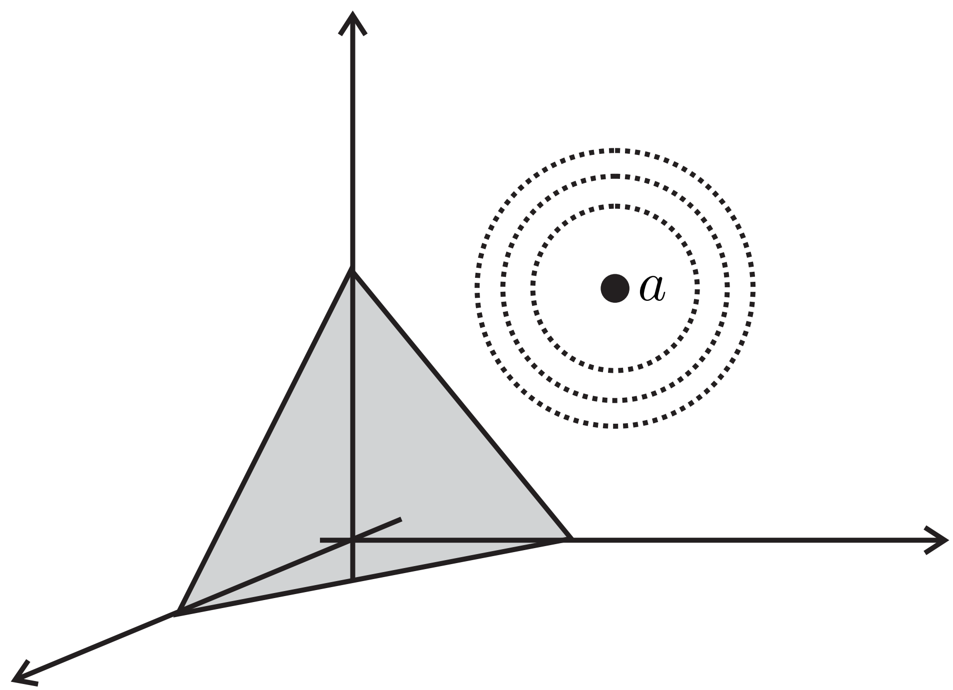
$$\begin{array}{ll} \text{minimize} & f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) \\ \text{subject to} & x_i \in X_i \\ & g_1(x_1) + g_2(x_2) + \cdots + g_n(x_n) \leq 0 \end{array}$$

Dual function is

$$L(\mu) = \sum_{i=1}^{n} \inf_{x_i \in X_i} f_i(x_i) + \mu^\top g(x_i)$$

Example: projection onto the probability simplex

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|x-a\|^2 \\ \text{subject to} & x \geq 0 \\ & 1_n^\top x = 1 \end{array}$$



KKT system:

$$\begin{cases} x - \mu = a + \lambda 1_n \\ x \geq 0 \\ \mu \geq 0 \\ x^\top \mu = 0 \\ 1_n^\top x = 1 \end{cases}$$

Fact: for $a, b, c \in \mathbb{R}^n$

$$\begin{cases} a - b = c \\ a \geq 0 \\ b \geq 0 \\ a^\top b = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} a = c^+ \\ b = c^- \end{cases}$$

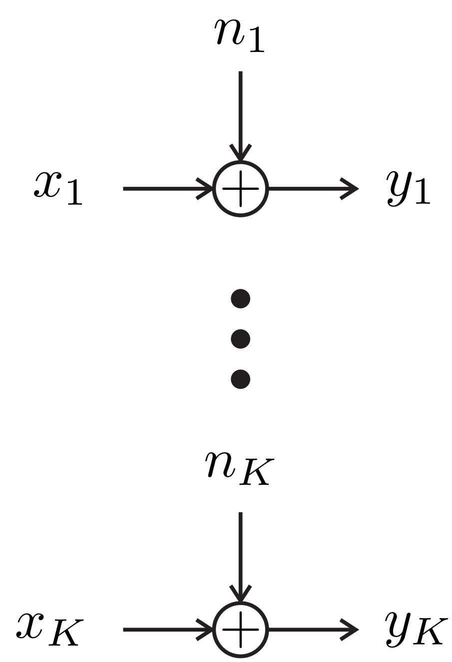
Simplification of KKT system:

$$\begin{cases} x = (a + \lambda 1_n)^+ \\ \mu = (a + \lambda 1_n)^- \\ 1_n^\top x = 1 \end{cases}$$

Solution:

- solve the piecewise linear equation $1_n^\top (a + \lambda^\star 1_n)^+ = 1$
- primal solution is $x^\star = (a + \lambda^\star 1_n)^+$

Example: capacity of K parallel Gaussian channels



$x_k \sim \mathcal{N}(0, P_k)$ and $n_k \sim \mathcal{N}(0, N_k)$

Power budget: $P_1 + P_2 + \cdots + P_K = P_0$

Goal: choose P_k's to maximize capacity

$$\begin{array}{ll} \text{maximize} & \sum_{k=1}^{K} \log\left(1 + \frac{P_k}{N_k}\right) \\ \text{subject to} & P \geq 0 \\ & 1_K^\top P = P_0 \end{array}$$

Optimization variable is $P = (P_1, P_2, \ldots, P_K)$

KKT system

$$\begin{cases} -\frac{1/N_k}{1+P_k/N_k} = \mu_k - \lambda,\ k = 1, 2, \ldots, K \\ P \geq 0,\ 1_K^\top P = P_0 \\ \mu \geq 0 \\ P^\top \mu = 0 \end{cases}$$

Top conditions ensure $\lambda > 0$ and defining $\eta := 1/\lambda$ yields

$$\begin{cases} \eta N_k \mu_k - P_k = N_k - \eta,\ k = 1, 2, \ldots, K \\ P \geq 0,\ 1_K^\top P = P_0 \\ \mu \geq 0 \\ P^\top \mu = 0 \end{cases}$$

Equivalent KKT system

$$\begin{cases} P_k - \eta N_k \mu_k = \eta - N_k \\ P_k \geq 0 \\ \eta N_k \mu_k \geq 0 \\ P_k(\eta N_k \mu_k) = 0 \\ 1_K^\top P = P_0 \end{cases}$$

Using the fact on page 52 yields

$$\begin{cases} P_k = (\eta - N_k)^+ \\ 1_K^\top P = P_0 \end{cases}$$

Solution:

- solve the piecewise linear equation $\mathbf{1}_n^\top(\eta^\star \mathbf{1}_K - N)^+ = P_0$
- primal solution is

$$P^\star = (\eta^\star \mathbf{1}_K - N)^+$$

This is known as a **water-filling** solution

Primal problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h_i(x) = 0 \quad i = 1, \ldots, p \\ & g(x) \preceq 0 \\ & x \in X \end{array}$$

Dual problem:

$$\begin{array}{ll} \text{maximize} & L(\lambda, \mu) \\ \text{subject to} & \mu \succeq_{K^*} 0 \end{array}$$

Associated KKT system is:

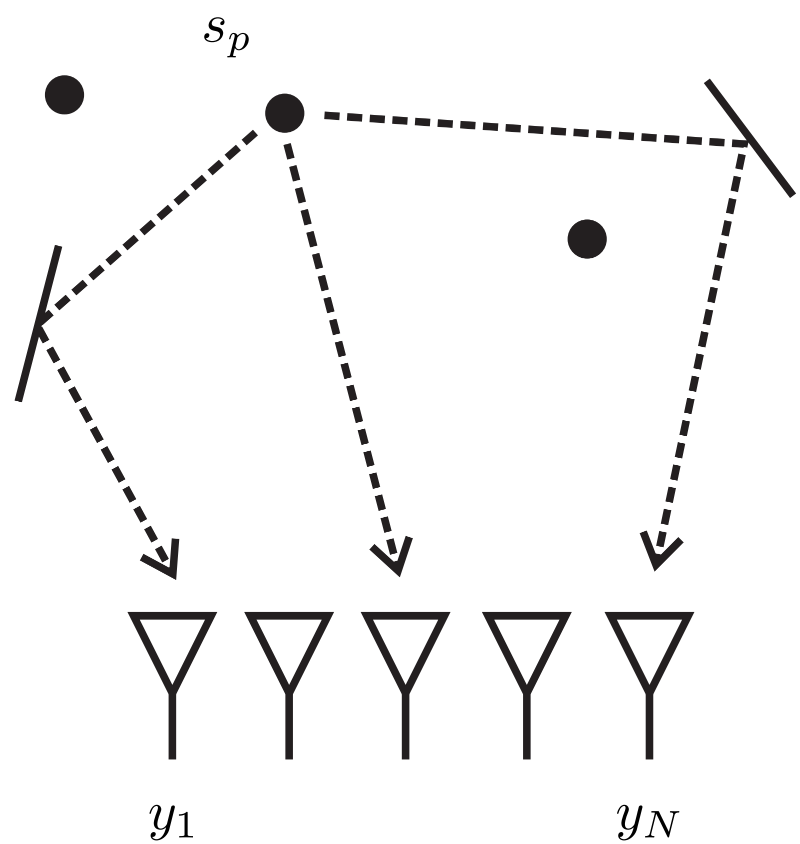
$$\begin{cases} x \in \arg\min \{L(y; \lambda, \mu) \,:\, y \in X\} & \text{(“stationarity”)} \\ h(x) = 0,\ g(x) \preceq_K 0 & \text{(primal feasibility)} \\ \mu \succeq_{K^*} 0 & \text{(dual feasibility)} \\ \langle g(x), \mu \rangle = 0 & \text{(complementary slackness)} \end{cases}$$

System of conditions posed on $(x, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m$

Theorem

$$\left\{\begin{array}{l} x^\star \text{ is a primal solution} \\ (\lambda^\star, \mu^\star) \text{ is a dual solution} \\ \text{Strong duality (SD) holds} \end{array}\right. \quad \Leftrightarrow \quad (x^\star, \lambda^\star, \mu^\star) \text{ solves the KKT system}$$

Example: analysis of a multiuser binary detector



P binary users and base station with N antennas

Data model is $y = Hs + v$

- $y \in \mathbb{R}^N$ is array snapshot
- $H \in \mathbb{R}^{N \times P}$ is full column-rank channel matrix (assumed known)
- $s \in \mathbb{R}^P$ is a binary vector
- Gaussian noise $v \sim \mathcal{N}(0, \sigma^2 I_N)$

ML detector solves

$$\begin{array}{ll} \text{minimize} & \|y - Hs\|^2 \\ \text{subject to} & s_p^2 = 1 \qquad\qquad p = 1, \ldots, P \end{array}$$

Optimization variable is $s = (s_1, s_2, \ldots, s_P)$

Complexity of ML detector is exponential in P

Equivalent problem:

$$\begin{array}{ll} \text{minimize} & \operatorname{tr}(AS) \\ \text{subject to} & \operatorname{diag}(S) = 1_{p+1} \\ & S \succeq 0 \\ & \operatorname{rank} S = 1 \end{array}$$

where

$$A := \begin{bmatrix} H^\top H & -H^\top y \\ -y^\top H & 0 \end{bmatrix}$$

Dropping the rank constraint yields the SDP detector

$$\begin{array}{ll} \text{minimize} & \mathrm{tr}\,(AS) \\ \text{subject to} & \mathrm{diag}(S) = 1_{p+1} \\ & S \succeq 0 \end{array}$$

Suppose $s^\star$ was transmitted. When is the SDP detector correct, i.e., when is

$$S^\star = \begin{bmatrix} s^\star \\ 1 \end{bmatrix} \begin{bmatrix} (s^\star)^\top & 1 \end{bmatrix}$$

a solution of the SDP ?

By the KKT conditions $S^\star$ is a solution iff there exists $(\lambda^\star, Z^\star)$ such that

$$\begin{cases} A = Z^\star + \mathsf{Diag}(\lambda^\star) \\ S^\star \succeq 0, \ \mathsf{diag}(S^\star) = 1_{P+1} \\ Z^\star \succeq 0 \\ \langle Z^\star, S^\star \rangle = 0 \end{cases}$$

Use the first condition to eliminate $Z^\star$ and get

$$\begin{cases} A - \mathsf{Diag}(\lambda^\star) \succeq 0 \\ \langle A - \mathsf{Diag}(\lambda^\star), S^\star \rangle = 0 \end{cases}$$

Thus, $S^\star$ is optimal iff there exist $\lambda^\star$ such that

$$\begin{cases} A - \mathsf{Diag}(\lambda^\star) \succeq 0 \\ (A - \mathsf{Diag}(\lambda^\star)) \begin{bmatrix} s^\star \\ 1 \end{bmatrix} = 0 \end{cases}$$

The second condition gives

$$\lambda^\star = \begin{bmatrix} \mathsf{Diag}(s^\star)^{-1} & 0 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} s^\star \\ 1 \end{bmatrix}$$

Using $y = Hs^\star + v$, the first condition is equivalent to

$$H^\top H + \mathsf{Diag}(s^\star)^{-1}\mathsf{Diag}(H^\top v) \succeq 0$$

(expected to hold with high probability at high SNR, i.e, small σ^2)

The solution $S^\star$ is unique if

$$H^\top H + \mathsf{Diag}(s^\star)^{-1}\mathsf{Diag}(H^\top v) \succ 0$$