Network Science
IST-CMU PhD course
Fall 2014
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Important: the homework is due the October 28. Send a scanned pdf file with your answers (or typed in LaTeX, if you prefer) to the TA's email.

Homework 2

1. *Basic properties of induced norms.* This problem and the next one establish a set of useful results for problem 3.

Let $P \in \mathbf{C}^{n\times n}$ be a hermitian (i.e., $P^H = P$) positive definite matrix. For a matrix A with complex-valued entries, the symbol A^H denotes the conjugate transpose of A. For example, if

$$A = \begin{bmatrix} 1+2i & 4-i & 5+2i \\ 2+2i & 7i & 3-i \end{bmatrix}$$

then

$$A^H = \begin{bmatrix} 1-2i & 2-2i \\ 4+i & -7i \\ 5-2i & 3+i \end{bmatrix}.$$

The eigenvalue decomposition of P can be written as $P = Q\Lambda Q^H$ where $Q \in \mathbf{C}^{n\times n}$ is an orthonormal matrix ($Q^HQ = QQ^H = I$) and

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

is a diagonal matrix containing the n (real) eigenvalues of P: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$. Note that there holds

$$m\,\|z\|^2 \leq z^H P z \leq M\,\|z\|^2$$

for all $z \in \mathbf{C}^n$ where $\|z\| = \sqrt{z^H z}$ is the usual euclidean norm.

(a) Define $P^{1/2} := Q\Lambda^{1/2}Q^H$ with

$$\Lambda^{1/2} := \begin{bmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{bmatrix}.$$

Show that $P^{1/2}$ is hermitian positive definite and that $P = P^{1/2}P^{1/2}$.

(b) Show that $\|\cdot\|_P : \mathbf{C}^n \to \mathbb{R}$,

$$\|z\|_P = \sqrt{z^H P z}, \tag{1}$$

is a norm—that is, show that (i) $\|z\|_P \geq 0$ for all $z \in \mathbf{C}^n$ with equality if and only if $z = 0$; (ii) $\|cz\|_P = |c| \, \|z\|_P$ for all $c \in \mathbf{C}$ and $z \in \mathbf{C}^n$; and (iii) $\|z + w\|_P \leq \|z\|_P + \|w\|_P$ for all $z, w \in \mathbf{C}^n$. Hint: relate $\|\cdot\|_P$ with $\|\cdot\|$ and use the fact that the latter function is a norm.

(c) Show that

$$\sqrt{\lambda_n} \, \|z\| \leq \|z\|_P \leq \sqrt{\lambda_1} \, \|z\|$$

for all $z \in \mathbf{C}^n$.

(d) Consider the function $||| \cdot |||_P : \mathbf{C}^{n \times n} \to \mathbb{R}$,

$$|||A|||_P = \sup \left\{ \|Az\|_P \, : \, \|z\|_P = 1 \right\}, \tag{2}$$

known as the induced norm of $\|\cdot\|_P$. It can be shown that the supremum in (2) is attained, i.e., for any given A there a exists z with $\|z\|_P = 1$ and $|||A|||_P = \|Az\|_P$ (you don't have to prove this[1]). Show that (2) is equivalent to

$$|||A|||_P = \sup_{z \neq 0} \frac{\|Az\|_P}{\|z\|_P}$$

and conclude that $||Az||_P \leq |||A|||_P ||z||_P$ for all $A \in \mathbf{C}^{n \times n}$ and $z \in \mathbf{C}^n$.

(e) Show that $||| \cdot |||_P$ is a matrix norm—that is, show that (i) $|||A|||_P \geq 0$ for all $A \in \mathbf{C}^{n \times n}$ with equality if and only if $A = 0$; (ii) $|||cA|||_P = |c| \, |||A|||_P$ for all $c \in \mathbf{C}$ and $A \in \mathbf{C}^{n \times n}$; (iii) $||||A + B|||_P \leq |||A|||_P + |||B|||_P$ for all $A, B \in \mathbf{C}^{n \times n}$; and (iv) $|||AB|||_P \leq |||A|||_P \, |||B|||_P$ for all $A, B \in \mathbf{C}^{n \times n}$.

(f) Let $||| \cdot |||_2 : \mathbf{C}^{n \times n} \to \mathbb{R}$ denote the matrix norm induced by the euclidean vector norm,

$$|||A|||_2 = \sup_{z \neq 0} \frac{\|Az\|}{\|z\|},$$

also known as the spectral norm (note that $|||A|||_2 = \sigma_{\max}(A)$). Show that

$$|||A|||_P \leq \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_n}} |||A|||_2$$

for all $A \in \mathbf{C}^{n \times n}$.

(g) Let $|| \cdot ||_F : \mathbf{C}^{n \times n} \to \mathbb{R}$ denote the Frobenius norm

$$\|A\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\operatorname{tr}(A^H A)}.$$

[1]It is a trivial consequence of the fact that $z \mapsto \|Az\|_P$ is a continuous function and $\left\{ z \, : \, \|z\|_P = 1 \right\}$ is a compact set.

Note that, from the singular value decomposition, we get $||A||_F = \sqrt{\sum_{i=1}^n (\sigma_i(A))^2}$ where $\sigma_i(A)$ denote the ith largest singular value of A. Show that

$$|||A|||_P \leq \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_n}} ||A||_F \tag{3}$$

for all $A \in \mathbf{C}^{n \times n}$.

(h) Suppose that sequence of matrices $A(t) \in \mathbf{C}^{n\times n}$ converges to the zero matrix as $t \to \infty$: $A(t) \to 0$ (equivalently, $A_{ij}(t) \to 0$ for all i, j where $A_{ij}(t)$ is the (i, j)th entry of $A(t)$). Show that $|||A(t)|||_P \to 0$. Hint: use (3).

(i) Show that $|||\cdot|||_P$ is a continuous function: if $A(t) \to A$ (equivalently, $A_{ij}(t) \to A_{ij}$ for all i, j) then $|||A(t)|||_P \to |||A|||_P$. Hint: show that $||||A|||_P - |||A(t)|||_P| \leq |||A - A(t)|||_P$ and use part (h).

2. *Convergent matrices.* Let the matrix $A \in \mathbf{C}^{n\times n}$ have sub-unit spectral radius, $\rho(A) < 1$ (such matrices are also called convergent) and let $C \in \mathbf{C}^{n\times n}$ be a hermitian positive definite matrix. The goal of this problem is to show that there exists an hermitian positive definite matrix $P \in \mathbf{C}^{n\times n}$ such that

$$P - A^H PA = C. \tag{4}$$

(As a quick sanity check, you should confirm right now that (4) holds for $n = 1$.)

(a) It is known that for any matrix $X \in \mathbf{C}^{n\times n}$ there holds

$$\rho(X) = \lim_{t\to\infty} \left(|||X^t|||\right)^{1/t}$$

where $|||\cdot|||$ denotes any matrix norm [2], cf. [1, corollary 5.6.14, pp. 299].
Let $|||\cdot|||$ be a matrix norm in $\mathbf{C}^{n\times n}$. Show that there exists a $0 \leq r < 1$ and $T \geq 1$ such that $t \geq T$ implies $|||A^t||| \leq r^t$ and $|||(A^H)^t||| \leq r^t$.

(b) Let $P(0) \in \mathbf{C}^{n\times n}$ be an arbitrary hermitian positive definite matrix and let $P(t+1) = C + A^H P(t)A$ for $t \geq 0$. Show that

$$P(t) = \left(A^H\right)^t P(0)A^t + \sum_{s=0}^{t-1} \left(A^H\right)^s CA^s,$$

for all $t \geq 1$.

(c) Show that the sequence $(P(t))_{t\geq 0}$ is bounded. Hint: choose a matrix norm $|||\cdot|||$ in $\mathbf{C}^{n\times n}$ and show that there exists $R \geq 0$ such that $|||P(t)||| \leq R$ for all $t \geq 0$ using parts (a) and (b).

(d) Define the residue sequence $\Delta(t) = \big(P(t) - A^H P(t)A\big) - C$, for $t \geq 0$. Show that

$$\Delta(t + 1) = A^H \Delta(t)A$$

and conclude that $\Delta(t) \to 0$.

[2] Recall that a norm in $\mathbf{C}^{n\times n}$ is a matrix norm if it satisfies the four properties in problem 1 (e).

(e) Since the sequence of matrices $(P(t))_{t\geq 0}$ is bounded in $\mathbf{C}^{n\times n}$ we can find a limit point, that is, there exists $P \in \mathbf{C}^{n\times n}$ and a sub-sequence $t(k)$ such that $P(t(k)) \to P$ as $k \to \infty$. Show that P satisfies (4) and is hermitian positive definite.

(f) Let $||\cdot||_P$ be the norm on $\mathbf{C}^n$ defined in (1). Show that there exists $0 \leq r < 1$ such that

$$\|Az\|_P \leq r\,\|z\|_P$$

for all $z \in \mathbf{C}^n$.

(g) Let $|||\cdot|||_P$ be the matrix norm on $\mathbf{C}^{n\times n}$ defined in (2). Show that $|||A|||_P < 1$.

3. *Rates of convergence for consensus algorithms.* The goal of this problem is to show that the consensus algorithms we saw in class converge exponentially fast.

(a) We begin with undirected graphs. Agent i holds θ_i. Agents initialize $x_i(0) = \theta_i$ and run the distributed algorithm

$$x(t+1) = Wx(t)$$

where W is a symmetric matrix with $W\mathbf{1} = \mathbf{1}$ and $\rho(W - J) < 1$, $J := \frac{1}{n}\mathbf{1}\mathbf{1}^T$ (also, $W_{ij} = 0$ whenever agents i and j cannot communicate). We know that

$$x(t) \to \overline{\theta}\mathbf{1}$$

where $\overline{\theta} = \frac{1}{n}\sum_{i=1}^n \theta_i$. Show that the convergence is exponentially fast, i.e., show that there exist $0 \leq r < 1$ and $A \geq 0$ such that

$$\left|x_i(t) - \overline{\theta}\right| \leq Ar^t$$

for all $t \geq 0$ and $i = 1, \ldots, n$.

(b) Now, we consider the push-sum algorithm for directed graphs. Agents initialize $x_i(0) = \theta_i$, $y_i(0) = 1$ and run the distributed algorithm

$$\begin{cases} x(t+1) &= Wx(t) \\ y(t+1) &= Wy(t) \end{cases}$$

where W is a primitive matrix with positive diagonal entries and $\mathbf{1}^T W = \mathbf{1}^T$ (also, $W_{ij} = 0$ whenever agent j cannot send information to agent i). We know that

$$z_i(t) := \frac{x_i(t)}{y_i(t)} \to \overline{\theta}$$

for all $i = 1, \ldots, n$. You will show that the convergence is exponentially fast, i.e., that there exist a $0 \leq r < 1$ and a constant A such that

$$\left|z_i(t) - \overline{\theta}\right| \leq Ar^t$$

for all $t \geq 0$ and $i = 1, \ldots, n$.

From the assumptions on W and the Perron-Frobenius theorem it follows that the Jordan decomposition of W is

$$W = \underbrace{\begin{bmatrix} v & U \end{bmatrix}}_{S} \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{1}^T \\ V^H \end{bmatrix}}_{S^{-1}} \tag{5}$$

for certain $U \in \mathbf{C}^{n\times(n-1)}$, $V \in \mathbf{C}^{n\times(n-1)}$ and $T \in \mathbf{C}^{(n-1)\times(n-1)}$ with $\rho(T) < 1$. Also, the vector v in (5) satisfies $v > 0$, $Wv = v$ and $1^T v = 1$. Thus,

$$W^t = v\mathbf{1}^T + UT^tV^H,$$

for $t \geq 0$.

Show that, for given $u, v \in \mathbf{C}^{n-1}$, there exist $0 \leq \widehat{r} < 1$ and a constant $\widehat{A} \geq 0$ such that

$$|u^H T^t v| \leq \widehat{A}\,\widehat{r}^t \tag{6}$$

for all $t \geq 0$. Hint: note that $\rho(T) < 1$. Use results from problems 1 and 2 and recall the Cauchy-Schwartz inequality: $|z^H w| \leq \|z\|\|w\|$ for $z, w \in \mathbf{C}^n$.

(c) Show that

$$x_i(t) = v_i\mathbf{1}^T\theta + e_i^T UT^tV^H\theta \tag{7}$$

where v_i is the ith entry of v, e_i is the ith column of the $n \times n$ identity matrix and $\theta = (\theta_1, \dots, \theta_n)$.

(d) Show that

$$y_i(t) = v_i n + e_i^T UT^tV^H\mathbf{1}. \tag{8}$$

(e) Note that $y(0) = \mathbf{1} > 0$ and, due to the properties of W, $y(t) = W^t\mathbf{1} > 0$ for all t. On the other hand, equation (8) implies each $y_i(t)$ converges to $v_i n$). We conclude that there exists a $m > 0$ such that $y(t) \geq m\mathbf{1}$ for all $t \geq 0$.

Show that

$$\left|\frac{x_i(t)}{y_i(t)} - \overline{\theta}\right| \leq \frac{\left|x_i(t) - \overline{\theta}\,y_i(t)\right|}{m}.$$

(f) Show that there exist $0 \leq r < 1$ and $A \geq 0$ such that

$$\left|\frac{x_i(t)}{y_i(t)} - \overline{\theta}\right| \leq Ar^t$$

for all $t \geq 0$ and $i = 1, \dots, n$.

References

[1] R. Horn and C. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.