# Codebook Design for Communication in Spread and Nonspread Space-Time Block Codes-based systems

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***Abstract*—The problem of space-time codebook design for communication in spread and nonspread multiple-antenna wireless systems is addressed and a new methodology for space-time codebook design is proposed. This optimizes the probability of error of the receiver's detector in the high signal-to-noise-ratio (SNR) regime, thus solving a nonlinear non-smooth optimization problem using an iterative method that exploits the Riemannian geometry imposed by the power constraints on the space-time codewords. Computer simulations demonstrate that, for the low SNR regime, our codebooks are marginally better than those provided by state-of-art known solutions. However, for the medium and high SNR regimes, our method provides codes that outperform other known codes.**

## I. Introduction

Space-time coding has received enormous attention as an efficient means that employs diversity to combat the effects of fading in wireless communication systems. This has been shown to provide a considerable increase in multiplexing and diversity gain in multiple-input/multiple-output (MIMO) systems [1], [2].

**Previous work.** Several space-time block code (STBC) schemes have been proposed over the recent years by imposing a certain structure on the codewords, such as orthogonal [3] or unitary group structures [4]. In the other direction, optimum minimum metric (OMM) codes [5] resulted from exhaustive computer search exhibit considerable performance improvement over prior structured approaches. In [6], a hybrid scheme employing limited computer searches together with a hierarchical codeset construction has been proved to enable construction of good high-rate codes in a computationally feasible manner, giving rise to codes that offer improved performance over previously known codes. In [7], the authors developed several upper bounds on the performance of STBCs. Rather than considering only the worst-case pairwise error probability (PEP), the bounds take the entire distance spectrum of the codes into consideration, resulting in improved performance assessment. The progressive union bound (PUB) as a performance index for STBCs was also proposed. It was observed that the PUB allows a tradeoff between numerical complexity and approximation accuracy. The authors demonstrated that code searches performed by optimizing the new criteria demonstrate improvement over worst-case designs.

**Contribution.** The main contribution of this paper is a new algorithm that systematically designs space-time codebooks for both spread and nonspread multiple-antena communication systems. Computer simulations show that the space-time codes obtained with our method perform marginally better than those already known in the low signal-to-noise-ratio (SNR) regime, but outperform them significantly in the medium and high SNR regime.

**Paper organization.** In section II, we formulate the problem addressed in this paper and discuss the selection of the codebook design criterium. In section III we propose a new algorithm that systematically designs space-time constellations for arbitrary spreading code correlation matrix and any $M$, $N$, $K$ and $T$, respectively, number of transmitter antennas, number of receiver antennas, size of codebook, and the channel block length. In Section IV, we present some codebook constructions and compare their performance with state-of-art solutions. Section V presents the main conclusions of our paper.

## II. Problem Formulation

**Assumptions and data model.** We work under the following assumptions:

1. The codeword $\boldsymbol{X}$ is chosen from a finite codebook $\mathcal{X} = \{\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_K\}$ known to the receiver, where $K$ is the size of the codebook. We impose the power constraint $\operatorname{tr}(\boldsymbol{X}_k^H \boldsymbol{X}_k) = TM$ for each codeword. Also, $\boldsymbol{X}_k \in \mathbb{C}^{T \times M}$;
2. For simplicity reasons, a single receive antenna is assumed, i.e., $N = 1$. (However, all of our methods are readily extensible to the $N > 1$ case) ;
3. An uncorrelated quasi-static narrowband flat[1] Rayleigh fading channel is assumed. That is, $\boldsymbol{h}(1) = \ldots = \boldsymbol{h}(T) \overset{\triangle}{=} \boldsymbol{h} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I}_M)^2$ where $\boldsymbol{h}(n)$ is the channel

[1] Our analysis is easily extendible to the frequency selective case.

[2] We use $\mathcal{CN}(\boldsymbol{p}, \boldsymbol{J})$ to denote a circularly symmetric complex Gaussian random vector with mean $\boldsymbol{p}$ and covariance matrix $\boldsymbol{J}$.

coefficient vector at time $n$. Also, the channel realizations are assumed to be known at the receiver but not at the transmitter;

4. Fixed spreading codes are used within one block $\boldsymbol{R}(1) = ... = \boldsymbol{R}(T) \stackrel{\triangle}{=} \boldsymbol{R}$ where $\boldsymbol{R}(n)$, for $n = 1, ..., T$, denotes the spreading code correlation matrix at time $n$ (for nonspread systems, $\boldsymbol{R}(n)$ takes the form of an all-ones matrix). Also, we assume equicorrelated spreading codes, i.e. $\boldsymbol{R}_{ij} = \rho \in \mathbb{R}$ for $1 \leq i \neq j \leq M$. Note that $0 < \rho \leq 1$ and $\boldsymbol{R}_{ii} = 1$, for $i = 1, ..., M$;

Assuming synchronous transmission, it can be shown that the received signal (output of a matched filter at the receiver) can be written as

$$\boldsymbol{y} = \sqrt{\sigma_t}\,(\boldsymbol{I}_T \otimes \boldsymbol{R})\underbrace{\left[\boldsymbol{D}(1)^T\,\boldsymbol{D}(2)^T\,...\,\boldsymbol{D}(T)^T\right]^T}_{\boldsymbol{D}}\boldsymbol{h} + \boldsymbol{m} \quad (1)$$

where $\otimes$ and $^T$ denote the Kronecker product and the transpose, respectively, $\boldsymbol{D}(n)$ is the $M \times M$ matrix obtained by the diagonalization of the $n$-th row of the codeword $\boldsymbol{X}$, $\sigma_t$ is a free parameter proportional to the SNR and $\boldsymbol{m} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I}_T \otimes \boldsymbol{R})$. For a proof see, e.g., [7].

**Receiver.** Under the above assumptions, the conditional probability density function of the received vector $\boldsymbol{y}$, given the transmitted matrix $\boldsymbol{X}$ and the channel $\boldsymbol{h}$, is given by

$$p(\boldsymbol{y}|\boldsymbol{X},\boldsymbol{h}) = \frac{\exp\{-||\boldsymbol{y} - \sqrt{\sigma_t}\,(\boldsymbol{I}_T \otimes \boldsymbol{R})\,\boldsymbol{D}\boldsymbol{h}||^2_{\boldsymbol{R}^{-1}}\}}{\pi^{MT}\,(\det(\boldsymbol{R}))^T},$$

where the notation $||\boldsymbol{z}||^2_{\boldsymbol{A}} = \boldsymbol{z}^H\boldsymbol{A}\boldsymbol{z}$ is used. We assume a maximum likelihood (ML) receiver which decides the index $k$ of the codeword as the index $\widehat{k}$ such that

$$\widehat{k} = \text{argmax}\{p(\boldsymbol{y}|\boldsymbol{X}_k,\boldsymbol{h}) : k = 1, 2, \ldots, K\}.$$

In words, the ML consists in a bank of $K$ parallel processors where the $k$-th processor computes the likelihood of the observation assuming the presence of the $k$-th codeword.

**Codebook design criterion.** In this work, the goal is to design a codebook $\mathcal{X} = \{\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_K\}$ of size $K$ for the current setup. One can see that a codebook $\mathcal{X}$ is a point in the space $\mathcal{M} = \{(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_K) : \text{tr}(\boldsymbol{X}_k^H\boldsymbol{X}_k) = TM\}$. Remark that $\mathcal{M}$ is the Cartesian product of $K$ spheres. First, we need to propose a merit function $f : \mathcal{M} \to \mathbb{R}$ which will "measure" the quality of each constellation $\mathcal{X}$. The average error probability for a specific $\mathcal{X}$ would be an intuitively appealing choice, but the theoretical analysis seems to be intractable. Instead, as usual [4], we rely on a PEP study to define our merit function. In this work we recall the Chernoff bound of the PEP in the high SNR regime, for arbitrary $\mathcal{X}$ and $\boldsymbol{R}$. Let $P_{\boldsymbol{X}_i \to \boldsymbol{X}_j}$ be the probability of the ML receiver deciding $\boldsymbol{X}_j$ when $\boldsymbol{X}_i$ is sent. It can be shown that

$$E_{\boldsymbol{h}}\left[P_{\boldsymbol{X}_i \to \boldsymbol{X}_j}\middle|\,\boldsymbol{h}\right] \leq \left(\frac{\sigma_t}{4}\right)^{-M}\frac{1}{\det(\boldsymbol{L}_{ij})} \quad (2)$$

with $\boldsymbol{L}_{ij} = (\boldsymbol{X}_i - \boldsymbol{X}_j)^H(\boldsymbol{X}_i - \boldsymbol{X}_j) \odot \boldsymbol{R}$, where $\odot$ denotes the Schur product. The proof can be found in, e.g., [7].

Since $\det(\boldsymbol{L}_{ij}) \geq (\lambda_{\min}(\boldsymbol{L}_{ij}))^M$ from (2) we can write

$$E_{\boldsymbol{h}}\left[P_{\boldsymbol{X}_i \to \boldsymbol{X}_j}\middle|\,\boldsymbol{h}\right] \leq \left(\frac{\sigma_t}{4}\right)^{-M}\frac{1}{(\lambda_{\min}(\boldsymbol{L}_{ij}))^M} \quad (3)$$

where $\lambda_{\min}(\boldsymbol{L}_{ij})$ is the minimum eigenvalue of the positive semidefinite matrix $\boldsymbol{L}_{ij}$. Although the bound in (3) is loose we will design codebooks aiming at maximizing $\lambda_{\min}(\boldsymbol{L}_{ij})$. The simulation results bellow will judge its effectiveness.

**Problem formulation.** Following the worst-case approach, we are motivated to define the merit function

$$f : \mathcal{M} \to \mathbb{R},\ \mathcal{X} = \{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_K\} \mapsto f(\mathcal{X})$$

as

$$f(\mathcal{X}) = \min\{f_{ij}(\mathcal{X}) : 1 \leq i \neq j \leq K\}$$

where

$$f_{ij}(\mathcal{X}) = \lambda_{\min}(\boldsymbol{L}_{ij}(\mathcal{X})).$$

Hence, constructing an optimal codebook $\mathcal{X} = \{\boldsymbol{X}_1, \boldsymbol{X}_2, ..., \boldsymbol{X}_K\}$ corresponds to solving the following non-linear and non-smooth optimization problem

$$\mathcal{X}^* = \underset{\mathcal{X} \in \mathcal{M}}{\arg\max}\ f(\mathcal{X}) \quad (4)$$

From (4) we see that, for $M = 1$ and $\rho = 1$, the problem of finding good codes coincides with the very well known packing problem of points on a sphere [8].

## III. Codebook Construction

Problem (4) requires the optimization of a non-smooth function over the smooth manifold $\mathcal{M}$ (Cartesian product of $K$ spheres). We propose an iterative method to tackle the optimization problem in (4). The method, which we call geodesic descent optimization algorithm (GDA), efficiently exploits the Riemannian geometry of the constraints. In the sequel, an overview of this iterative scheme is given (more details can be found in [9]).

Let $\mathcal{X}_k$ be the $k$-th iterate (the initialization $\mathcal{X}_0$ is randomly generated). First, the index set $\mathcal{A}$ of "active" constraint pairs (i,j), i.e., $\mathcal{A} = \{(i,j) : f(\mathcal{X}_k) = f_{ij}(\mathcal{X}_k)\}$ is identified. Then, we verify if there is an ascent direction $\boldsymbol{d}_k$ simultaneously for all functions $f_{ij}$ with $(i,j) \in \mathcal{A}$. We search for the ascent direction $\boldsymbol{d}_k$ within $T_{\mathcal{X}_k}\mathcal{M}$, the tangent space to $\mathcal{M}$ at $\mathcal{X}_k$. (Note that the search is not computationally demanding since it consists in solving a linear program. In order to solve the linear problem we need to determine the gradient $\nabla f_{ij}$. In Appendix, we give its respective expression.) If there are no such ascent direction, the algorithm stops. Otherwise, an Armijo search for $f(\mathcal{X})$ along the geodesic $\boldsymbol{\gamma}_k(t)$ which emanates from $\mathcal{X}_k$ in the direction $\boldsymbol{d}_k$ is performed; see figure 1. This Armijo search determines $\mathcal{X}_{k+1}$ and we repeat the loop. A geodesic is the generalization of a straight line in Euclidean space to a curved surface [10]. In words, GDA closely assimilates a sub-gradient method and consequently, the algorithm usually converges slowly near local minimizers. Nonetheless, since the codebooks are generated off-line this is not a serious shortcoming.

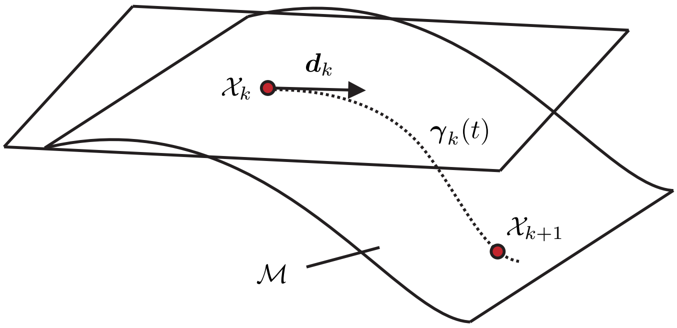


Fig. 1. Optimization of a non-smooth function on a smooth manifold

## IV. Results

We have constructed codes for three different cases of the size of the codebook $K = 8, 16$ and 32. We shall compare our codes with the schemes presented in [7]. In all simulations the solid-plus (blue) and dashed-circle (green) curves represent performances of codes constructed by our method, and codes proposed in [7], respectively. In either cases, the ML receiver is implemented. Also, we assumed a Rayleigh fading model for the channel, i.e., $\boldsymbol{h} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I}_M)$. We considered the case where $T = 3$, $M = 3$ transmit antennas, $N = 1$ receive antennas, $\rho = 0.3$ and $K = 8$. In figure 2, we show the symbol error rate (SER) versus

$$\text{SNR} = \frac{\mathsf{E}\{||\sqrt{\sigma_t}\,(\boldsymbol{I}_T \otimes \boldsymbol{R})\,\boldsymbol{D}\boldsymbol{h}||^2\}}{\mathsf{E}\{||\boldsymbol{m}||^2\}}$$

defined at the output of the matched filter. As we can see, our codebook construction is only marginally better for this particular case. Figure 3 plots the result of the experiment for $T = 3$, $M = 3$, $N = 1$, $\rho = 0.3$ and $K = 16$. It can be seen that our codes demonstrate a gain that increases with SNR. Figure 4 plots the result of the experiment for $T = 8$, $M = 3$, $N = 1$, $\rho = 0.3$ and $K = 32$. For SER $= 7 \cdot 10^{-2}$, our codes demonstrate a gain of 2dB gain when compared with the codes presented in [7]. [3]

Based on the results presented in figures 2– 4 we conclude that, when the size of the codebook $K$ increases, our codes are marginally better than those presented in [7] in the low SNR regime, but significantly outperform them in the medium and high SNR regime. The main advantage of our approach is that, in contrast to [7] where each entry of the codeword is constrained to phase-shift keying constellations, it exploits all the design degrees of freedom without restricting the codewords to have a specific structure.

As a final remark, we remind that it was shown that for the case $M = 1$ and $\rho = 1$ (nonspread systems) the problem of finding good codes coincides with the very well known packing problem of points on a sphere. By using our tool we have recovered the best known packings given by Sloane [8]. [4]

[3]One can note that there is a 4dB "shift" to the left when comparing the performances of codes proposed in [7] herein and in [7]. It is due to a different way of defining the SNR in our work and in [7].

[4]All codes presented in this work, together with some sphere packings, are available at http://users.isr.ist.utl.pt/∼marko/Publications.html

Although a novelty of the work is a tool that constructs codebooks for arbitrary $M$ and $\rho$, nevertheless, we compared our approach with that in [8] just to check what is the value of our tool for this special scenario, i.e., the simulations were carried out for this particular case just to perform a "sanity check".

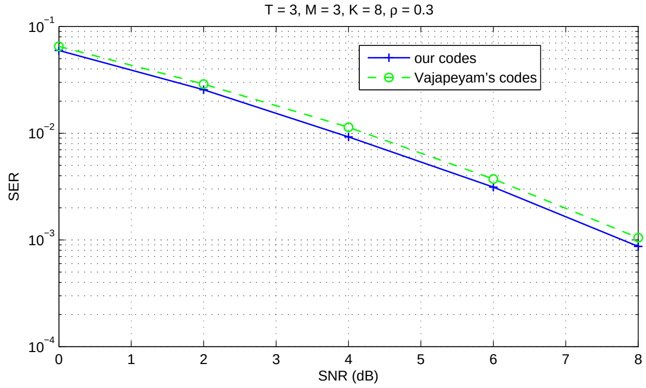


Fig. 2. $T = 3$, $M = 3$, $N = 1$, $\rho = 0.3$ and $K = 8$. Solid-plus signed curve–our codes, dashed-circled curve–codes presented in [7]. ML receiver is implemented.

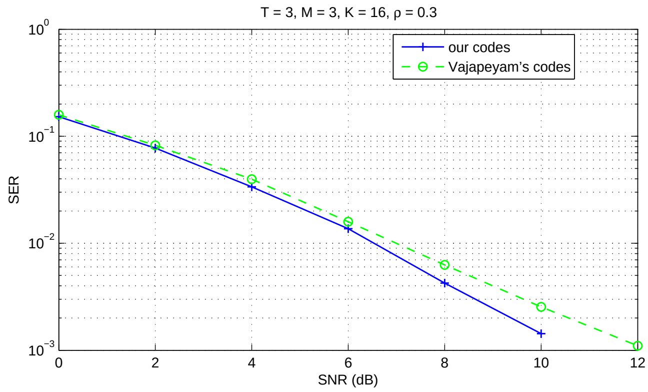


Fig. 3. $T = 3$, $M = 3$, $N = 1$, $\rho = 0.3$ and $K = 16$. Solid-plus signed curve–our codes, dashed-circled curve–codes presented in [7]. ML receiver is implemented.

## V. Conclusions

We addressed the problem of space–time codebook construction for both spread and nonspread multiple-antenna wireless systems. A new methodology for designing space-time codebooks for this setup, taking the probability of error of the detector in the high SNR regime as the code design criterion was proposed. The method, called a geodesic descent optimization algorithm, solves the resulting nonlinear and non-smooth optimization problem by efficiently exploiting the Riemannian geometry of the constraints. New codebooks are obtained by this method and their performance is shown to outperform previous state-of-art solutions. This shows the relevance of the codebook construction tool proposed herein.

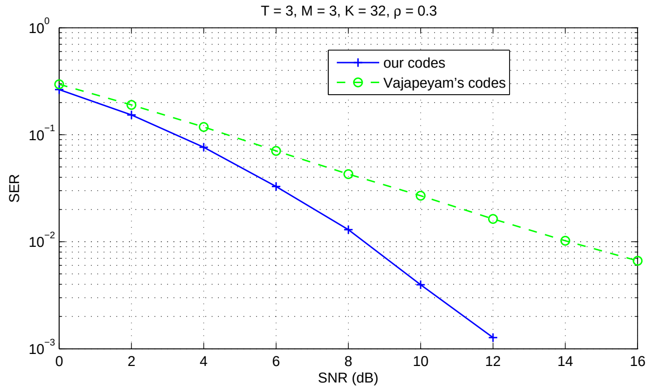


Fig. 4. $T = 3$, $M = 3$, $N = 1$, $\rho = 0.3$ and $K = 32$. Solid-plus signed curve–our codes, dashed-circled curve–codes presented in [7]. ML receiver is implemented.

## VI. Appendix

In this appendix, we calculate gradient to be used in the GDA. Although the function $f_{ij}$ assumes complex valued entries, that is

$$f_{ij} : \underbrace{\mathbb{C}^{T\times M} \times ... \times \mathbb{C}^{T\times M}}_{K} \to \mathbb{R},$$

$f_{ij}(\boldsymbol{X}_1, ..., \boldsymbol{X}_K) = \lambda_{\min}(\boldsymbol{L}_{ij})$ where $\boldsymbol{L}_{ij} = (\boldsymbol{X}_i - \boldsymbol{X}_j)^H (\boldsymbol{X}_i - \boldsymbol{X}_j) \odot \boldsymbol{R}$, we treat $f_{ij}$ as a function of the real and imaginary components of $\boldsymbol{X}_1, \boldsymbol{X}_2, ..., \boldsymbol{X}_K$, i.e.

$$f_{ij} : \underbrace{\mathbb{R}^{T\times M} \times ...\mathbb{R}^{T\times M}}_{2K} \to \mathbb{R},$$

$$f_{ij}\left(\Re\{\boldsymbol{X}_1\}, \Im\{\boldsymbol{X}_1\}, ..., \Re\{\boldsymbol{X}_K\}, \Im\{\boldsymbol{X}_K\}\right) = \lambda_{\min}(\boldsymbol{L}_{ij}).$$

Let $\lambda_{\min}$ be a simple eigenvalue of the Hermitian matrix $\boldsymbol{L}_{ij}(\mathcal{C}_0)$, and let $\boldsymbol{u}_0$ be an associated unit-norm eigenvector, so that $\boldsymbol{L}_{ij}(\mathcal{C}_0)\boldsymbol{u}_0 = \lambda_{\min}\left(\boldsymbol{L}_{ij}(\mathcal{C}_0)\right)\boldsymbol{u}_0$. The differential $df_{ij}$, computed at the point $\mathcal{C}_0$, is given by, pp. 162 in [12]

$$df_{ij} = \boldsymbol{u}_0^H d\boldsymbol{L}_{ij}\boldsymbol{u}_0.$$

where $d\boldsymbol{L}_{ij}$ denotes the differential of the map $\mathcal{C} \mapsto \boldsymbol{L}_{ij}(\mathcal{C})$ computed at the point $\mathcal{C}_0$. Thus, it can be easily seen that

$$df_{ij} = \operatorname{tr}\left( d\left( (\boldsymbol{X}_i - \boldsymbol{X}_j)^H (\boldsymbol{X}_i - \boldsymbol{X}_j)\right) \underbrace{\boldsymbol{U}_0^H \boldsymbol{R}^T \boldsymbol{U}_0^H}_{\boldsymbol{C}} \right)$$

where $\boldsymbol{U}_0$ is the matrix obtained by the diagonalization of $\boldsymbol{u}_0$. Now, it is straightforward to identify the gradient. Hence, the gradient is given by

$$\nabla f_{ij}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{0}_{(i-1)c\times 1} \\ \Re\{\text{vec}(\boldsymbol{C}_i)\} \\ \Im\{\text{vec}(\boldsymbol{C}_i)\} \\ \boldsymbol{0}_{(j-i-1)c\times 1} \\ \Re\{\text{vec}(\boldsymbol{C}_j)\} \\ \Im\{\text{vec}(\boldsymbol{C}_j)\} \\ \boldsymbol{0}_{(K-j)c\times 1} \end{bmatrix}_{2KTM\times 1}$$

for $1 \le i \ne j \le K$, where $c = 2TM$, $\boldsymbol{C}_i = 2(\boldsymbol{X}_i - \boldsymbol{X}_j)\boldsymbol{C}$, $\boldsymbol{C}_j = -\boldsymbol{C}_i$ and

$$\boldsymbol{x} = \left[\Re\{\boldsymbol{x}_1\}^T \; \Im\{\boldsymbol{x}_1\}^T \; \dots \; \Re\{\boldsymbol{x}_K\}^T \; \Im\{\boldsymbol{x}_K\}^T\right]^T_{2KTM\times 1}$$

with $\boldsymbol{x}_i = \text{vec}(\boldsymbol{X}_i)$.[5]

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[5]the vector obtained by stacking the columns of $\boldsymbol{X}_i$ on top of each other, from left to right.