Nonlinear Optimization (18799 B, PP)
IST-CMU PhD course
Spring 2017

Instructor: João Xavier (`jxavier@isr.ist.utl.pt`)
TA: Shanghang Zhang (`shzhang.pku@gmail.com`)

The homework is due July 14.

Homework 6

Problem A. *(Isotonic regression)* A signal $u = (u_1, u_2, \ldots, u_n) \in \mathbf{R}^n$ is said be be non-decreasing if $u_1 \leq u_2 \leq \cdots \leq u_n$. Suppose that the non-decreasing signal $u \in \mathbf{R}^n$ is measured in additive noise:

$$y = u + w.$$

Here, $y \in \mathbf{R}^n$ is the measurement (known vector), and $w \in \mathbf{R}^n$ is the noise (unknown vector).

To estimate the non-decreasing signal u from the measurement y, we look for the closest non-decreasing signal:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \frac{1}{2}\|y - x\|^2 \\ \text{subject to} & x_i \leq x_{i+1} \quad i = 1, 2, \ldots, n-1. \end{array} \tag{1}$$

In problem (1), the optimization variable is $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$.

Show how to solve (1) via ADMM. That is, cast (1) in a canonical ADMM form and write explicitly the ADMM updates. Code your algorithm in Matlab, at the end of the file `homework6.m`. Compare your solution with the solution obtained via CVX (plot it on the figure). Figure 1 illustrates an example,

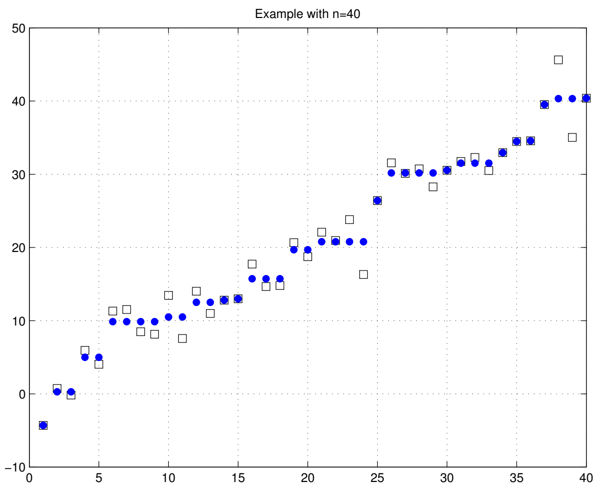

Figure 1: Measurement vector y (black, square) and solution of (1) (blue, dot).

Hint 1: Start by rewriting (1) as

$$\underset{x}{\text{minimize}} \sum_{i=1}^{n-1} \underbrace{i_C(x_i, x_{i+1})}_{f_i(x_i, x_{i+1})} + \underbrace{\frac{1}{2}\|y - x\|^2}_{f_n(x_1, \ldots, x_n)}.$$

where i_C is the indicator of the convex set $C = \{(x, y) \in \mathbf{R}^2 \,:\, x \leq y\}$. Recall that

$$i_C(x, y) = \begin{cases} 0 & , \text{ if } (x, y) \in C \\ +\infty & , \text{ otherwise.} \end{cases}$$

Now, introduce extra variables ($x_i^{(k)}$ is the copy of x_i in the function f_k):

$$
\begin{array}{ll}
\underset{x_i, x_i^k}{\text{minimize}} & \sum_{i=1}^{n-1} f_i\left(x_i^{(i)}, x_{i+1}^{(i)}\right) + f_n\left(x_1^{(n)}, \ldots, x_n^{(n)}\right) \\
\text{subject to} & x_1 = x_1^{(1)}, \quad x_1 = x_1^{(n)} \\
& x_2 = x_2^{(1)}, \quad x_2 = x_2^{(2)}, \quad x_2 = x_2^{(n)} \\
& x_3 = x_3^{(2)}, \quad x_3 = x_3^{(3)}, \quad x_3 = x_3^{(n)} \\
& \vdots \\
& x_{n-1} = x_{n-1}^{(n-2)}, \quad x_{n-1} = x_{n-1}^{(n-1)}, \quad x_{n-1} = x_{n-1}^{(n)} \\
& x_n = x_n^{(n-1)}, \quad x_n = x_n^{(n)}.
\end{array} \tag{2}
$$

Finally, apply ADMM to problem (2) by viewing $\left\{x_i^{(k)}\right\} = \left(x_1^{(1)}, x_1^{(n)}, \ldots, x_n^{(n-1)}, x_n^{(n)}\right)$ as one block of variables, and $\{x_i\} = (x_1, \ldots, x_n)$ as the other.

Hint 2: Let C be a closed, convex set. The solution of the problem

$$
\underset{x}{\text{minimize}} \; \tfrac{1}{2} \|x - y\|^2 + \iota_C(x)
$$

is the projection of y onto the set C, *i.e.*, the solution $x^\star$ of the optimization problem is $x^\star = p_C(y)$.

Hint 3: Consider the half-space $C = \{(x, y) \in \mathbf{R} \times \mathbf{R} \, : \, x \leq y\}$. Note that C is a set in $\mathbf{R} \times \mathbf{R}$. It can be shown that the projector onto C is given by

$$
p_C(p, q) = (p, q) - \frac{1}{2}(p - q)_+(1, -1),
$$

for any $(p, q) \in \mathbf{R} \times \mathbf{R}$.