

**Nonlinear Optimization (18799 B, PP)**  
**IST-CMU PhD course**  
**Spring 2018**

Instructor: João Xavier (jxavier@isr.ist.utl.pt)  
 TA: Hung Tuan (hung.seadc@gmail.com)

The homework is due July 19.

## Homework 5

**Problem A.** (*Isotonic regression*) A signal  $u = (u_1, u_2, \dots, u_n) \in \mathbf{R}^n$  is said to be non-decreasing if  $u_1 \leq u_2 \leq \dots \leq u_n$ . Suppose that the non-decreasing signal  $u \in \mathbf{R}^n$  is measured in additive noise:

$$y = u + w.$$

Here,  $y \in \mathbf{R}^n$  is the measurement (known vector), and  $w \in \mathbf{R}^n$  is the noise (unknown vector). To estimate the non-decreasing signal  $u$  from the measurement  $y$ , we look for the closest non-decreasing signal:

$$\begin{aligned} & \underset{x}{\text{minimize}} && \frac{1}{2} \|y - x\|^2 \\ & \text{subject to} && x_i \leq x_{i+1} \quad i = 1, 2, \dots, n-1. \end{aligned} \tag{1}$$

In problem (1), the optimization variable is  $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ .

Show how to solve (1) via ADMM. That is, cast (1) in a canonical ADMM form and write explicitly the ADMM updates. Code your algorithm in Matlab, at the end of the file `homework5.m`. Compare your solution with the solution obtained via CVX (plot it on the figure). Figure 1 illustrates an example,

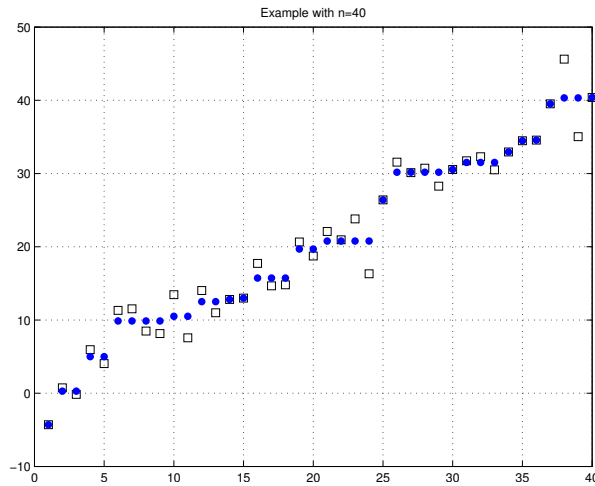


Figure 1: Measurement vector  $y$  (black, square) and solution of (1) (blue, dot).

Hint 1: Start by rewriting (1) as

$$\underset{x}{\text{minimize}} \sum_{i=1}^{n-1} \underbrace{i_C(x_i, x_{i+1})}_{f_i(x_i, x_{i+1})} + \underbrace{\frac{1}{2} \|y - x\|^2}_{f_n(x_1, \dots, x_n)}.$$

where  $i_C$  is the indicator of the convex set  $C = \{(x, y) \in \mathbf{R}^2 : x \leq y\}$ . Recall that

$$i_C(x, y) = \begin{cases} 0 & , \text{ if } (x, y) \in C \\ +\infty & , \text{ otherwise.} \end{cases}$$

Now, introduce extra variables  $(x_i^{(k)})$  is the copy of  $x_i$  in the function  $f_k$ :

$$\begin{aligned}
 & \underset{x_i, x_i^k}{\text{minimize}} && \sum_{i=1}^{n-1} f_i \left( x_i^{(i)}, x_{i+1}^{(i)} \right) + f_n \left( x_1^{(n)}, \dots, x_n^{(n)} \right) \\
 & \text{subject to} && x_1 = x_1^{(1)}, \quad x_1 = x_1^{(n)} \\
 & && x_2 = x_2^{(1)}, \quad x_2 = x_2^{(2)}, \quad x_2 = x_2^{(n)} \\
 & && x_3 = x_3^{(2)}, \quad x_3 = x_3^{(3)}, \quad x_3 = x_3^{(n)} \\
 & && \vdots \\
 & && x_{n-1} = x_{n-1}^{(n-2)}, \quad x_{n-1} = x_{n-1}^{(n-1)}, \quad x_{n-1} = x_{n-1}^{(n)} \\
 & && x_n = x_n^{(n-1)}, \quad x_n = x_n^{(n)}.
 \end{aligned} \tag{2}$$

Finally, apply ADMM to problem (2) by viewing  $\{x_i^{(k)}\} = (x_1^{(1)}, x_1^{(n)}, \dots, x_n^{(n-1)}, x_n^{(n)})$  as one block of variables, and  $\{x_i\} = (x_1, \dots, x_n)$  as the other.

Hint 2: Let  $C$  be a closed, convex set. The solution of the problem

$$\underset{x}{\text{minimize}} \quad \frac{1}{2} \|x - y\|^2 + \iota_C(x)$$

is the projection of  $y$  onto the set  $C$ , *i.e.*, the solution  $x^*$  of the optimization problem is  $x^* = p_C(y)$ .

Hint 3: Consider the half-space  $C = \{(x, y) \in \mathbf{R} \times \mathbf{R} : x \leq y\}$ . Note that  $C$  is a set in  $\mathbf{R} \times \mathbf{R}$ . It can be shown that the projector onto  $C$  is given by

$$p_C(p, q) = (p, q) - \frac{1}{2}(p - q)_+(1, -1),$$

for any  $(p, q) \in \mathbf{R} \times \mathbf{R}$ .

Hint 4: take a look at the manuscript “Distributed optimization and statistical learning via the alternating direction method of multipliers,” by S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein (available at <http://web.stanford.edu/~boyd>).