Nonlinear Optimization (18799 B, PP)
IST-CMU PhD course
Spring 2018

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The homework is due June 25.

Homework 4

Problem A. *(Robust fusion of estimates)* Let $\widehat{\theta}_i \in \mathbf{R}^n$, $i = 1, \dots, k$ be estimates of some unknown parameter $\theta \in \mathbf{R}^n$. Assume that each estimate is unbiased ($\mathbf{E}(\widehat{\theta}_i) = \theta$ for all i) and has known error covariance matrix

$$P_i := \mathbf{E}\left((\widehat{\theta}_i - \theta)(\widehat{\theta}_i - \theta)^T\right).$$

Assume that each P_i is constant (*i.e.*, each P_i is independent of θ); furthermore, consider that all estimates are uncorrelated. For example, this might represent a setup where k independent localization systems give us estimates $\widehat{\theta}_i$ of an unknown target position θ. In such a setup, each P_i being constant means that the statistical accuracy of each one of the localization systems is the same regardless of the target position.

We want to fuse linearly the estimates $\widehat{\theta}_i$ to produce the final estimate

$$\widehat{\theta} = W_1\widehat{\theta}_1 + W_2\widehat{\theta}_2 + \cdots + W_k\widehat{\theta}_k,$$

where $W_i \in \mathbf{R}^{n\times n}$ denote weighting matrices to be found.

It can be shown that, if we want to make $\widehat{\theta}$ an unbiased estimate with the lowest mean square error (MSE),

$$\mathbf{E}\left(\left\|\widehat{\theta} - \theta\right\|^2\right),$$

then we must choose the weighting matrices as

$$\begin{array}{ll} \underset{W_1,\dots,W_k}{\text{minimize}} & \sum_{i=1}^k \operatorname{tr}\left(W_i P_i W_i^T\right) \\ \text{subject to} & \sum_{i=1}^k W_i = I_n, \end{array} \tag{1}$$

where $I_{n\times n}$ is the $n \times n$ identity matrix.

Now, assume that the error covariance matrices P_i giving the accuracy of the estimates are uncertain. We know only lower and upper bounds of P_i, *i.e.*, in compact notation, $L_i \leq P_i \leq U_i$, where $L_i, U_i \in \mathbf{R}^{n\times n}$ are known. Here, the notation $L_i \leq P_i$ means that each entry of P_i is greater than or equal to the corresponding entry in L_i; the meaning of $P_i \leq U_i$ is similar.

To protect the final estimate $\widehat{\theta}$ against this uncertainty, we change problem (1) to

$$\begin{array}{ll} \underset{W_1,\dots,W_k}{\text{minimize}} & \sup\left\{\sum_{i=1}^k \operatorname{tr}\left(W_i P_i W_i^T\right) \,:\, L_i \leq P_i \leq U_i, P_i \succeq 0 \text{ for all } i\right\} \\ \text{subject to} & \sum_{i=1}^k W_i = I_n. \end{array} \tag{2}$$

The objective function in (2) computes — for given $W_1, \dots, W_k$ — the worst MSE in the uncertainty region (the constraint $P_i \succeq 0$ is included because covariance matrices are semidefinite positive). Note that the objective function is the optimal value of an optimization problem formulated over the variables $P_1, \dots, P_k$. You can assume that, for each i, there exists a $P_i \succ 0$ such that $L_i \leq P_i \leq U_i$.

Formulate (2) as a SDP. You can add more variables.

Hint: you need to dualize the objective function of (2).