Nonlinear Optimization (18799 B, PP)
IST-CMU PhD course
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The homework is due April 29.

Homework 3

Problem A. (*Open loop fault-tolerant control*) Consider a controlled dynamical system

$$x(t) = Ax(t-1) + Bu(t-1), \quad t = 1, 2, \ldots, T, \tag{1}$$

where $x(t) \in \mathbf{R}^n$ and $u(t) \in \mathbf{R}^p$ denote the system state and the control input at time t, respectively. The initial state is zero ($x(0) = 0$) and the system matrices $A \in \mathbf{R}^{n \times n}$ and $B \in \mathbf{R}^{n \times p}$ are given.

Suppose that we want to design a control sequence $u(0), u(1), \ldots, u(T-1)$ in order to minimize a linear function of the terminal state: $c^T x(T)$ where $c \in \mathbf{R}^n$ is given. (For example, the dynamical system may represent a controlled robot moving in $\mathbf{R}^2$. The state is $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ with $(x_1(t), x_2(t))$ and $(x_3(t), x_4(t))$ denoting its position and speed at time t, respectively. If we want to move the robot as far as possible to the right in T time steps, we would minimize $c^T x(T)$ with $c = (-1, 0, 0, 0)$: this corresponds to maximize $x_1(T)$.)

The control sequence must comply with the magnitude bound $\|u(t)\|_\infty \leq 1$ for all t.

Since (1) implies that the terminal state is given by

$$x(T) = \sum_{t=0}^{T-1} A^{T-t-1} B u(t) \tag{2}$$

we would formulate the optimization problem

$$\begin{array}{ll} \underset{u(0), u(1), \ldots, u(T-1)}{\text{minimize}} & c^T \left(\sum_{t=0}^{T-1} A^{T-t-1} B u(t) \right) \\ \text{subject to} & \|u(t)\|_\infty \leq 1, \quad t = 0, 1, \ldots, T-1. \end{array} \tag{3}$$

Now, assume that one of the p actuators in the control vector

$$u(t) = (u_1(t), u_2(t), \ldots, u_p(t)) \in \mathbf{R}^p$$

fails; if the ith actuator fails ($i \in \{1, 2, \ldots, p\}$), the corresponding entry in $u(t)$ becomes zero: $u_i(t) = 0$ for $t = 0, 1, \ldots, T-1$. In that case, the terminal state is no longer given by (2). Instead,

$$x(T) = \sum_{t=0}^{T-1} A^{T-t-1} B Z_i u(t)$$

where Z_i is a $p \times p$ diagonal matrix with all diagonal entries equal to 1 except for the (i, i) entry which is equal to 0. For example, for $p = 4$:

$$Z_3 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 1 \end{bmatrix}.$$

The product $Z_i u(t)$ simply sets the ith component of $u(t)$ to zero thus modeling the malfunction of actuator i in the control vector $u(t)$. For convenience, we also define Z_0 as the $p \times p$ identity matrix (which corresponds to no actuator failures).

The goal of this problem is to design a control sequence $u(0), u(1), \ldots, u(T-1)$ that minimizes the worst damage caused by failure of, at most, one actuator:

$$\begin{array}{ll} \underset{u(0),u(1),\ldots,u(T-1)}{\text{minimize}} & \underbrace{\max\left\{ c^T \left(\sum_{t=0}^{T-1} A^{T-t-1} B Z u(t) \right) \, : \, Z \in \{Z_0, Z_1, \ldots, Z_p\} \right\}}_{f(u(0),u(1),\ldots,u(T-1))} \\ \text{subject to} & \|u(t)\|_\infty \leq 1, \quad t = 0, 1, \ldots, T-1. \end{array} \tag{4}$$

The objective function f in (4) receives a control sequence $u(0), u(1), \ldots, u(T-1)$ and returns its worst performance under failure of, at most, one actuator.

Formulate (4) as a linear program (LP).

Problem B. *(Geometric approximation)* We are given hyperplanes

$$H_{s_k, r_k} = \{x \in \mathbf{R}^n \, : \, s_k^T x = r_k\}$$

for $k = 1, \ldots, K$. We want to place a ball $B(c, R) = \{x \in \mathbb{R}^n \, : \, \|x - c\| \leq R\}$ with given radius $R \geq 0$ near the hyperplanes. More precisely, we want to solve

$$\underset{c}{\text{minimize}} \quad \sum_{k=1}^{K} d\left(B(c, R), H_{s_k, r_k}\right) \tag{5}$$

where

$$d(\mathcal{A}, \mathcal{B}) = \inf\{\|a - b\| \, : \, a \in \mathcal{A}, b \in \mathcal{B}\}$$

denotes the distance between sets $\mathcal{A}$ and $\mathcal{B}$.

For example, in the special case $R = 0$, the ball $B(c, R)$ is just the point c. In this case, (5) looks for the point c that minimizes the sum of the distances from c to all given hyperplanes.

(a) Formulate problem (5) as a second-order cone program (SOCP).

(b) Generate an instance of this problem in MATLAB for $n = 2$, *i.e.*, generate some straight lines H_{s_k, r_k} and fix a radius $R > 0$. Then, solve numerically your instance through the software CVX (available from `http://cvxr.com/cvx`). Plot the optimal placement for the ball along with the straight lines. (You should also submit to us the MATLAB code you wrote.)