Nonlinear Optimization (18799 B, PP)
IST-CMU PhD course
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The homework is due May 14.

Homework 3

Instructions: read sections 4.1, 4.2, 4.3, and 4.4 of [1].

Problem A. *(Optimal filter input)* Consider a discrete-time system

$$y(t) = H(0)x(t) + H(1)x(t-1) + \cdots + H(D)x(t-D),$$

where $(x(t))_{t\geq 0}$ and $(y(t))_{t\geq 0}$ denote the input and output, respectively.

Assume $x(t) \in \mathbf{R}^n$ and $y(t) \in \mathbf{R}^m$, with $x(t) = 0$ for $t \leq 0$, i.e., the system is at rest. Moreover, all the inputs must be upper bounded in magnitude by a given constant $U > 0$: $\|x(t)\|_\infty \leq U$ for all t. Hereafter, we refer to $H = (H(0), H(1), \ldots, H(D))$ as the system.

We are given a desired output for a finite time horizon T, say,

$$y_{\text{des}} = (y_{\text{des}}(1), y_{\text{des}}(2), \ldots, y_{\text{des}}(T))$$

with $y_{\text{des}}(t) \in \mathbf{R}^m$. Thus, y_{des} – the stacking of all outputs – is a vector in $\mathbf{R}^{mT}$.

For an input segment $x = (x(1), x(2), \ldots, x(T))$, it is straightforward to see that the output $y = (y(1), y(2), \ldots, y(T))$ is given by $y = \mathcal{H}x$ where

$$\mathcal{H} = \begin{bmatrix} H(0) & & & & & \\ H(1) & H(0) & & & & \\ \vdots & \ddots & \ddots & & & \\ H(D) & \cdots & H(1) & H(0) & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & H(D) & \cdots & H(1) & H(0) \end{bmatrix} \in \mathbf{R}^{mT \times nT}$$

(missing blocks are zero).

If we assume the system is known and we adopt the ℓ_∞ norm to measure distance in the output space, the design of the optimal input translates into solving

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \|\mathcal{H}x - y_{\text{des}}\|_\infty \\ \text{subject to} & \|x\|_\infty \leq U. \end{array}$$

Suppose now that the system H is randomly drawn from a finite set $\{H_1, H_2, \ldots, H_K\}$ with known *a priori* probabilities $\mathbb{P}(H = H_k)$, for $k = 1, 2, \ldots, K$. Assume all the possible systems have the same (known) order D. The system realization H_k is unknown to us at the start, but it is revealed at the end of time $S < T$ (the instant S is known from the beginning). Thus, we operate from $t = 1$ up to $t = S$ in a blind mode (ignoring the underlying system) but with full system knowledge from $t = S + 1$ to $t = T$.

We want to design the input that minimizes the average ℓ_∞ distance between the corresponding output and the desired one. This means that we want to find both the optimal input "head" $(x(1), \ldots, x(S))$ (the input segment that we use in the blind mode) and the optimal input "tails" $(x_k(S + 1), \ldots, x_k(T))$ for $k = 1, \ldots, K$ (the input segments that we use after being told the system realization is H_k for $k = 1, \ldots, K$).

Formulate this optimization problem as a linear program.

Hint: read carefully problem 4.64, page 211, from [1].

Problem B. *(Geometric approximation)* We are given hyperplanes

$$H_{s_k,r_k} = \{x \in \mathbf{R}^n \,:\, s_k^T x = r_k\}$$

for $k = 1, \ldots, K$. We want to place a ball $B(c, R) = \{x \in \mathbb{R}^n \,:\, \|x - c\| \leq R\}$ with known radius R near the hyperplanes; more precisely, we want to solve

$$\underset{c}{\text{minimize}} \quad \textstyle\sum_{k=1}^{K} d\left(B(c,R), H_{s_k,r_k}\right) \tag{1}$$

where

$$d(\mathcal{A}, \mathcal{B}) = \inf\{\|a - b\| \,:\, a \in \mathcal{A}, b \in \mathcal{B}\}$$

denotes the distance between sets $\mathcal{A}$ and $\mathcal{B}$.

(a) Formulate problem (1) as a second-order cone program. You can add more variables (in fact, you will need to do so.)

(b) Generate an instance of this problem in MATLAB for $n = 2$, i.e., generate some straight lines H_{s_k,r_k} and fix a radius $R > 0$. Then, solve numerically your instance through the software CVX (available from `http://cvxr.com/cvx`). Plot the optimal placement for the ball along with the straight lines. (You should turn in the MATLAB code used to solve the instance.)

References

[1] S. Boyd and L. Vandenberghe. *Convex optimization.* Cambridge University Press, 2004.