Nonlinear Optimization (18799 B, PP)
IST-CMU PhD course
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The homework is due April 19.

Homework 2

Instructions: read section 3.2 of [1].

Problem A. (*Bhattacharyya distance*) Let $\Delta_n = \left\{p \in \mathbf{R}^n \,:\, p \geq 0, \mathbf{1}^T p = 1\right\}$ be the probability simplex of size n. That is, a point in Δ_n is a vector $p = (p_1, p_2, \ldots, p_n)$ that can represent a probability mass function over an alphabet of size n: $p_i \geq 0$ for all i, and $p_1 + p_2 + \cdots + p_n = 1$.

The Bhattacharyya distance between two points p and q in Δ_n is defined as

$$\mathrm{BC}\,(p, q) = -\log\left(\sum_{i=1}^{n} \sqrt{p_i}\sqrt{q_i}\right).$$

Let q be a fixed point in Δ_n. Show that the function $f \,:\, \Delta_n \to \mathbf{R}$, $f(p) = \mathrm{BC}(p, q)$ is convex.

Problem B. (*VaR and CVaR*) In financial risk management, it's important to measure how risky is a given loss vector $L = (L_1, L_2, \ldots, L_n)$, where L_i is the loss we suffer when scenario i happens.

(a) A popular risk measure is the value-at-risk (VaR). Assume scenario i happens with probability π_i. The VaR of a given loss vector $L = (L_1, L_2, \ldots, L_n)$ is defined as

$$\mathrm{VaR}(L) = \min\left\{l \,:\, \sum_{i=1}^{n} \pi_i 1_{L_i \leq l} \geq \alpha\right\},$$

where α is a fixed confidence level (usually, $\alpha = 0.9$ or $\alpha = 0.99$). The symbol $1_{a \leq b}$ means 1 if $a \leq b$, and 0 if $a > b$.

In words, $\mathrm{VaR}(L)$ is the least number l for which the probability of a loss happening in the interval $[0, l]$ is greater than or equal to α. For example, take $n = 5$, $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = (0.1, 0.4, 0.2, 0.25, 0.05)$, and $\alpha = 0.9$. Then $\mathrm{VaR}\,(2, 5, 3, 2, 9) = 5$ and $\mathrm{VaR}\,(2, 9, 3, 2, 5) = 9$.

Is VaR a convex function?

(b) Another popular risk measure is the conditional value-at-risk (CVaR). The CVaR of a given loss vector $L = (L_1, L_2, \ldots, L_n)$ is defined as

$$\mathrm{CVaR}(L) = \min\left\{l + \frac{1}{1-\alpha}\sum_{i=1}^{n} \pi_i\,(L_i - l)_+ \,:\, l \in \mathbf{R}\right\},$$

where $a_+ = \max\{a, 0\}$.

Is CVaR a convex function?

References

[1] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge University Press, 2004.