Nonlinear Optimization (18799 B, PP)
IST-CMU PhD course
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The homework is due April 3.

Homework 1

Problem A. *(Placing a base station)* You want to place a base station to serve N wireless users in a given area. The position of user $n = 1, \ldots, N$ is given, and is denoted by $u_n \in \mathbf{R}^2$ for $n = 1, \ldots, N$. The position of the base station is to be decided, and is denoted by $b \in \mathbf{R}^2$.

In this problem, you'll investigate the convexity of some sets related to the placement of the base station.

(a) Suppose that the base station has a given wireless range $R > 0$. This means that the base station can only serve users whose distance from b is less than or equal to R. Let $A \subset \mathbf{R}^2$ be the set of base station positions that serve all users, that is,

$$A = \left\{ b \in \mathbf{R}^2 \mid \|u_n - b\| \leq R, \text{ for } n = 1, \ldots, N \right\}.$$

Is A a convex set? Justify your answer (If you think A is convex, you should prove it; if you think A is not convex, you should give a specific instance—i.e., give specific N and $u_1, \ldots, u_N$—for which A is not convex.)

(b) In general, users don't want to have a base station very close to them; specifically, suppose users don't want to have a base station at a distance less than a given $\epsilon > 0$. Let $B \subset \mathbf{R}^2$ be the set of base station positions whose distance from any user is greater than or equal to ϵ, that is,

$$B = \left\{ b \in \mathbf{R}^2 \mid \|b - u_n\| \geq \epsilon, \text{ for } n = 1, \ldots, N \right\}.$$

Is the set B convex? Justify you answer.

(c) Suppose that the quality of service provided by the base station to an user degrades as the distance from the base station to the user increases (because of signal power loss). Suppose the set of users is partitioned in two subsets: subset 1 contains users 1 to M; subset 2 contains users $M + 1$ to N.
Let $C \subset \mathbf{R}^2$ be the set of base station positions that serve any user in the subset 1 better than any user in subset 2, that is,

$$C = \left\{ b \in \mathbf{R}^2 \mid \|b - u_i\| \leq \|b - u_j\|, \text{ for } i = 1, \ldots, M \text{ and } j = M + 1, \ldots, N \right\}.$$

Is C a convex set? Justify your answer.

Problem B. *(Controlling a dynamical system)* Consider a controlled dynamical system

$$x(t) = Ax(t-1) + Bu(t), \quad t = 1, \ldots, T,$$

where $x(t) \in \mathbf{R}^n$ is the state of the system at time t, $u(t) \in \mathbf{R}^p$ is the control signal we apply at time t, and $A \in \mathbf{R}^{n \times n}$ and $B \in \mathbf{R}^{n \times p}$ are given matrices. For example, $x(t)$ can represent the state of a robot (position and speed) at time t, and $u(t)$ can be the force we apply to the robot at time t via some thrusters.

Let $x_0 \in \mathbf{R}^n$ be the initial state of the system, that is, $x(0) = x_0$.

(a) (*Warm up*) Show that the final state of the system (the state at the end of the time horizon T) is given by

$$x(T) = A^T x_0 + \sum_{t=1}^{T} A^{T-t} B u(t). \tag{1}$$

(Equation (1) shows how the final state depends on the initial state and the whole control sequence $u = (u(1), u(2), \ldots, u(T)) \in \mathbf{R}^{Tp}$.)

(b) Suppose that we desire the final state of the system to be a given vector $x_{\text{des}} \in \mathbf{R}^n$. Also, suppose that the controls should be bounded in magnitude; more specifically, suppose that any control $u(t)$ that we want to apply to the system should obey $\|u(t)\|_\infty \leq U$, for given $U > 0$. (This is a natural restriction in practice because electronics saturate, thrusters have finite power, and so on.)

Let $A \subset \mathbf{R}^{Tp}$ be the set of control sequences that move the system from the initial state x_0 to the desired state x_{des} and respect the bound constraint, that is,

$$A = \left\{ u = (u(1), \ldots, u(T)) \in \mathbf{R}^{Tp} \mid x(T) = x_{\text{des}} \text{ and } \|u(t)\|_\infty \leq U \text{ for } t = 1, \ldots, T \right\}.$$

(Recall that the final state $x(T)$ is a function of the control sequence $u = (u(1), \ldots, u(T))$, as equation (1) shows.)

Show that A is a convex set.

(c) Consider now the set B of initial states for which it is possible to attain the desired state x_{des} with a bounded control, that is,

$$B = \left\{ x_0 \in \mathbf{R}^n \mid A^T x_0 + \sum_{t=1}^{T} A^{T-t} B u(t) = x_{\text{des}}, \text{ for some } (u(1), \ldots, u(T)) \in B_\infty(0, U) \right\}.$$

Show that B is a convex set.