Nonlinear Optimization (18799 B, PP)
IST-CMU PhD course
Spring 2018
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The homework is due April 5.

Homework 1

Problem A. (*Chekhov's riffle*) A function $\phi : \mathbf{R}^n \to \mathbf{R}$ is said to be a norm if it is

1. positive-definite: $\phi(x) \geq 0$ for all $x \in \mathbf{R}^n$, and $\phi(x) = 0$ only if $x = 0$;
2. homogeneous: $\phi(ax) = |a|\phi(x)$, for all $a \in \mathbf{R}$ and $x \in \mathbf{R}^n$;
3. subadditive: $\phi(x + y) \leq \phi(x) + \phi(y)$ for all $x, y \in \mathbf{R}^n$.

For example, the function $\phi(x) = \|x\|_2 = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}$ is a norm—the usual Euclidean norm.

Let $A \in \mathbf{R}^{n \times n}$ be an invertible matrix. Show that the function $f(x) = \|Ax\|_2$ is a norm; that is, show that f has the three properties listed above.

Why this title for the problem? Anton Chekov, a famous playwright, said that "One must never place a loaded rifle on the stage if it isn't going to go off. It's wrong to make promises you don't mean to keep." Consider this problem a riffle that you may use in one of the next problems.

Problem B. *(Setting prices)* There are two kinds of items, denoted $i = 1, 2$, that we can buy (from a larger supplier) and sell (to many clients). Each unit of item i bought from the supplier costs us $c_i > 0$ euros. We want to decide the prices, denoted p_i, at which we will sell the items.

Earlier market research has shown that the demand for item i depends (not surprisingly) on its selling price p_i. Specifically, the demand declines with increasing price as follows:

$$d_i(p_i) = r_i - s_i p_i,$$

where $r_i > 0$ and $s_i > 0$ are given.

Thus, if we sell at prices p_1 and p_2, our profit is

$$P(p_1, p_2) = (p_1 - c_1)d_1(p_1) + (p_2 - c_2)d_2(p_2).$$

Let $P_{\text{target}} > 0$ be a given target profit. Show that the set of prices that ensure a profit greater than or equal to P_{target} is convex. That is, show that the set

$$S = \left\{(p_1, p_2) \in \mathbf{R}^2 \,:\, P(p_1, p_2) \geq P_{\text{target}}, p_1 \geq 0,\, p_2 \geq 0\right\}$$

is convex. (You may assume that the set is non-empty.)

Hint: the set S can be written as a norm ball intersected with a polyhedron.

Problem C. *(Interference patterns)* A jammer broadcasts random symbols from each of its N antennas. Let $s_n \in \mathbf{R}$ be the random symbol sent through the nth antenna. We assume that s_n is a gaussian random variable with zero mean and variance $\sigma_n^2 > 0$. We don't know the variances, but we know that the average variance is bounded by a known $\overline{\sigma}^2 > 0$:

$$\frac{1}{N} \sum_{n=1}^{N} \sigma_n^2 \leq \overline{\sigma}^2.$$

The N signals broadcasted by the jammer reach K receivers. Let $h_{kn} \in \mathbf{R}$ denote the channel gain between the nth jammer antenna and the kth receiver. The signal read at the kth receiver is

$$y_k = \sum_{n=1}^{N} h_{kn} s_n,$$

which can be shown to be a gaussian random variable with zero mean and variance

$$I_k(\sigma) := \sum_{n=1}^{N} h_{kn}^2 \sigma_n^2.$$

Note that I_k depends on $\sigma = (\sigma_1, \ldots, \sigma_N)$. Intuitively, $I_k(\sigma)$ is the amount of disturbance that the jammer creates at the kth receiver for a given σ.

We call the vector $I(\sigma) = (I_1(\sigma), \ldots, I_K(\sigma)) \in \mathbf{R}^K$ the interference pattern across the K receivers. We're interested in the set of all possible interference patterns that may occur:

$$S = \left\{ I(\sigma) \in \mathbf{R}^K \ : \ \sigma \geq 0, \frac{1}{N} \sum_{n=1}^{N} \sigma_n^2 \leq \overline{\sigma}^2 \right\}.$$

Show that S is a convex set.