Nonlinear Optimization (18799 B,PP)
IST-CMU PhD course
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Important: The homework is due the 2nd of February

Homework 1

Problem A. *(Discrete-time signals)* Let $\{x[t] \, : \, t = 1, 2, \ldots, N\}$ be a discrete-time signal with finite duration N. We can interpret it as a vector $x \in \mathbb{R}^N$ by stacking its samples as in

$$x = \begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[N] \end{bmatrix}.$$

Which of the following sets of signals are convex in $\mathbb{R}^N$? Which ones are convex cones ?

(a) The set of signals x whose energy is upper-bounded by a given constant $K \geq 0$. Note: the energy of a signal x is defined as $\sum_{t=1}^{N} x[t]^2$. Sketch this set in $\mathbb{R}^N$ for $N = 2$ and $K = 4$.

(b) The set of signals x which "intersect" a given reference signal r, that is,

$$x[t] = r[t] \quad \text{for some} \quad t = 1, \ldots, N.$$

Sketch this set in $\mathbb{R}^N$ for $N = 2$ and $r = (r[1], r[2]) = (3, 1)$.

(c) The set of signals x whose envelope is bounded by a given reference signal r, that is,

$$|x[t]| \leq r[t] \quad \text{for all} \quad t = 1, \ldots, N.$$

Sketch this set in $\mathbb{R}^N$ for $N = 2$ and $r = (r[1], r[2]) = (3, 1)$.

(d) The set of signals with "even" symmetry, that is,

$$x[t] = x[N - t + 1] \quad \text{for all} \quad t = 1, 2, \ldots, N/2.$$

Assume that N is even. Sketch this set in $\mathbb{R}^N$ for $N = 2$.

Problem B. *(Joint probability mass functions)* Let X and Y be discrete random variables taking values in the finite alphabets $\mathcal{X} = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}$ and $\mathcal{Y} = \{y_1, y_2, \ldots, y_n\} \subset \mathbb{R}$, respectively. Let the matrix $P_{XY} \in \mathbb{R}^{n \times n}$ encode their joint probability mass function, that is, the (i, j)th entry of P_{XY} is $(P_{XY})_{ij} = \mathrm{Prob}(X = x_i, Y = y_j)$. Naturally, such a matrix belongs to the set of $n \times n$ matrices with nonnegative entries summing up to 1. That is, $P_{XY} \in \Delta$ where

$$\Delta := \{P \in \mathbb{R}^{n \times n} \, : \, P_{ij} \geq 0 \text{ for all } i, j \text{ and } \sum_{i,j=1}^{n} P_{ij} = 1\}.$$

In the sequel, the symbol $\mathrm{E}\{\cdot\}$ denotes the expectation operation with respect to P_{XY}. For example, $\mathrm{E}\left\{X^2 \cos(Y)\right\} = \sum_{i,j=1}^{n} x_i^2 \cos(y_j) \, (P_{XY})_{ij}$. Note that the expectation is a function of P_{XY}.

(a) Show that Δ is a convex set in $\mathbb{R}^{n\times n}$.

(b) Consider the set of matrices $P_{XY} \in \Delta$ such that $a \leq \mathrm{E}\{XY\} \leq b$ and $c \leq \mathrm{E}\{X\} \leq d$, where $a, b, c, d \in \mathbb{R}$ are given constants. Is this set convex in $\mathbb{R}^{n\times n}$?

(c) Recall that X and Y are independent random variables if and only if

$$\mathrm{Prob}(X = x_i, Y = y_j) = \mathrm{Prob}(X = x_i)\mathrm{Prob}(Y = y_j) \text{ for all } i, j.$$

Consider the set of matrices $P_{XY} \in \Delta$ that lead to independent random variables X and Y. Is this set convex in $\mathbb{R}^{n\times n}$?

Problem C. *(Irreducible matrices)* An $n \times n$ matrix P is called a permutation matrix if exactly one entry in each column and row is equal to 1 and all the other entries are 0. For example,

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a 3×3 permutation matrix. An $n \times n$ matrix A ($n \geq 2$) is said to be reducible if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

where B and D are square matrices and 0 is a zero matrix. A matrix is said to be irreducible if it is not reducible.

Show that the set of $n \times n$ matrices ($n \geq 2$) which are nonnegative (i.e., all entries are nonnegative) and irreducible is convex in $\mathbb{R}^{n\times n}$.

Note: the concept of irreducible matrices shows up in graph theory and Markov chains. For example, in graph theory, it is known that a directed graph is strongly connected (i.e., there is a directed path from any vertex to any other vertex) if and only if its adjacency matrix is irreducible (if the graph has n nodes, its adjacency matrix is an $n \times n$ matrix such that $A_{ij} = 1$ if there is an edge from vertex i to vertex j, and $A_{ij} = 0$ otherwise).