Nonlinear Optimization

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14 January 2011

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Optimization problem

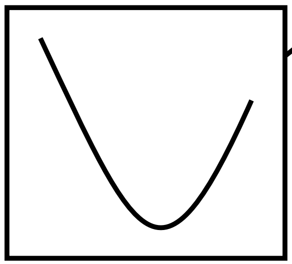
- This course is about solving optimization problems:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \Omega \end{array}$$

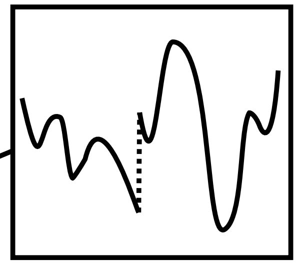
- $x \in \mathbb{R}^n$ is the optimization variable
- $f : \mathbb{R}^n \to \mathbb{R}$ is the cost function
- $\Omega \subset \mathbb{R}^n$ is the constraint set

Key-point

“The great watershed in optimization isn’t between linearity and nonlinearity, but between convexity and non-convexity” RT Rockefellar



Convex programs (“easy”)	**Nonconvex programs (“difficult”)**
• example: LP’s	• example: combinatorial



Course overview

- The focus of this course is on **convex** programs
  - How to recognize them ?
  - How to solve them ?

- The course is organized in 4 parts:
  - formulation of convex optimization problems
  - conditions for optimality and duality theory
  - numerical algorithms
  - nonsmooth optimization

Course overview

- Main bibliography:
  - [CO] *Convex optimization* by S. Boyd and L. Vandenberghe
    Available for download at:

    `http://www.stanford.edu/~boyd/cvxbook/`

  - [NO] *Numerical optimization*, 2nd ed., by J. Nocedal and S. Wright

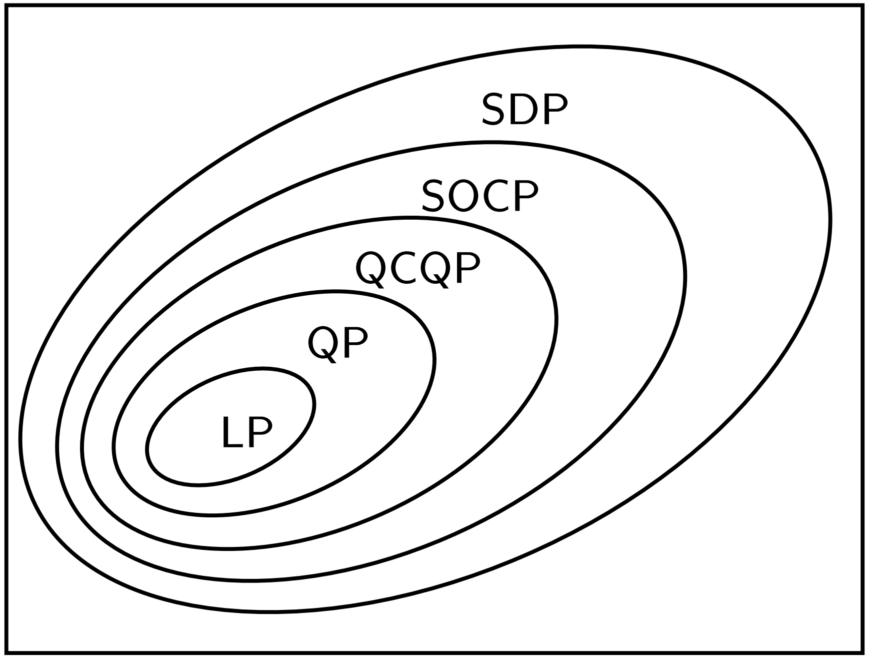
- Secondary bibliography:
  - *Lectures on modern convex optimization*, Aharon Ben-Tal and Arkadi Nemirovski, 2001, MPS-SIAM Series on Optimization
  - *Nonlinear programming*, 2nd ed., Dimitri Bertsekas, 1999, Athena Scientific

Part I: formulation of convex optimization problems

- Convex and nonconvex programs can look similar !
- How to detect and formulate convex programs ?

- Hierarchy of the most popular convex optimization programming classes

Convex programs



LP	=	linear programming
QP	=	quadratic programming
QCQP	=	quadratically constrained quadratic programming
SOCP	=	second-order cone programming
SDP	=	semidefinite programming

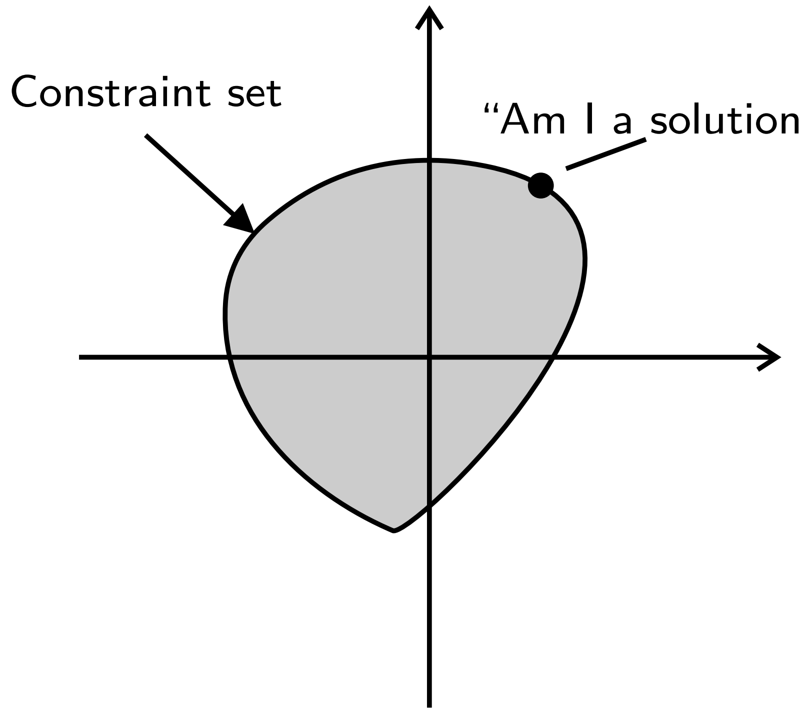
- There are efficient algorithms for each class (e.g. in MATLAB)
- Many applications: control, communications, pattern recognition, image processing, graphs, networks, statistics, etc

Part I: formulation of convex optimization problems

- Will cover:
  - [CO] chapter 2: convex sets
  - [CO] chapter 3: convex functions
  - [CO] chapter 4: convex optimization problems
  - [CO] selected parts from chapters 6, 7, 8 (applications)

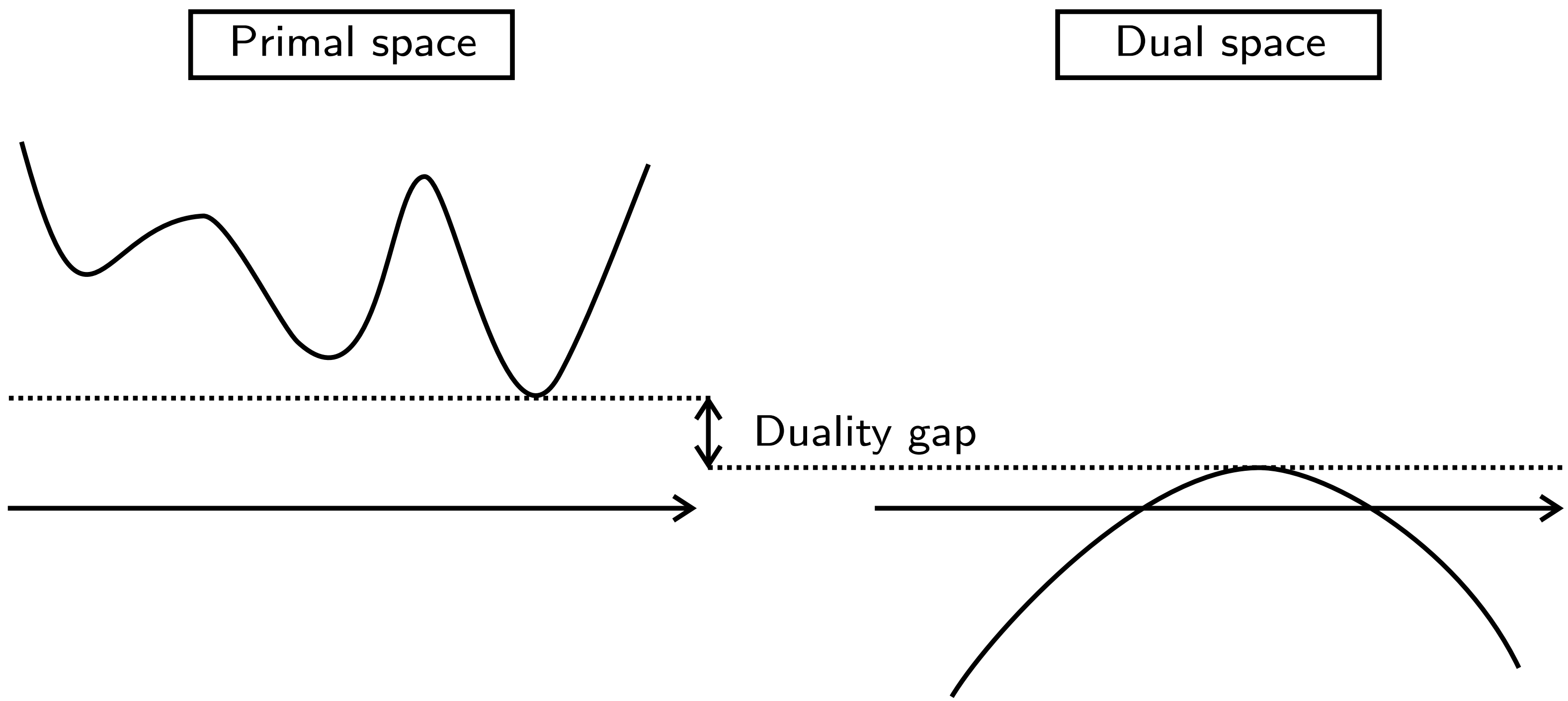
Part II: conditions for optimality and duality theory

- Pinpointing the solutions: the Karush-Kuhn-Tucker (KKT) conditions



- Applications of KKT conditions:
  - sometimes, can provide closed-form solution
  - or suggest a simple (usually finite-step) algorithm
  - form the basis for more complex iterative algorithms (e.g. primal-dual interior-point methods)

- Duality theory:
  - each optimization problem has a convex “twin” brother (in fact, many)
  - geometrical interpretation



  - Under mild conditions for convex programs: duality gap = zero

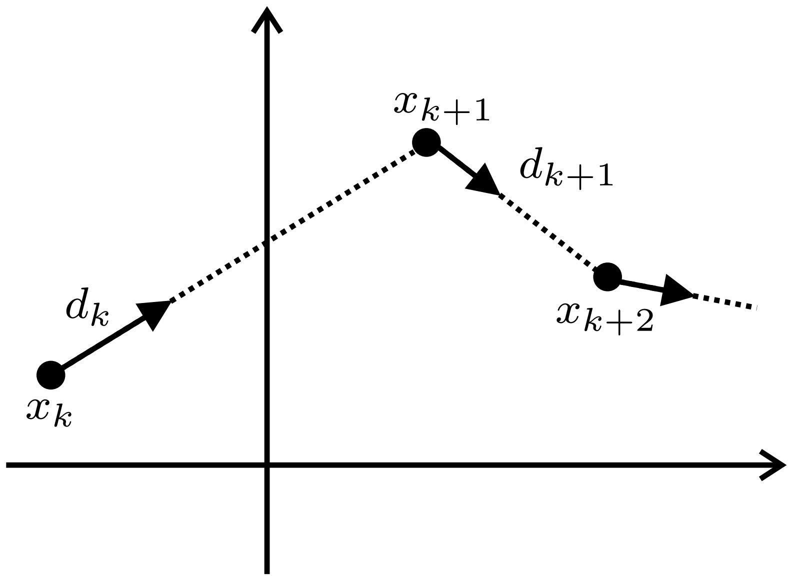
- Applications of duality theory:
  - the dual might be easier to solve
  - the dual provides certified lower bounds on the primal problem
  - duality provides non-heuristic stopping criteria for numerical algorithms
  - duality provides "unexpected" results (e.g. max-flow = min-cut in directed graphs)
  - duality can simplify problem formulation (e.g. remove exponential number of constraints)
  - duality enables problem decomposition (separation into smaller subproblems)
  - duality offers convex relaxations for nonconvex problems (e.g. max-cut)
  - duality is important for algorithm development

Part II: conditions for optimality and duality theory

- Will cover:
  - [CO] chapter 5: duality

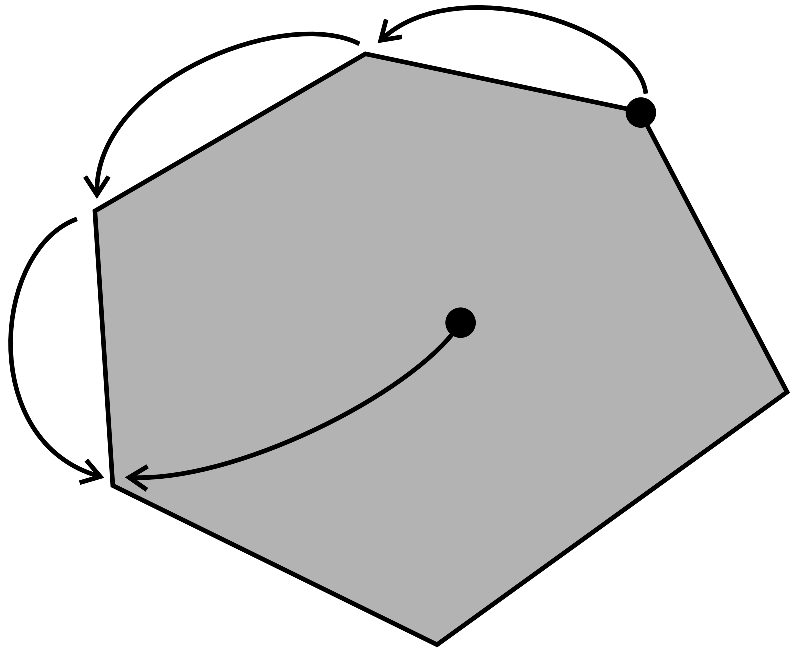
Part III: numerical algorithms

- Line-search algorithms for unconstrained optimization



  - Armijo's rule and Newton direction
  - how fast do they converge ?

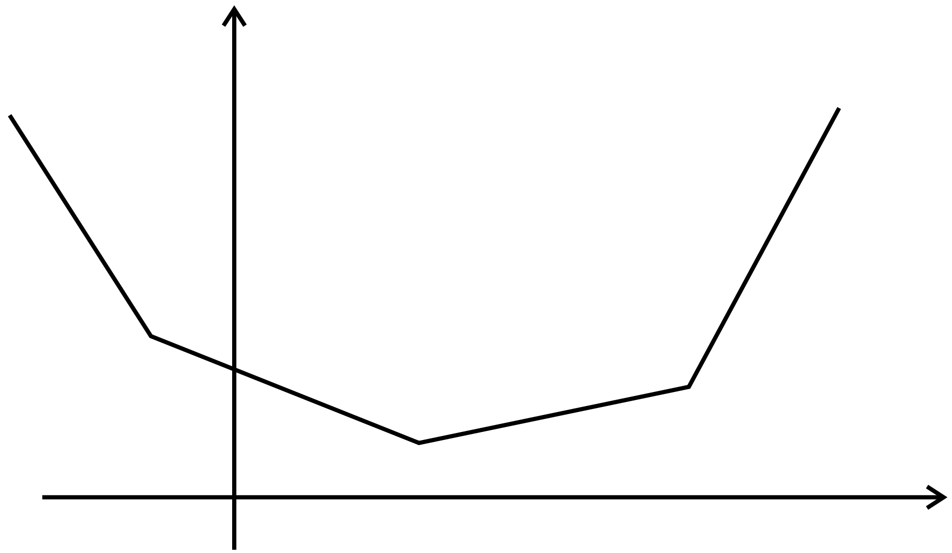
- Algorithms for constrained convex optimization: interior-point algorithms



Part III: numerical algorithms

- Will cover:
  - [CO] chapter 9: unconstrained optimization
  - [CO] chapter 10: equality constrained minimization
  - [CO] chapter 11: interior-point methods
  - [NO] selected parts from chapters 6, 17 and 19

Part IV: nonsmooth optimization



- Nonsmooth convex optimization appears a lot:
  - many problems are naturally nonsmooth. Example:

$$\text{minimize} \quad \|Ax - b\|^2 + \beta\left(|x_1| + |x_2| + \cdots + |x_n|\right)$$

  - a smooth reformulation is usually possible, but new variables/constraints enter the problem
  - solving the dual problem is often a nonsmooth optimization problem
  - lots of applications in sensor networks

Grading

- Grade = Homework (50%) + 24h take home exam (40%) + 1h oral exam (10%)
- Homework (**tentative** schedule):

#	Due (11pm Lisbon = 6pm Pittsburgh)
1	February, 2
2	February, 16
3	March, 2
4	March, 16
5	March, 30
6	April, 13
7	April, 27

- 24h take exam is on May, 3
- 1h oral examination between May, 4 and May, 6
- Office hours: TBD