Nonlinear Signal Processing
2006-2007

Riemannian Geodesics

(Ch.5, "Riemannian Manifolds", J. Lee, Springer-Verlag)

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Lecture's key-points

☐ A Riemannian manifolds has a "natural" connection

□ **Definition [Connection and metric compatibility]** Let M be a Riemannian manifold with metric $g \equiv \langle \cdot, \cdot \rangle$. A linear connection ∇ is said to be compatible with the metric if for all smooth vector fields $X, Y, Z \in \mathcal{T}(M)$ there holds

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

□ **Lemma [Characterization of metric compatibility]** Let (M, g) be a Riemannian manifold. The following conditions are equivalent:

(a) the linear connection ∇ is compatible with g

(b) if V, W are smooth vector fields along a smooth curve γ, then

$$\frac{d}{dt}\langle V, W\rangle = \langle D_t V, W\rangle + \langle V, D_t W\rangle$$

(c) parallel translation $P_{s\to t} : T_{\gamma(s)}M \to T_{\gamma(t)}M$ is an isometry

☐ **Definition [Lie bracket of vector fields]** Let M be a smooth manifold and let $X, Y \in \mathcal{T}(M)$ be smooth vector fields. Their Lie bracket is the smooth vector field $[X, Y] \in \mathcal{T}(M)$ defined by

$$f \in C^\infty(M) \mapsto [X, Y]_p f = X_p(Yf) - Y_p(Xf)$$

for any $p \in M$

☐ **Definition [Torsion-free connections]** Let ∇ be a linear connection on the smooth manifold M. The connection ∇ is said to be torsion-free (or symmetric) if

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

for all $X, Y \in \mathcal{T}(M)$

☐ **Theorem [Fundamental theorem of Riemannian geometry]** Let (M, g) be a Riemannian manifold. There exists a unique linear connection ∇ which is torsion-free and compatible with g.

This connection is called the Riemannian or the Levi-Civita connection

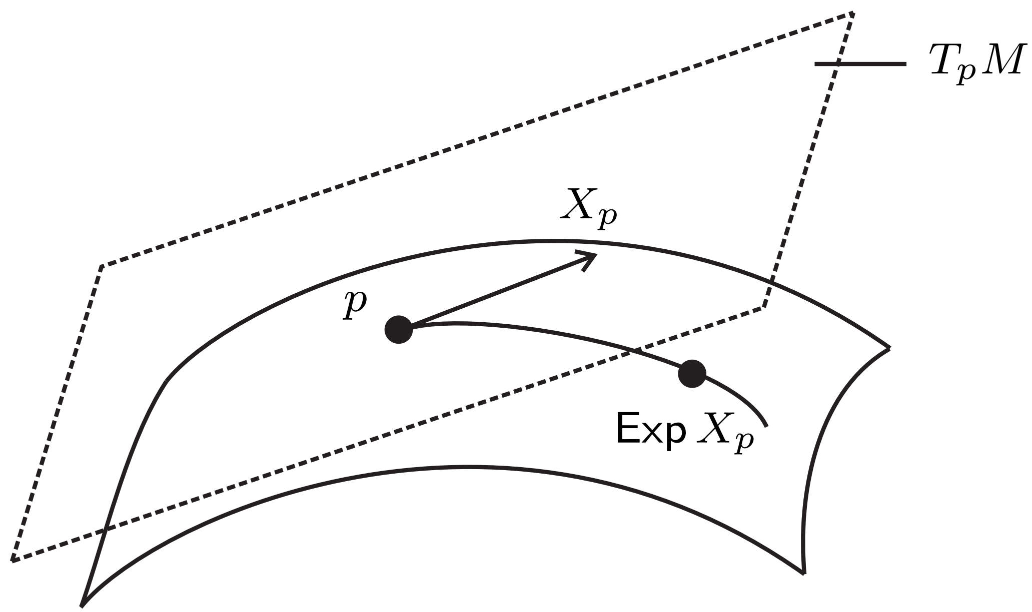
☐ **Lemma [Riemannian geodesics]** All Riemannian geodesics are constant-speed curves

☐ **Definition [Exponential map]** Let M be a Riemannian manifold. Let

$\mathcal{E} = \{V \in TM$: there is a geodesic γ_V whose domain contain $[0,1]$ and $\dot{\gamma}_V(0) = V\}$.

The exponential map is defined as

$$\mathsf{Exp} : \mathcal{E} \subset TM \to M \qquad V \mapsto \mathsf{Exp}(V) = \gamma_V(1).$$



□ **Proposition [Properties of the exponential map]** Let (M, g) be a Riemannian manifold. The following conditions are equivalent:

(a) $\mathcal{E}$ is an open subset of TM, and each set $\mathcal{E}_p = \mathcal{E} \subset T_pM$ is star-shaped with respect to 0

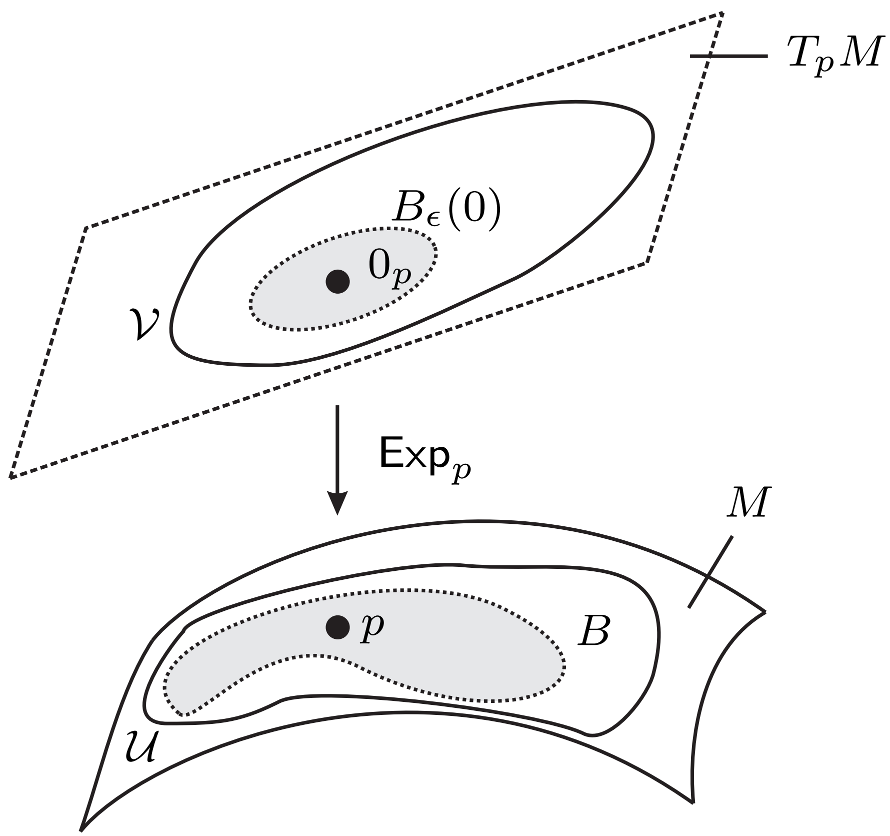
(b) for each $V \in TM$, the geodesic γ_V is given by

$$\gamma_V(t) = \mathsf{Exp}(tV)$$

for all t such that either side is defined

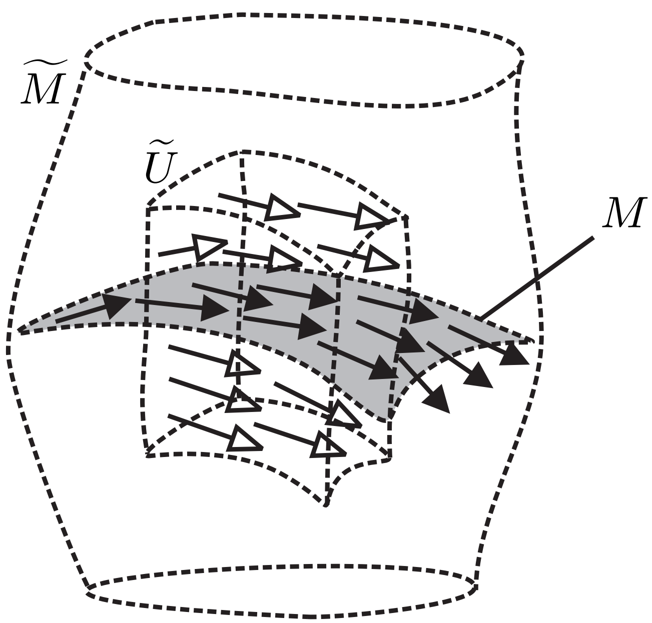
(c) the exponential map is smooth

☐ **Lemma [Normal neighborhood lemma]:** For any $p \in M$, there is a neighborhood $\mathcal{V}$ of the origin in T_pM and a neighborhood $\mathcal{U}$ of p such that $\text{Exp}_p : \mathcal{V} \to \mathcal{U}$ is a diffeomorphism



☐ A geodesic ball is a set $B = \text{Exp}_p(B_\epsilon(0))$ such that Exp_p is a diffeomorphism on $B_\epsilon(0)$

☐ **Definition [Local extension]** Let M be an embedded submanifold of $\widetilde{M}$. Let X be a smooth vector field on M. A local extension of X around p is a smooth vector field $\widetilde{X}$ on a open neighborhood $\widetilde{U} \subset \widetilde{M}$ of p, which agrees with X on $M \cap \widetilde{U}$

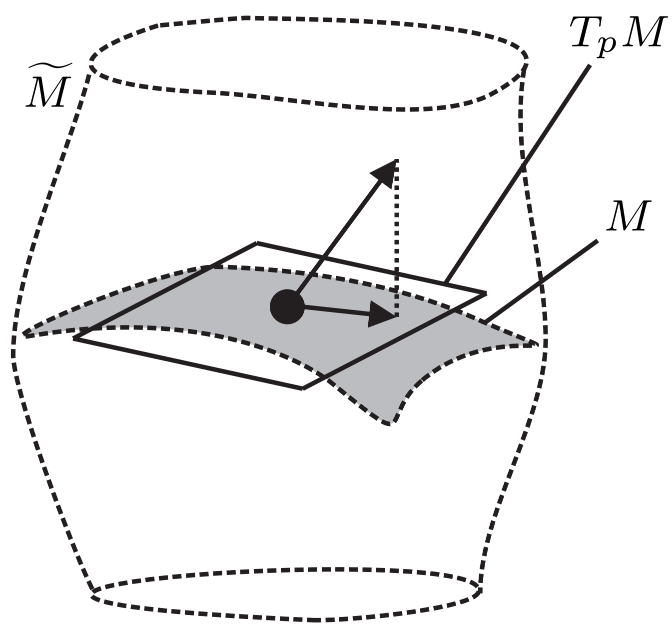


☐ **Lemma [Existence of local extensions]** Let M be an embedded submanifold of $\widetilde{M}$. Let X be a smooth vector field on M. Then, for any $p \in M$, there exists a local extension of X around p

□ **Lemma [Characterization of Riemannian connections on Riemannian submanifolds]** Let $\widetilde{M}$ be a Riemannian manifold and M a Riemannian submanifold of M. Let $\widetilde{\nabla}$ and ∇ denote the respective Riemannian connections. Let X, Y denote smooth vector fields on M. Then, for any $p \in M$, we have

$$\nabla_{X_p} Y = \left(\widetilde{\nabla}_{\tilde{X}_p} \widetilde{Y} \right)^{\top}$$

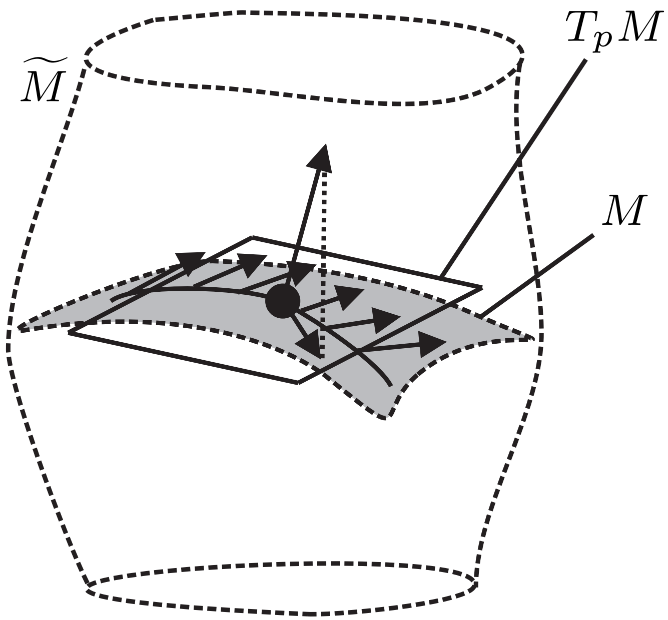
where $\widetilde{X}, \widetilde{Y}$ denote local extensions of X, Y around p and $(\cdot)^{\top} : T_p \widetilde{M} \to T_p M$ corresponds to orthogonal projection



□ **Lemma [Characterization of covariant derivative on Riemannian submanifolds]**
Let $\widetilde{M}$ be a Riemannian manifold and M a Riemannian submanifold of M. Let $\gamma\ :]a,b[\subset \mathbb{R} \to M$ be a smooth curve, $\widetilde{\gamma} = \gamma \circ \iota\ :]a,b[\to \widetilde{M}$, and let $\widetilde{D}_t$ and D_t denote the covariant derivative operators along γ and $\widetilde{\gamma}$, respectively. Let V denote a vector field along γ. Then, for any $t_0 \in]a,b[$, we have

$$D_t V(t_0) = \left(\widetilde{D}_t V(t_0)\right)^\top$$

where $(\cdot)^\top\ :\ T_p\widetilde{M} \to T_p M$ corresponds to orthogonal projection

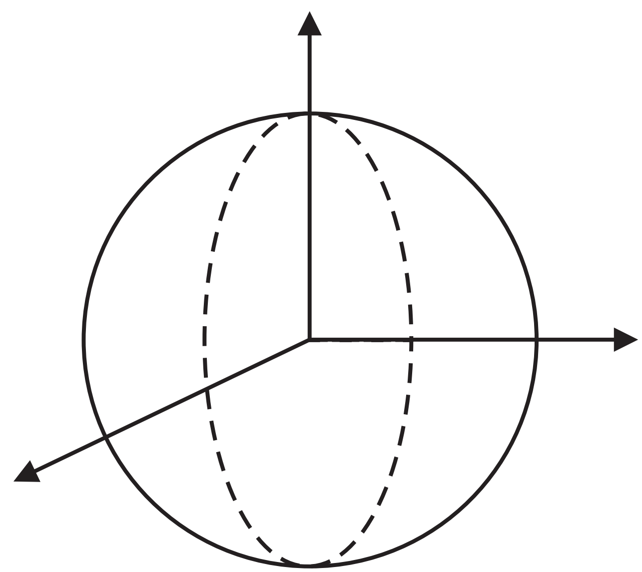


☐ **Lemma [Characterization of geodesics on Riemannian submanifolds]** Let $\widetilde{M}$ be a Riemannian manifold and M a Riemannian submanifold of $\widetilde{M}$. A smooth curve $\gamma : I \subset \mathbb{R} \to M$ is a geodesic in M if and only if

$$\widetilde{D}_t \dot{\gamma}(t) \quad \perp \quad T_{\gamma(t)} M$$

for all $t \in I$. Here $\widetilde{D}_t$ denotes the covariant derivative operator on $\widetilde{M}$

☐ **Example (geodesics on the unit-sphere):** the geodesics on the unit-sphere $\mathsf{S}^{n-1}(\mathbb{R}) = \{x \in \mathbb{R}^n : \|x\| = 1\}$ are the great circles



The geodesic emanating from $p \in \mathsf{S}^{n-1}(\mathbb{R})$ in the non-zero direction $\delta \in T_p\mathsf{S}^{n-1}(\mathbb{R})$ is given by

$$\gamma_{p,\delta}(t) = p\cos\left(\|\delta\|\, t\right) + \frac{\delta}{\|\delta\|}\sin\left(\|\delta\|\, t\right)$$

for all $t \in \mathbb{R}$

☐ **Example (parallel transport on the unit-sphere):** parallel transport along the unit-speed ($\|d\| = 1$) geodesic $\gamma_{p,d}(t)$ is given by

$$\tau_{0\to t} \;:\; T_p\mathsf{S}^{n-1}(\mathbb{R}) \to T_{\gamma_{p,d}(t)}\mathsf{S}^{n-1}(\mathbb{R}) \qquad \delta \mapsto \left[I_n + \left(\dot{\gamma}_{p,d}(t) - d\right) d^\top\right]\delta$$

☐ **Example (geodesics on the orthogonal group):** the geodesics on the orthogonal group $\mathsf{O}(n)$ (viewed as a Riemannian submanifold of $\mathbb{R}^{n\times n}$) which emanate from $Q \in \mathsf{O}(n)$ in the tangent direction $\Delta = QK$, $K \in \mathsf{K}(n, \mathbb{R})$ are given by

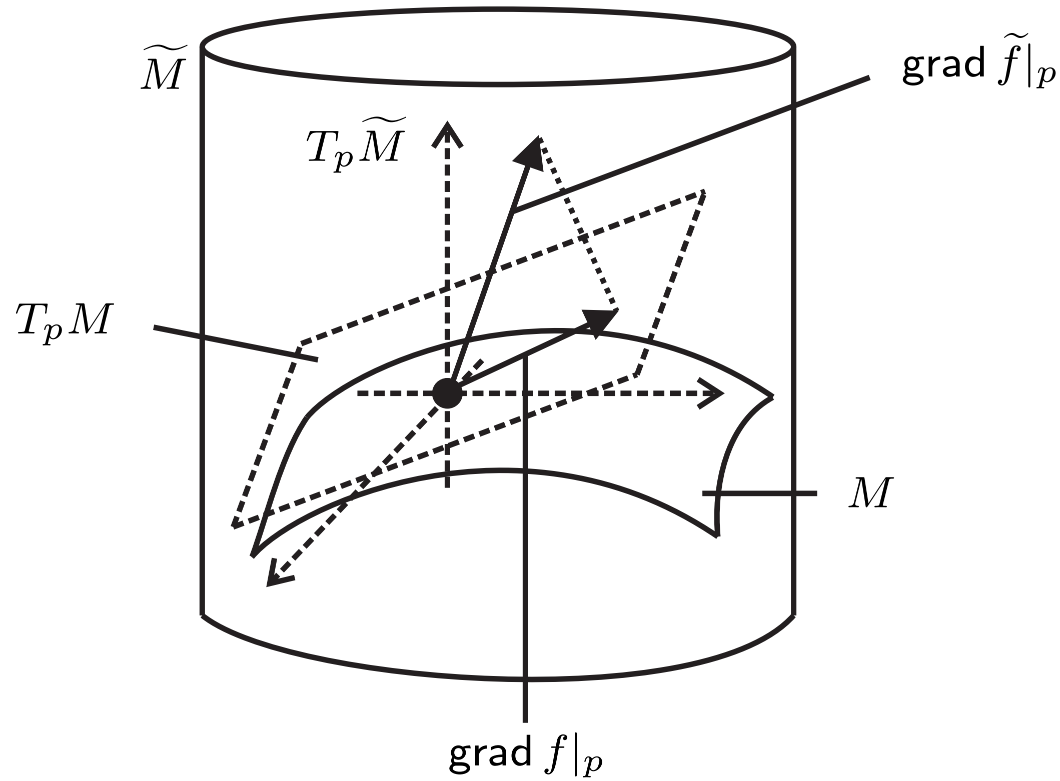
$$\gamma(t) = Qe^{Kt}$$

for all $t \in \mathbb{R}$

☐ **Lemma [Characterization of gradient for Riemannian submanifolds]** Let $\widetilde{M}$ be a Riemannian manifold and M a Riemannian submanifold of $\widetilde{M}$. Let $\widetilde{f} : \widetilde{M} \to \mathbb{R}$ be a smooth function and $f : M \to \mathbb{R}$, $f = \widetilde{f}|_M$. Then, for any $p \in M$, we have

$$\text{grad } f|_p = \left(\text{grad } \widetilde{f}|_p\right)^\top$$

where $(\cdot)^\top : T_p\widetilde{M} \to T_p\widetilde{M}$ denotes orthogonal projection onto T_pM



☐ **Lemma [Characterization of covariant Hessian]** Let M be a Riemannian manifold and ∇ its Riemannian connection. Let $f : M \to \mathbb{R}$ be a smooth function.

(a) For any smooth vector fields $X, Y \in \mathcal{T}(M)$ we have

$$\nabla^2 f(X, Y) = \langle \nabla_X \mathsf{grad}\, f, Y \rangle = \langle \nabla_Y \mathsf{grad}\, f, X \rangle$$

(b) For any $X_p \in T_p M$, we have

$$\nabla^2 f(X_p, X_p) = \frac{d^2}{dt^2} f(\gamma(t))|_{t=0}$$

where γ denotes the geodesic emanating from p in the tangent direction X_p

☐ **Example (gradient and Hessian on the unit-sphere):** let $f : \mathsf{S}^{n-1}(\mathbb{R}) \to \mathbb{R}$, $f(x) = c^\top x$. The (intrinsic) gradient and Hessian of f are given by

$$\begin{aligned} \mathsf{grad}\, f|_p &= \left(I_n - pp^\top\right) c \\ \nabla^2 f(\delta, \delta) &= -(c^\top p)\, \delta^\top \delta \quad \text{for } \delta \in T_p \mathsf{S}^{n-1}(\mathbb{R}) \end{aligned}$$

□ **Example (geometrical interpretation of 2nd order KKT conditions):** let $M = \{x \in \mathbb{R}^n \,:\, F(x) = 0\}$ be an embedded submanifold of $\mathbb{R}^n$ where $F : \mathbb{R}^n \to \mathbb{R}^p$ has rank p over M. The classical sufficient 2nd order KKT conditions for $p \in M$ to be a local minimizer for

$$\min_{x \in M} \phi(x)$$

are equivalent to

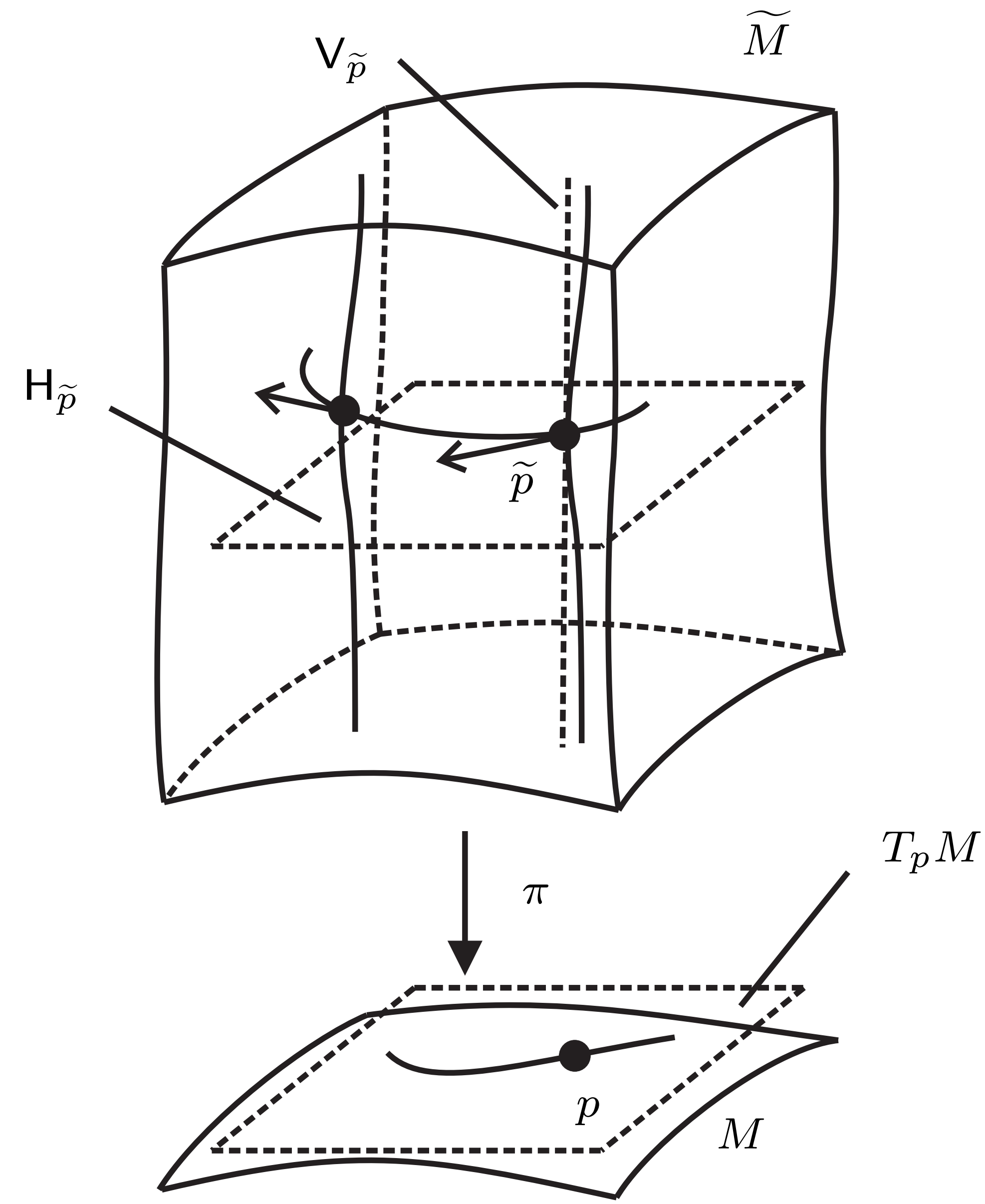
$$\begin{cases} \text{grad}\,\phi|_p = 0 & \text{(stationary point)} \\ \nabla^2\phi(\delta,\delta) > 0 \text{ for any } \delta \in T_pM - \{0\} & \text{(Hessian is definite positive)} \end{cases}$$

□ **Proposition [Geodesics and Riemannian submersions]** Let $\pi : \widetilde{M} \to M$ be a Riemannian submersion.

(a) Let $\widetilde{\gamma} : I \subset \mathbb{R} \to \widetilde{M}$ be a geodesic of $\widetilde{M}$. If $\dot{\widetilde{\gamma}}(t_0)$ is horizontal for some $t_0 \in I$, then $\dot{\widetilde{\gamma}}(t)$ is horizontal for all $t \in I$

(b) Let $\widetilde{\gamma} : I \subset \mathbb{R} \to \widetilde{M}$ be a geodesic of $\widetilde{M}$. Then $\gamma = \pi \circ \widetilde{\gamma}$ is a geodesic of M

(c) Let $p = \pi(\widetilde{p})$ and $\gamma : I \to M$ be a geodesic of M such that $\gamma(t_0) = p$, $t_0 \in I$. Then, there is an open interval $J \subset I$ containing t_0 and an horizontal geodesic $\widetilde{\gamma} : J \to \widetilde{M}$ such that $\widetilde{\gamma}(t_0) = \widetilde{p}$ and $\gamma = \pi \circ \widetilde{\gamma}$. Such $\widetilde{\gamma}$ is unique


$\widetilde{M}$
$\mathsf{V}_{\widetilde{p}}$
$\mathsf{H}_{\widetilde{p}}$
$\widetilde{p}$
π
T_pM
p
M

☐ **Example (horizontal lifts can only be local):** consider the Riemannian manifolds $\widetilde{M} = \mathbb{R}^2 - \{(1,0)\}$, $M = \mathbb{R}$ and the Riemannian submersion $\pi : \widetilde{M} \to M$, $\pi(x,y) = x$.

The geodesic $\gamma(t) = t$ on M is defined for all $t \in \mathbb{R}$. The (horizontal) geodesic $\widetilde{\gamma}(t)$ on $\widetilde{M}$ is only defined for $t < 1$

☐ **Example (geodesics on the Stiefel manifold):** let the Stiefel manifold

$$\mathsf{O}(n,p) = \{X \in \mathbb{R}^{n \times p} \, : \, Q^\top Q = I_p\}$$

be identified as the quotient space $\mathsf{O}(n)/\mathsf{O}(n-p)$ where $\mathsf{O}(n-p)$ acts on $\mathsf{O}(n)$ as

$$\underbrace{R}_{(n-p)\times(n-p)} \cdot [\ \underbrace{Q_1}_{n \times p} \quad \underbrace{Q_2}_{n \times (n-p)}\ ] = \begin{bmatrix} Q_1 & Q_2 R^\top \end{bmatrix}$$

▷ The usual metric is assumed for $\mathsf{O}(n)$, i.e., if $\Delta_1 = QK_1$, $\Delta_2 = QK_2$ denote tangent vectors in $T_Q\mathsf{O}(n)$ with $K_1, K_2 \in \mathsf{K}(n, \mathbb{R})$ then

$$\langle \Delta_1, \Delta_2 \rangle = \operatorname{tr}\left(\Delta_2^\top \Delta_1\right) = \operatorname{tr}\left(K_2^\top K_1\right)$$

▷ The vertical space at $Q = [\, Q_1 \; Q_2 \,] \in \mathsf{O}(n)$ is given by

$$\mathsf{V}_Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \mathsf{K}(n-p, \mathbb{R}) \end{bmatrix}$$

▷ The horizontal space at $Q = [\, Q_1 \; Q_2 \,] \in \mathsf{O}(n)$ is given by

$$\mathsf{H}_Q = \left\{ Q \begin{bmatrix} A & -B^\top \\ B & 0 \end{bmatrix} \; : \; A \in \mathsf{K}(p, \mathbb{R}), \; B \in \mathbb{R}^{(n-p)\times p} \right\}$$

▷ The geodesic of $\mathsf{O}(n)$ emanating from Q in the horizontal direction

$$\Delta = Q \begin{bmatrix} A & -B^\top \\ B & 0 \end{bmatrix}$$

is given by

$$\gamma(t) = Q e^{\begin{bmatrix} A & -B^\top \\ B & 0 \end{bmatrix} t}$$

□ **Example (geodesics on the Grassmann manifold):** let the Grassmann manifold $\mathsf{G}(n,p)$ be realized as the coset space

$$\mathsf{G}(n,p) = \mathsf{O}(n) / \left(\mathsf{O}(p) \times \mathsf{O}(n-p)\right).$$

▷ The vertical space at $Q \in \mathsf{O}(n)$ is given by

$$\mathsf{V}_Q = Q \begin{bmatrix} \mathsf{K}(p,\mathbb{R}) & 0 \\ 0 & \mathsf{K}(n-p,\mathbb{R}) \end{bmatrix}$$

▷ The horizontal space at $Q \in \mathsf{O}(n)$ is given by

$$\mathsf{H}_Q = \left\{ Q \begin{bmatrix} 0 & -A^\top \\ A & 0 \end{bmatrix} \, : \, A \in \mathbb{R}^{(n-p)\times p} \right\}$$

▷ The geodesic of O(n) emanating from Q in the horizontal direction

$$\Delta = Q \begin{bmatrix} 0 & -A^\top \\ A & 0 \end{bmatrix}$$

is given by

$$\gamma(t) = Q e^{\begin{bmatrix} 0 & -A^\top \\ A & 0 \end{bmatrix} t}$$

□ **Proposition [Cartesian product of Riemannian manifolds]** Let (M, g_M) and (N, g_N) denote Riemannian manifolds. Then $(M \times N, g_{M\times N})$ is a Riemannian manifold where $g_{M\times N} = \pi_M^* g_M + \pi_N^* g_N$. Furthermore

(a) the projection maps π_M and π_N are Riemannian submersions

(b) if $\gamma_M : I \to M$ and $\gamma_N : I \to N$ are geodesics then

$$\gamma : I \to M \times N \quad \gamma(t) = (\gamma_M(t), \gamma_N(t))$$

is a geodesic on $M \times N$