Nonlinear Signal Processing
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Connections

(Ch.4, "Riemannian Manifolds", J. Lee, Springer-Verlag)

Instituto Superior Técnico, Lisbon, Portugal

João Xavier

jxavier@isr.ist.utl.pt

Lecture's key-points

☐ A connection permits the differentiation of vector fields

□ **Definition [Connection]** Let M be a smooth manifold. A linear connection on M is a map

$$\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M) \qquad (X, Y) \mapsto \nabla_X Y$$

such that

(a) $\nabla_X Y$ is $C^\infty(M)$-linear with respect to X:

$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y \quad \text{for } f_1, f_2 \in C^\infty(M),\ X_1, X_2, Y \in \mathcal{T}(M)$$

(b) $\nabla_X Y$ is $\mathbb{R}$-linear with respect to Y:

$$\nabla_X (a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2 \quad \text{for } a_1, a_2 \in \mathbb{R},\ X, Y_1, Y_2 \in \mathcal{T}(M)$$

(c) ∇ satisfies the rule:

$$\nabla_X (fY) = (Xf)Y + f \nabla_X Y \quad \text{for } f \in C^\infty(M),\ X, Y \in \mathcal{T}(M)$$

□ **Example (Euclidean connection):** let $M = \mathbb{R}^n$. For given smooth vector fields $X = X^i\partial_i, Y = Y^i\partial_i \in \mathcal{T}(\mathbb{R}^n)$ define

$$\nabla_X Y = (XY^i)\partial_i.$$

Then, ∇ is a linear connection on $\mathbb{R}^n$, also called the Euclidean connection

□ **Lemma [A linear connection is a local object]** Let ∇ be a linear connection on M. Then, $\nabla_X Y$ at $p \in M$ only depends on the values of Y in a neighborhood of p and the value of X at p

□ **Definition [Christoffel symbols]** Let $\{E_1, E_2, \ldots, E_n\}$ be a local frame on an open subset $U \subset M$ (i.e., each E_i is a smooth vector field on U and $\{E_{1p}, E_{2p}, \ldots, E_{np}\}$ is a basis for T_pM for each $p \in U$).

For any $1 \leq i, j \leq n$, we have the expansion

$$\nabla_{E_i} E_j = \Gamma_{ij}^k \, E_k.$$

The n^3 functions $\Gamma_{ij}^k : U \to \mathbb{R}$ defined this way are called the Christoffel symbols of ∇ with respect to $\{E_1, E_2, \ldots, E_n\}$

□ **Example (Christoffel symbols for the Euclidean connection):** let $M = \mathbb{R}^n$ and consider the (global) frame $\{\partial_1, \partial_2, \ldots, \partial_n\}$ on M. The Christoffel symbols corresponding to the Euclidean connection vanish identically with respect to this frame

□ **Definition [Covariant derivative of smooth covector fields]** Let ∇ be a linear connection on M and let ω be a smooth covector field on M. The covariant derivative of ω with respect to X is the smooth covector field $\nabla_X \omega$ given by

$$(\nabla_X \omega)(Y) = X\omega(Y) - \omega(\nabla_X Y) \qquad \text{for } Y \in \mathcal{T}(M)$$

□ **Lemma [An inner-product on V establishes an isomorphism $V \simeq V^*$]** Let $\langle \cdot, \cdot \rangle$ denote an inner-product on the n-dimensional vector space V. To each $X \in V$ corresponds the covector $X^\flat \in V^*$ given by $X^\flat = \langle \cdot, X \rangle$, that is,

$$X^\flat(Y) = \langle Y, X \rangle \quad \text{for } Y \in V.$$

The map $V \to V^*$, $X \mapsto X^\flat$ is an isomorphism. Its inverse is $V^* \to V$, $\omega \mapsto \omega^\sharp$

□ **Definition [Gradient and Hessian of a smooth function]** Let M be a Riemannian manifold and let f be a smooth function on M.

▷ The gradient of f, written $\text{grad}\, f$, is the smooth vector field defined pointwise as

$$\text{grad}\, f|_p = (df|_p)^\sharp \,,$$

for all $p \in M$. Thus, for any tangent vector $X_p \in T_pM$, we have

$$X_p f = (df)_p(X_p) = \langle X_p, \text{grad}\, f|_p \rangle$$

▷ Let ∇ be a linear connection on M. The Hessian of f with respect to ∇, written $\nabla^2 f$, is the smooth tensor field of order 2 on M defined as

$$\nabla^2 f(X,Y) = (\nabla_Y df)(X) = Y(Xf) - (\nabla_Y X)\, f, \quad \text{for } X, Y \in \mathcal{T}(M)$$

□ **Example (gradient and Hessian of a smooth function in (flat) $\mathbb{R}^n$):** let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Thus,

$$df = \partial_1 f dx^1 + \partial_2 f dx^2 + \cdots + \partial_n f dx^n.$$

Consider the usual Riemannian metric on $\mathbb{R}^n$:

$$g\left(\partial_i|_p, \partial_j|_p\right) = \delta_i^j.$$

The gradient of f at p is given by

$$\text{grad}\, f(p) = \partial_1 f(p)\, \partial_1|_p + \partial_2 f(p)\, \partial_2|_p + \cdots + \partial_n f(p)\, \partial_n|_p.$$

Let ∇ be the Euclidean connection. The Hessian of f at p is given by

$$\nabla^2 f(X_p, Y_p) = X^i Y^j \partial_{ij}^2 f(p) \quad \text{for } X_p = X^i \partial_i|_p, Y_p = Y^j \partial_j|_p.$$

□ **Example (gradient and Hessian of a smooth function in $\mathbb{R}^n$):** let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function.

Consider the Riemannian metric on $\mathbb{R}^n$:

$$g = e^{2x+yz}\, dx \otimes dx + (2 - \cos(z))\, dy \otimes dy + (y^2 + 1)\, dz \otimes dz.$$

The gradient of f at p is given by

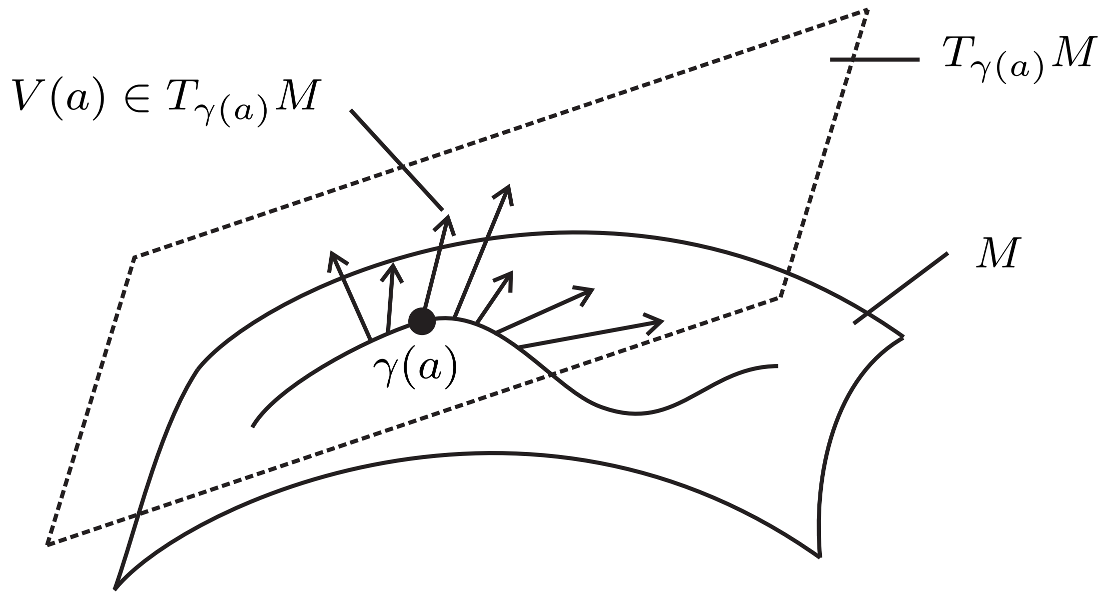
$$\text{grad}\, f(p) = \frac{\partial_x f(p)}{e^{2x+yz}}\, \partial_x|_p + \frac{\partial_y f(p)}{2 - \cos(z)}\, \partial_y|_p + \frac{\partial_z f(p)}{y^2 + 1}\, \partial_z|_p.$$

Let ∇ be the Euclidean connection. The Hessian of f at p is given by

$$\nabla^2 f(X_p, Y_p) = X^i Y^j \partial^2_{ij} f(p) \quad \text{for } X_p = X^i \partial_i|_p, Y_p = Y^j \partial_j|_p.$$

☐ **Definition [Vector fields along curves]** Let M be a smooth manifold and let $\gamma : I \subset \mathbb{R} \to M$ be a smooth curve (I is an interval).

A vector field along γ is a smooth map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for all $t \in I$.



The space of vector fields along γ is denoted by $\mathcal{T}(\gamma)$.

A vector field along γ is said to be extendible if there exists a smooth vector field $\widetilde{V}$ defined on an open set U containing $\gamma(I) \subset M$ such that $V(t) = \widetilde{V}_{\gamma(t)}$ for all $t \in I$

□ **Lemma [Covariant derivatives along curves]** A linear connection ∇ on M determines, for each smooth curve $\gamma : I \to M$, a unique operator

$$D_t : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$$

such that:

(a) [linearity over $\mathbb{R}$]

$$D_t(aV + bW) = a\, D_t V + b\, D_t W \quad \text{for } a, b \in \mathbb{R},\ V, W \in \mathcal{T}(\gamma);$$

(b) [product rule]

$$D_t(fV) = \dot{f}\, V + f\, D_t V \quad \text{for } f \in C^\infty(I),\ V \in \mathcal{T}(\gamma);$$

(c) [compatibility with ∇]

$$D_t V(a) = \nabla_{\dot{\gamma}(a)} \widetilde{V}$$

whenever $\widetilde{V}$ is an extension of V.

The symbol $D_t V$ is termed the covariant derivative of V along γ.

☐ **Example (the canonical covariant derivative in $\mathbb{R}^n$):** let $M = \mathbb{R}^n$ and ∇ denote the Euclidean connection. Let $\gamma : I \to \mathbb{R}^n$ be a smooth curve and

$$V(t) = V^i(t)\,\partial_i|_{\gamma(t)}$$

be a smooth vector field along γ. Then,

$$D_t V(t) = \dot{V}^i(t)\,\partial_i|_{\gamma(t)}.$$

☐ **Definition [Acceleration of curves, geodesics]** Let ∇ be a linear connection on M and γ a smooth curve. The acceleration of γ is the smooth vector field along γ given by $D_t\dot{\gamma}$.

A smooth curve γ is said to be a geodesic if $D_t\dot{\gamma} = 0$.

☐ **Example (the geodesics in flat $\mathbb{R}^n$):** Let $M = \mathbb{R}^n$ and ∇ denote the Euclidean connection. Let

$$\gamma : I \to \mathbb{R}^n \qquad \gamma(t) = \big(\gamma^1(t), \gamma^2(t), \ldots, \gamma^n(t)\big)$$

be a smooth curve.

The acceleration of γ is given by

$$D_t\dot{\gamma} = \ddot{\gamma}^i(t)\partial_i|_{\gamma(t)}.$$

Thus, γ is a geodesic if and only if

$$\gamma(t) = a + tb$$

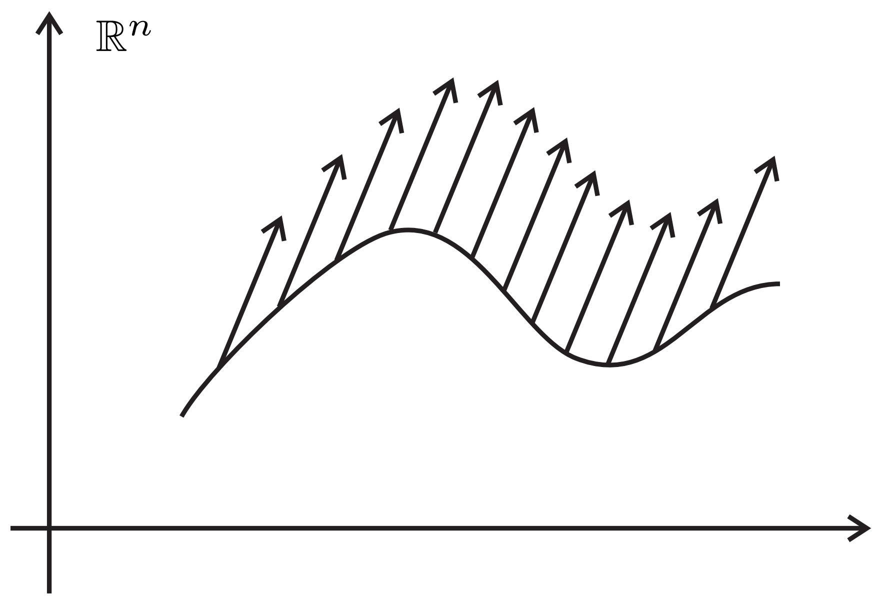
for some $a, b \in \mathbb{R}^n$.

Note that the curve $c(t) = (t^2, t^2, \ldots, t^2)$ is <u>not</u> a geodesic.

□ **Theorem [Existence and uniqueness of geodesics]** Let M be a manifold with a linear connection ∇. For any $X_p \in T_pM$ there is an $\epsilon > 0$ and a geodesic $\gamma\ :]-\epsilon,\epsilon[\to M$ such that $\gamma(0) = p$, $\dot{\gamma}(0) = X_p$.

If $\sigma\ :]-\epsilon,\epsilon[\to M$ is another geodesic such that $\sigma(0) = p$, $\dot{\sigma}(0) = X_p$, then $\sigma \equiv \gamma$

☐ **Definition [Parallel vector fields along curves]** Let M be a manifold with a linear connection ∇, and $\gamma : I \subset \mathbb{R} \to M$ a smooth curve. The smooth vector field V along γ is said to be parallel along γ if $D_t V \equiv 0$



☐ **Example (parallel vector field in $\mathbb{R}^n$):** let $M = \mathbb{R}^n$ and ∇ denote the Euclidean connection. Let $\gamma : I \to \mathbb{R}^n$ be a smooth curve and

$$V(t) = V^i(t)\partial_i|_{\gamma(t)}$$

be a smooth vector field along γ. Then V is parallel if and only if $V^i(t) = \text{const}$.

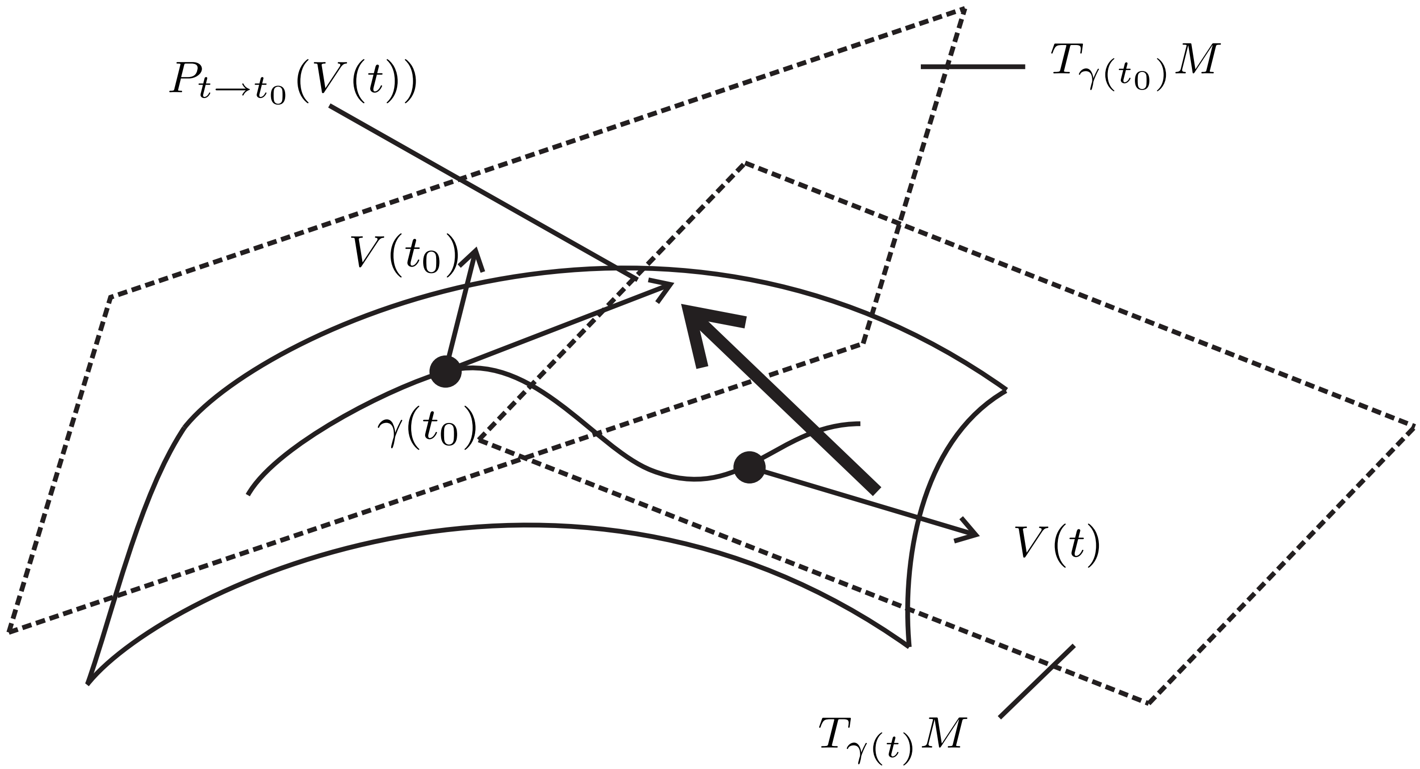
☐ **Theorem [Existence and uniqueness of parallel vector fields along curves]** Let M be a manifold with a linear connection ∇ and $\gamma : I \subset \mathbb{R} \to M$ a smooth curve. Given $t_0 \in I$ and $V_0 \in T_{\gamma(0)}M$, there is a unique parallel vector field V along γ such that $V(t_0) = V_0$

☐ **Lemma [Parallel translation]** Let M be a manifold with linear connection ∇ and $\gamma : I \subset \mathbb{R} \to M$ a smooth curve. For $s, t \in I$, let

$$P_{s \to t} : T_{\gamma(s)}M \to T_{\gamma(t)}M$$

denote the linear parallel transport map. Then, for any smooth vector field V along γ,

$$D_t V(t_0) = \lim_{t \to t_0} \frac{P_{t \to t_0} V(t) - V(t_0)}{t - t_0}$$


$P_{t \to t_0}(V(t))$
$T_{\gamma(t_0)}M$
$V(t_0)$
$\gamma(t_0)$
$V(t)$
$T_{\gamma(t)}M$