Nonlinear Signal Processing
2006-2007

Submanifolds

(Ch.5, "Introduction to Smooth Manifolds", J. Lee, Springer-Verlag)

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Lecture's key-points

- ☐ The implicit function theorem is an useful tool

- ☐ Two main approaches for creating embedded submanifolds:
  - ▷ image of smooth embedding
  - ▷ pre-image of smooth constant rank map

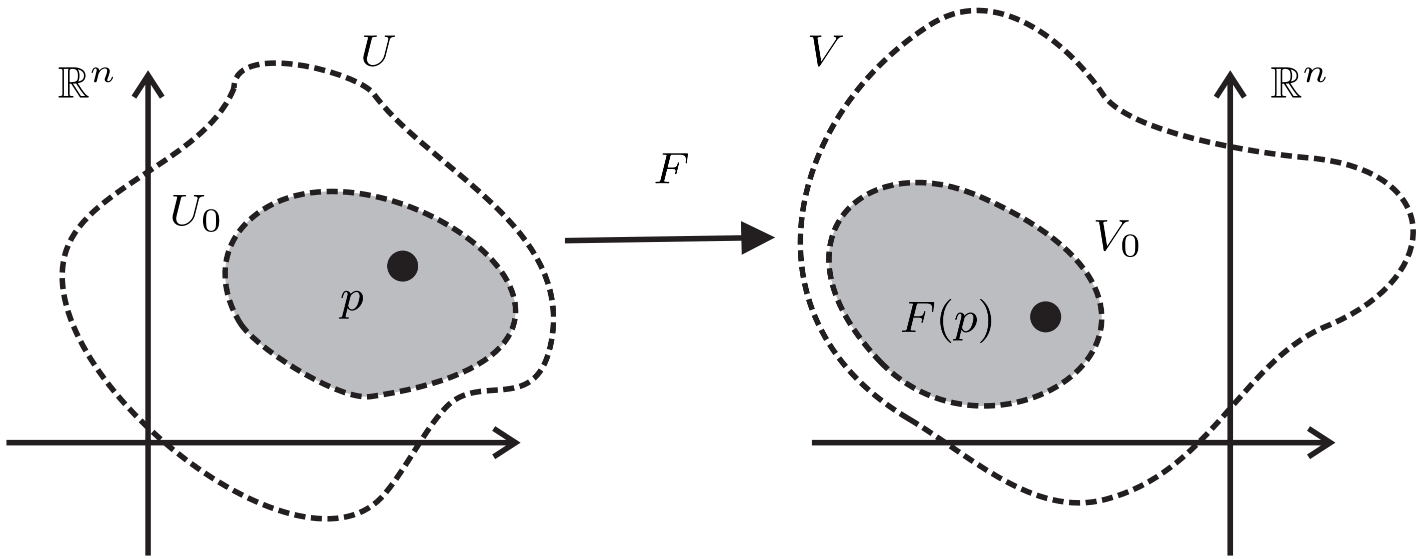
□ **Theorem [Inverse Function Theorem]** Let U and V be open subsets of $\mathbb{R}^n$ and $F : U \to V$ a smooth map. Let $p \in U$. If

$$DF(p) = \begin{bmatrix} \frac{\partial F^1}{\partial x^1}(p) & \frac{\partial F^1}{\partial x^2}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \frac{\partial F^2}{\partial x^1}(p) & \frac{\partial F^2}{\partial x^2}(p) & \cdots & \frac{\partial F^2}{\partial x^n}(p) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial F^n}{\partial x^1}(p) & \frac{\partial F^n}{\partial x^2}(p) & \cdots & \frac{\partial F^n}{\partial x^n}(p) \end{bmatrix}$$

is nonsingular, then there exist neighborhoods $U_0 \subset U$ of p and $V_0 \subset V$ of $q = F(p)$ such that $F : U_0 \to V_0$ is a diffeomorphism. Furthermore, we have

$$DF^{-1}(y_0) = (DF(x_0))^{-1}$$

where $x_0 = F^{-1}(y_0)$ for each $y_0 \in V_0$



$*$ *Intuition: the bijectivity of $DF(p)$ carries over* <u>*locally*</u> *to F*

□ **Example (Inverse function theorem as a generalization of the linear case):** let

$$F : \mathbb{R}^n \to \mathbb{R}^n \qquad F(x) = Ax$$

where $A : n \times n$ is nonsingular. By simple linear algebra, the linear map F is (globally) bijective. Note that

$$DF(p) = A$$

for any $p \in \mathbb{R}^n$

□ **Example (simple illustration):** consider the smooth map

$$F : \mathbb{R}^2 \to \mathbb{R}^2 \qquad F(x,y) = \left(x^2 + y^2, xy\right) .$$

Then,

$$DF(1,0) = \begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix}_{(x,y)=(1,0)} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

is non-singular, which means that F is a diffeomorphism near $(1,0)$. Note that F is not a bijective map: $F(1,1) = F(-1,-1)$

☐ **Theorem [Implicit Function Theorem]** Let $W \subset \mathbb{R}^n \times \mathbb{R}^k$ be an open set and $F : W \to \mathbb{R}^k$,

$$(x, y) = \left(x^1, \ldots, x^n, y^1, \ldots, y^k\right) \quad \stackrel{F}{\longmapsto} \quad F(x, y) = \left(F^1(x, y), \ldots, F^k(x, y)\right)$$

a smooth map. Let $(p, q) = \left(p^1 \ldots, p^n, q^1, \ldots, q^k\right) \in W$ with $F(p, q) = 0$ and suppose that

$$D_y F(p, q) = \begin{pmatrix} \frac{\partial F^1}{\partial y^1}(p,q) & \frac{\partial F^1}{\partial y^2}(p,q) & \cdots & \frac{\partial F^1}{\partial y^k}(p,q) \\ \frac{\partial F^2}{\partial y^1}(p,q) & \frac{\partial F^2}{\partial y^2}(p,q) & \cdots & \frac{\partial F^2}{\partial y^k}(p,q) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial F^k}{\partial y^1}(p,q) & \frac{\partial F^k}{\partial y^2}(p,q) & \cdots & \frac{\partial F^k}{\partial y^k}(p,q) \end{pmatrix}$$

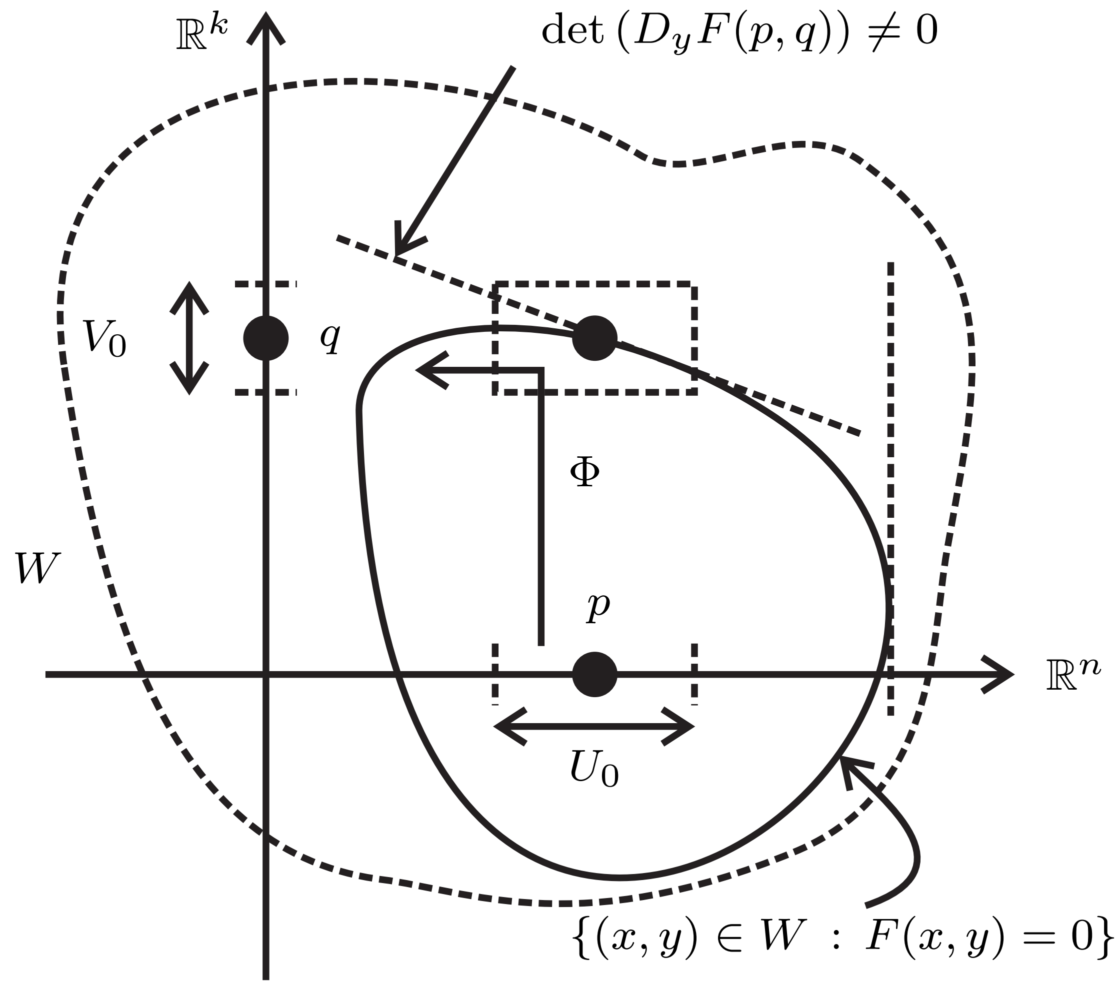
is nonsingular.

Then, there exist neighborhoods U_0 of p and V_0 of q and a smooth map $\Phi : U_0 \to V_0$ such that $U_0 \times V_0 \subset W$ and

$$(x, y) \in U_0 \times V_0,\ F(x, y) = 0 \quad \text{if and only if} \quad y = \Phi(x).$$

Furthermore, we have $D\Phi(p) = -\left(D_y F(p, q)\right)^{-1} D_x F(p, q)$, where

$$D_x F(p, q) = \begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p, q) & \frac{\partial F^1}{\partial x^2}(p, q) & \cdots & \frac{\partial F^1}{\partial x^n}(p, q) \\ \frac{\partial F^2}{\partial x^1}(p, q) & \frac{\partial F^2}{\partial x^2}(p, q) & \cdots & \frac{\partial F^2}{\partial x^n}(p, q) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial F^k}{\partial x^1}(p, q) & \frac{\partial F^k}{\partial x^2}(p, q) & \cdots & \frac{\partial F^k}{\partial x^n}(p, q) \end{pmatrix}.$$


$\det\left(D_y F(p,q)\right) \neq 0$
$\mathbb{R}^k$
V_0
q
Φ
W
p
$\mathbb{R}^n$
U_0
$\{(x,y) \in W \,:\, F(x,y) = 0\}$

☐ **Example (Implicit function theorem as a generalization of the linear case):** let

$$F : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k \qquad F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

where $A : k \times n$ and the matrix $B : k \times k$ is nonsingular.

By simple linear algebra,

$$F(x, y) = 0 \quad \text{if and only if} \quad y = -B^{-1}Ax.$$

That is,

$$F(x, y) = 0 \quad \text{if and only if} \quad y = \Phi(x),$$

where

$$\Phi : \mathbb{R}^n \to \mathbb{R}^k \qquad \Phi(x) = -B^{-1}Ax.$$

Note that

$$B = D_y F(p, q) \quad \text{and} \quad D_x F(p, q) = A$$

for all $(p, q) \in \mathbb{R}^n \times \mathbb{R}^k$

☐ **Example (simple eigenvalues are smooth):** Let $X_0 \in \mathbb{R}^{n \times n}$ be a symmetric matrix and u_0 be an unit-norm eigenvector associated with the simple eigenvalue λ_0:

$$X_0 u_0 = \lambda_0 u_0 \ \text{ and } \ u_0^\top u_0 = 1.$$

Then, there exists:

- ▷ a neighborhood $U_0 \subset \mathbb{R}^{n \times n}$ of X_0
- ▷ a neighborhood $V_0 \subset \mathbb{R}^n \times \mathbb{R}$ of (u_0, λ_0)
- ▷ a smooth map

$$\Phi \, : \, U_0 \to V_0 \qquad \Phi(X) = (u(X), \lambda(X))$$

such that $u(X_0) = u_0$, $\lambda(X_0) = \lambda_0$, and

$$X u(X) = \lambda(X) u(X), \quad u(X)^\top u(X) = 1 \qquad \text{for all } X \in U_0.$$

The derivative of the map Φ at X_0 is given by

$$D\Phi(X_0) = \begin{bmatrix} u_0^\top \otimes (\lambda_0 I_n - X_0)^+ \\ u_0^\top \otimes u_0^\top \end{bmatrix}$$

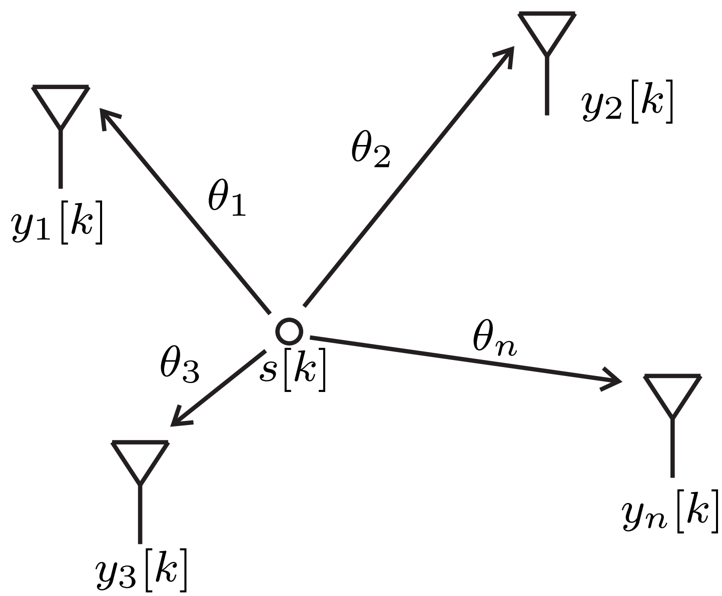
□ **Example (signal processing application - asymptotic performance analysis):**

▷ Data model: $y[k] = \theta s[k] + w[k] \qquad k = 1, 2, \ldots, K$

- $y[k] = (y_1[k], y_2[k], \ldots, y_n[k]) \in \mathbb{R}^n$ = observation vector
- $\theta \in \mathsf{S}_+^{n-1}(\mathbb{R})$ = **unknown** deterministic parameter (channel)

$$\mathsf{S}_+^{n-1}(\mathbb{R}) = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, \|x\| = 1 \text{ and } x_n > 0\}$$

- $s[k] \in \mathbb{R}$ = zero-mean, unit-power Gaussian random process
- $w[k] \in \mathbb{R}^n$ = random Gaussian process $\sim \mathcal{N}(0, \sigma^2 I_n)$



▷ The maximum-likelihood (ML) estimate of θ,

$$\widehat{\theta}_K = \underset{\theta \in \mathsf{S}_+^{n-1}(\mathbb{R})}{\text{argmax}} \; p(y_1, \ldots, y_K; \theta),$$

is easily seen to be given by the unit-norm eigenvector (with last coordinate positive) which is associated with the maximum eigenvalue of the sample covariance matrix

$$\widehat{R}_K = \frac{1}{K} \sum_{k=1}^{K} y[k] y[k]^\top.$$

In the sequel, we write $\widehat{\theta}_K = \phi(\widehat{R}_K)$, where ϕ stands for the map just described

▷ We are interested in evaluating the mean-square error (MSE) of the estimate

$$\mathsf{MSE} = \mathsf{E}\left\{ \left\| \widehat{\theta}_K - \theta \right\|^2 \right\}.$$

Since it is difficult to obtain the exact distribution of the statistic $\widehat{\theta}_K$, we resort to an asymptotic analysis ($K \to +\infty$)

▷ A fundamental tool in asymptotic analysis is the δ-method: let $x_K \in \mathbb{R}^n$ denote a sequence of random vectors satisfying

$$\sqrt{K}\,(x_K - \mu) \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

where $\xrightarrow{d}$ means convergence in distribution (as $K \to +\infty$), and let $f : \mathbb{R}^n \to \mathbb{R}^m$ denote a map which is of class C^1 near μ. Then,

$$\sqrt{K}\,(f(x_K) - f(\mu)) \xrightarrow{d} \mathcal{N}\left(0, Df(\mu)\Sigma Df(\mu)^\top\right)$$

where $Df(\mu)$ stands for the derivative of f at the point μ

▷ In our context, it can be shown (trivial application of the central limit theorem) that

$$\sqrt{K}\left(\mathsf{vec}\left(\widehat{R}_K\right) - \mathsf{vec}(R)\right) \xrightarrow{d} \mathcal{N}\left(0, \Sigma\right)$$

for a certain covariance matrix Σ (not shown here) and where

$$R = \mathsf{E}\left\{y[k]y[k]^\top\right\} = \theta\theta^\top + \sigma^2 I_n$$

denotes the correlation matrix corresponding to our data model. Furthermore, note that the maximum eigenvalue of R is $\lambda_{\mathsf{max}} = 1 + \sigma^2$

$\triangleright$ The previous example has shown that ϕ is smooth in a neighborhood of R and its derivative is given by

$$D\phi(R) = \theta^\top \otimes (\lambda_{\mathsf{max}} I_n - R)^+.$$

Thus, we have

$$\sqrt{K}\left(\widehat{\theta}_K - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, D\phi(R)\Sigma D\phi(R)^\top\right)$$

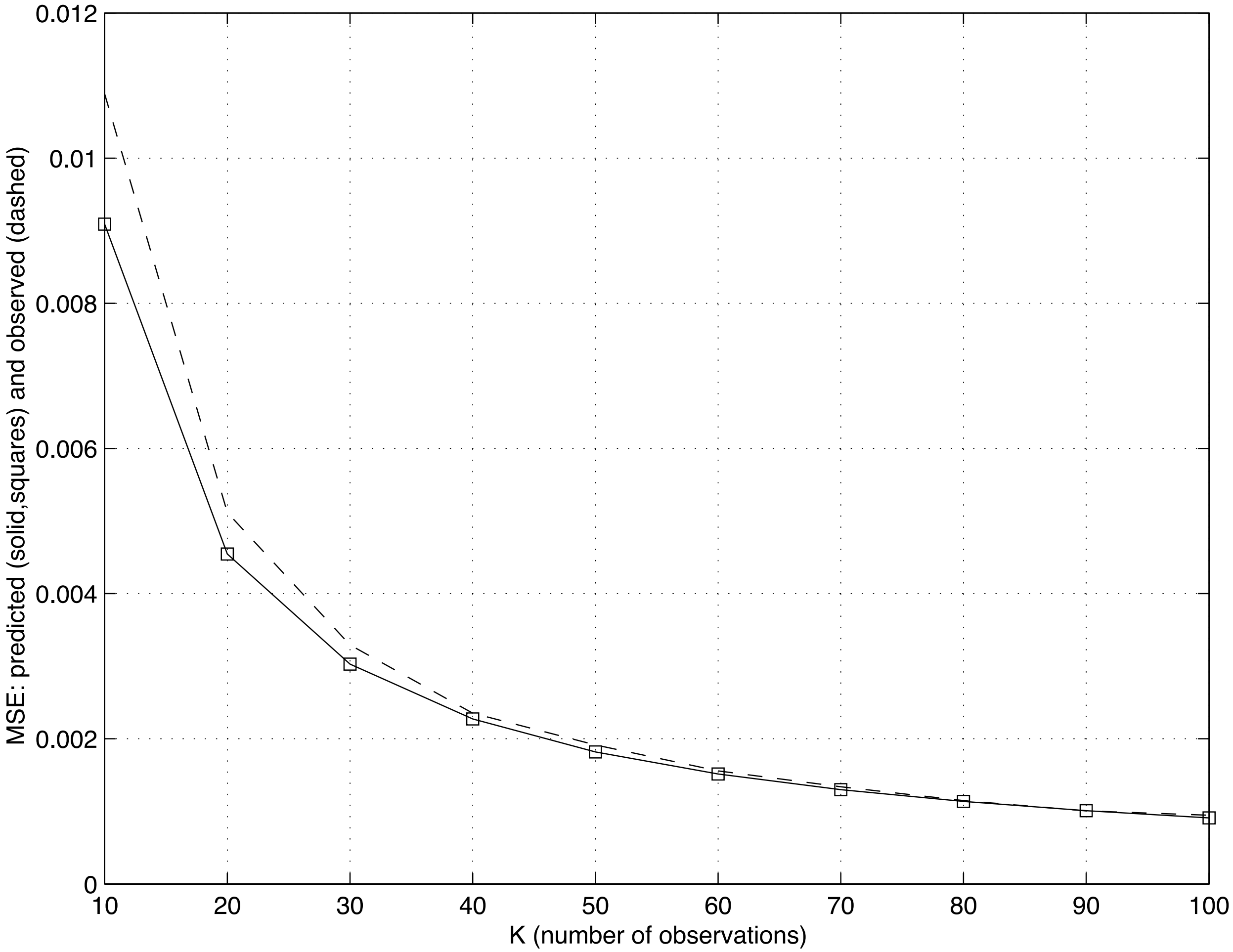
from which follows the approximation (for a given K)

$$\widehat{\theta}_K - \theta \sim \mathcal{N}\left(0, \frac{1}{K} D\phi(R)\Sigma D\phi(R)^\top\right).$$

That is,

$$\begin{aligned} \mathsf{MSE} &= \mathsf{E}\left\{\left\|\widehat{\theta}_K - \theta\right\|^2\right\} \\ &= \mathsf{tr}\left(\mathsf{E}\left\{\left(\widehat{\theta}_K - \theta\right)\left(\widehat{\theta}_K - \theta\right)^\top\right\}\right) \\ &\simeq \frac{1}{K}\mathsf{tr}\left(D\phi(R)\Sigma D\phi(R)^\top\right) \end{aligned}$$

▷ Simulation example: observation y has dimension $n = 10$, $\theta = (1/\sqrt{n}, \ldots, 1/\sqrt{n})$, the signal-to-noise ratio SNR $= \mathsf{E}\left\{\|\theta s\|^2\right\} / \mathsf{E}\left\{\|w\|^2\right\} = 1/(n\sigma^2)$ is fixed at 10 dB and the sample size K is varied between $K_{\mathsf{min}} = 10$ and $K_{\mathsf{max}} = 100$

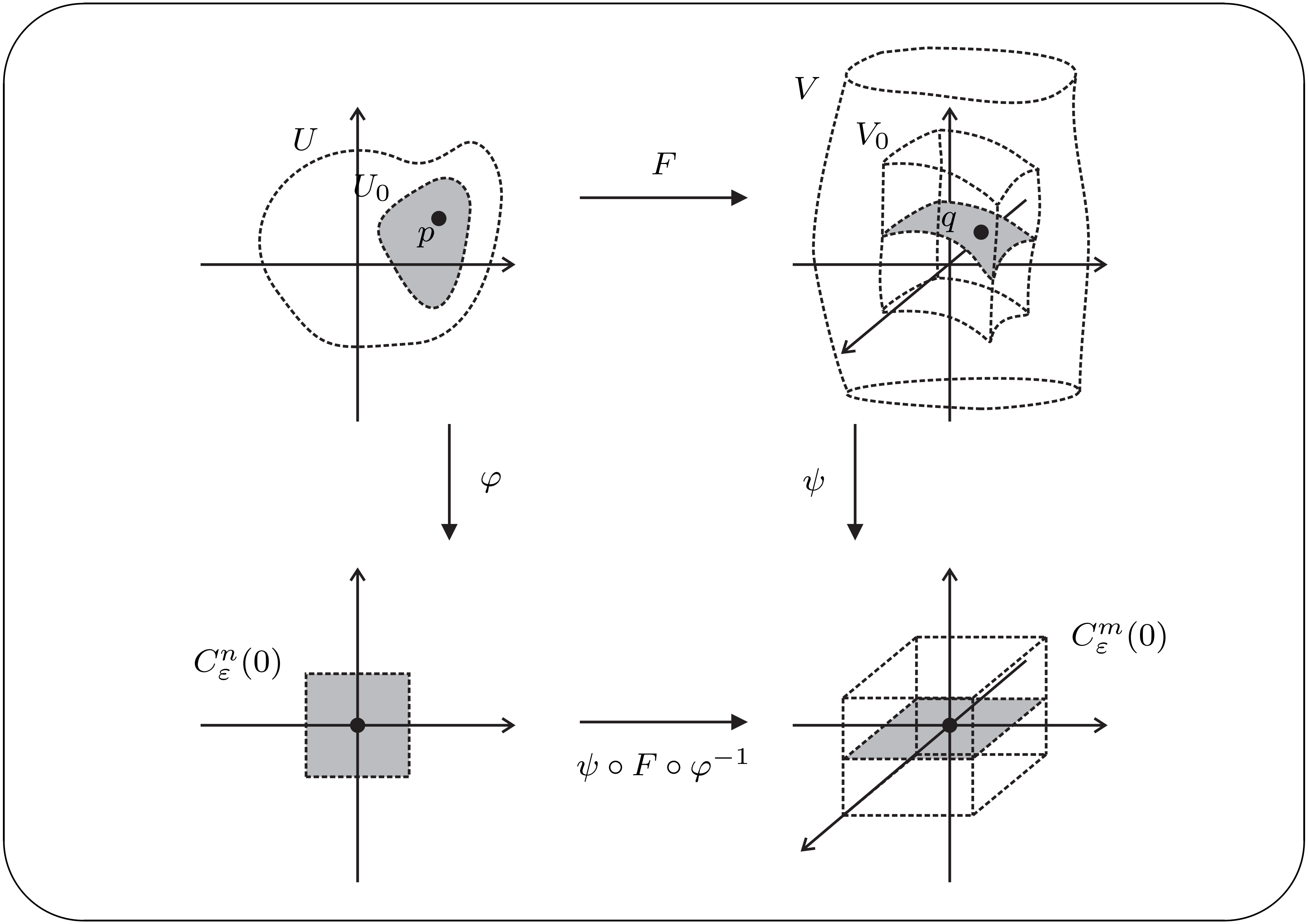


☐ **Theorem [Rank theorem]** Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets, and $F : U \to V$ a smooth map with constant rank k, that is, $\operatorname{rank}(DF(x)) = k$ for each $x \in U$. Let $p \in U$. Then, there exist neighborhoods $U_0 \subset U$ of p and $V_0 \subset V$ of $q = F(p)$ and diffeomorphisms $\varphi : U_0 \to \widehat{U}_0$ and $\psi : V_0 \to \widehat{V}_0$ such that

$$\psi \circ F \circ \varphi^{-1}\left(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n\right) = \left(x^1, \ldots, x^k, 0, \ldots, 0\right).$$

The neighborhoods U_0 and V_0 can be chosen such that: **(i)** $\widehat{U}_0 = C^n_\varepsilon(0)$ and $\widehat{V}_0 = C^m_\varepsilon(0)$ or **(ii)** $\widehat{U}_0 = B^n_\varepsilon(0)$ and $\widehat{V}_0 = B^m_\varepsilon(0)$, for any chosen $\varepsilon > 0$

$*$ *Intuition: looks like a nonlinear generalization of the SVD for linear maps*



U
U_0
p
F
V
V_0
q
φ
ψ
$C^n_\varepsilon(0)$
$\psi \circ F \circ \varphi^{-1}$
$C^m_\varepsilon(0)$

☐ **Definition [Rank of a smooth map, immersions, submersions]** Let $F : M \to N$ be a smooth map between smooth manifolds. The rank of F at $p \in M$ is the dimension of the linear subspace $\operatorname{Im} F_*(T_pM) \subset T_{F(p)}N$. Equivalently, it is the rank of the Jacobian matrix $\operatorname{rank} D\widehat{F}(\varphi(p))$ in any smooth chart.

We say that F has constant rank k if the rank of F at any $p \in M$ is k.

▷ the smooth map $F : M \to N$ is called an immersion if F_* is injective at every point. Equivalently, if $\operatorname{rank} F = \dim M$ at every point

▷ the smooth map $F : M \to N$ is called a submersion if F_* is surjective at every point. Equivalently, if $\operatorname{rank} F = \dim N$ at every point

□ **Example (an immersion of the unit-sphere):** consider the map

$$F : \mathsf{S}^{n-1}(\mathbb{R}) \to \mathbb{R}^{n \times n} \qquad F(u) = uu^\top.$$

We already know that F is smooth. The map F is also an immersion

□ **Example (a submersion onto the unit-sphere):** consider the map

$$F : \mathbb{R}^n - \{0\} \to \mathsf{S}^{n-1}(\mathbb{R}) \qquad F(x) = \frac{x}{\|x\|}.$$

We already know that F is smooth. The map F is also a submersion

□ **Example (product manifolds):** let M and N be smooth manifolds.

For fixed $q \in N$, the inclusion map

$$\iota_q : M \to M \times N \qquad \iota(p) = (p, q)$$

is an immersion. The projection map

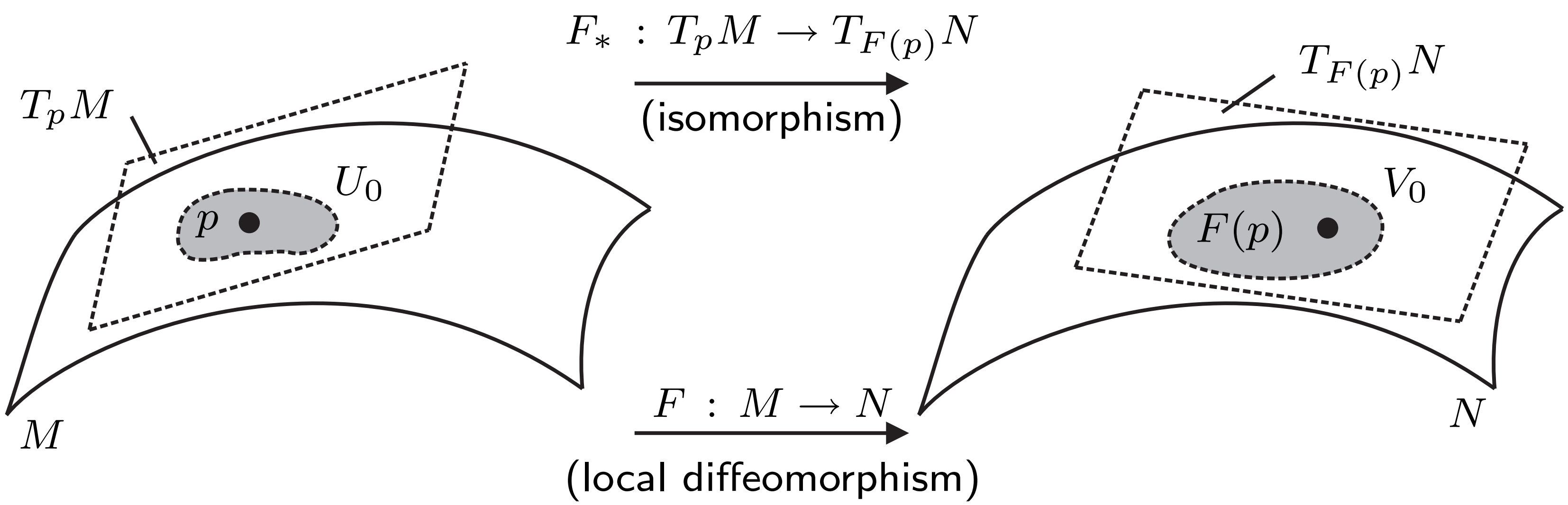
$$\pi_M : M \times N \to M \qquad \pi_M(p, q) = p$$

is a submersion

☐ **Lemma [Composition of immersions and submersions]** The composition of immersions is an immersion. The composition of submersions is a submersion

☐ **Theorem [Inverse function theorem for manifolds]** Let $F : M \to N$ be a smooth map between manifolds. Let $p \in M$ and suppose $F_* : T_pM \to T_{F(p)}N$ is an isomorphism (equivalently, a bijective linear map). Then there exist neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism

$*$ *Intuition: the bijectivity of F_* transpires locally to F*



▷ *Remark that the inverse map $F^{-1} : V_0 \to U_0$ is smooth*

☐ **Example (Cholesky decomposition is a diffeomorphism):**

▷ The Cholesky decomposition asserts that for any $P \in \mathsf{P}(n, \mathbb{R})$ there is an unique $L \in \mathsf{L}^+(n, \mathbb{R})$ such that

$$P = LL^\top.$$

Thus, we can define a map

$$\text{Cholesky} : \mathsf{P}(n, \mathbb{R}) \to \mathsf{L}^+(n, \mathbb{R})$$

which, given a positive-definite P, computes its Cholesky factor L s.t. $P = LL^\top$.

The purpose of this example is to show that the map Cholesky is smooth

▷ We already know that the map

$$F : \mathsf{L}^+(n, \mathbb{R}) \to \mathsf{P}(n, \mathbb{R}) \qquad F(L) = LL^\top$$

is bijective (linear algebra) and smooth. Remark that the map Cholesky is the inverse map of F

▷ Also, by exploiting the isomorphisms

$$T_{L_0}\mathsf{L}^+(n,\mathbb{R}) \simeq \mathsf{L}(n,\mathbb{R}) \quad \text{and} \quad T_{F(L_0)}\mathsf{P}(n,\mathbb{R}) \simeq \mathsf{S}(n,\mathbb{R}),$$

we have computed a representation of the push-forward map
$F_* : T_{L_0}\mathsf{L}^+(n,\mathbb{R}) \to T_{F(L_0)}\mathsf{P}(n,\mathbb{R})$ as

$$F_* : \mathsf{L}(n,\mathbb{R}) \to \mathsf{S}(n,\mathbb{R}) \qquad F_*(\Delta) = \Delta L_0^\top + L_0\Delta^\top$$

▷ If we show that F_* is an isomorphism, we can use the last theorem to conclude that Cholesky $= F^{-1}$ is smooth (because it is smooth on a neighborhood of any given point $P_0 = F(L_0) \in \mathsf{P}(n,\mathbb{R})$)

▷ To prove that the linear map F_* is bijective it suffices to prove that F_* is injective because $\dim \mathsf{L}(n,\mathbb{R}) = \dim \mathsf{S}(n,\mathbb{R})$

▷ To prove that F_* is injective, we must show that $\operatorname{Ker} F_* = \{0\}$. So, let $\Delta \in \mathsf{L}(n, \mathbb{R})$ satisfy $F_*(\Delta) = 0$, that is,

$$\Delta L_0^\top + L_0 \Delta^\top = 0.$$

Pre-multiplying by L_0^{-1} and post-multiplying by $\left(L_0^\top\right)^{-1}$ both sides of the equation yields

$$\left(L_0^{-1}\Delta\right) + \left(L_0^{-1}\Delta\right)^\top = 0.$$

Note that L_0^{-1} is a lower-triangular matrix and $\Psi = L_0^{-1}\Delta$ also (product of two lower-triangular matrices). But,

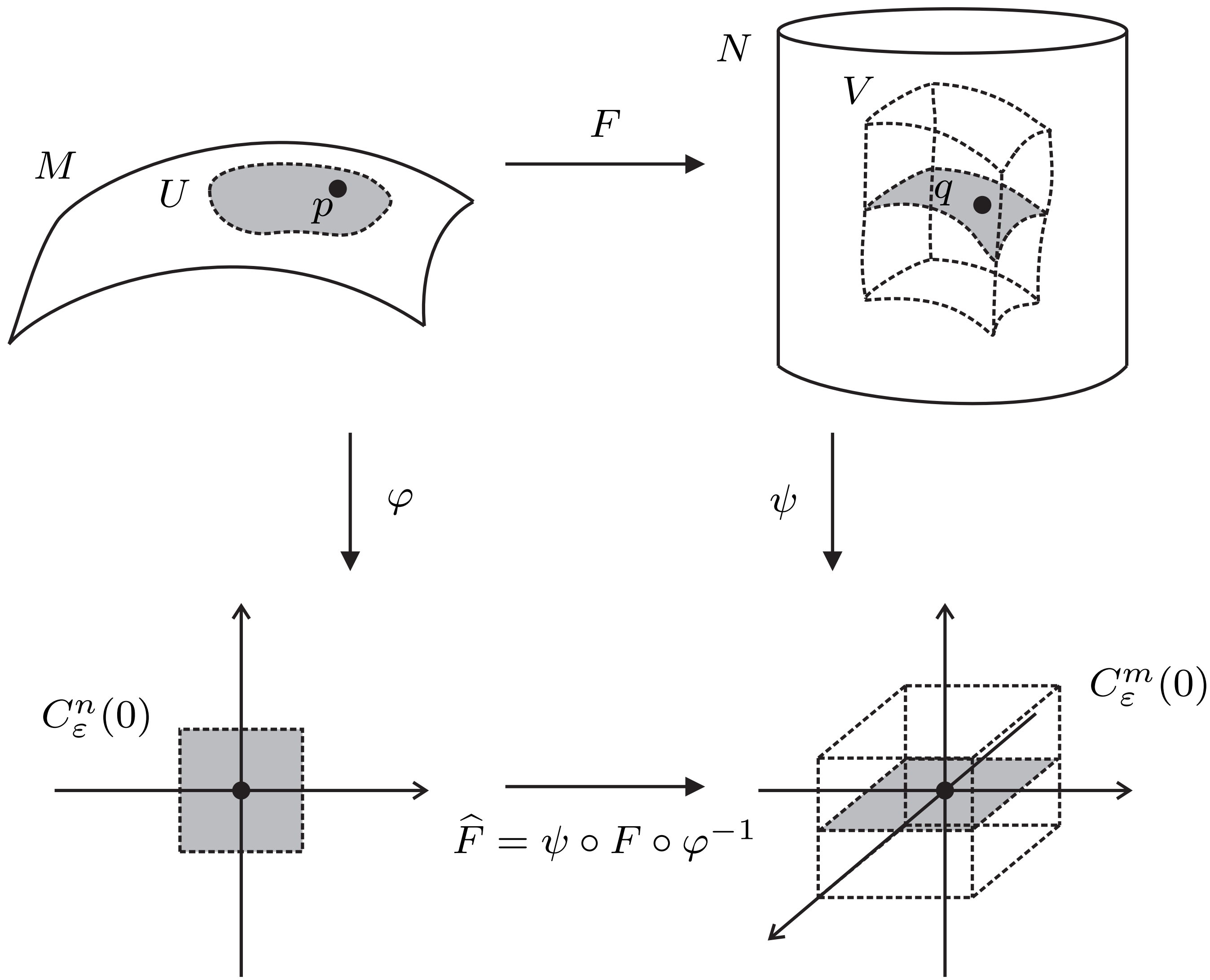
$$\Psi + \Psi^\top = 0 \quad \text{and} \quad \Psi : \text{lower-triangular} \quad \Rightarrow \quad \Psi = 0.$$

As a consequence, $\Delta = L_0 \Psi = 0$.

We conclude that the map Cholesky is smooth. In fact, it is a diffeomorphism (because its inverse F is also smooth)

☐ **Theorem [Rank theorem for manifolds]** Suppose that the smooth map $F : M \to N$ has constant rank k, with dim $M = m$ and dim $N = n$. Then, for any given $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that the coordinate representation $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is given by

$$\widehat{F}\left(x^1, x^2, \ldots, x^k, x^{k+1}, \ldots, x^m\right) = \Bigg(x^1, x^2, \ldots, x^k, \underbrace{0, \ldots, 0}_{n-k \text{ zeros}} \Bigg).$$


M
U
p
F
N
V
q
φ
ψ
$C^n_\varepsilon(0)$
$\widehat{F} = \psi \circ F \circ \varphi^{-1}$
$C^m_\varepsilon(0)$

☐ **Theorem [Constant rank and immersions, submersions and diffeomorphisms]** Let $F : M \to N$ be a smooth map of constant rank.

(a) If F is injective, then it is an immersion

(b) If F is surjective, then it is a submersion

(c) If F is bijective, then it is a diffeomorphism

☐ **Example (an immersion of the unit-circle):** consider the map

$$F : \mathsf{S}^1(\mathbb{R}) \to \mathbb{R}^{2\times 2} \qquad F(u) = \begin{bmatrix} u & Ju \end{bmatrix}$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

▷ The map F is smooth (why?). The goal of this example is to show that F is an immersion, without computing in coordinates

▷ Note that F is injective. If we prove that F has constant rank, we are done (see last theorem). Let $p, q \in \mathsf{S}^1(\mathbb{R})$. We must show that the two linear maps

$$F_{*p} \,:\, T_p\mathsf{S}^1(\mathbb{R}) \to T_{F(p)}\mathbb{R}^{2\times 2} \quad \text{and} \quad F_{*q} \,:\, T_q\mathsf{S}^1(\mathbb{R}) \to T_{F(q)}\mathbb{R}^{2\times 2}$$

have the same rank

▷ The trick consists in noting that, for any fixed rotation $Q \in \mathsf{SO}(2)$, we have $F \circ L_Q = \widehat{L}_Q \circ F$ or, equivalently, the commutative diagram:

$$\begin{tikzcd} \mathsf{S}^1(\mathbb{R}) \arrow[r, "L_Q"] \arrow[d, "F"'] & \mathsf{S}^1(\mathbb{R}) \arrow[d, "F"] \\ \mathbb{R}^{2\times 2} \arrow[r, "\widehat{L}_Q"'] & \mathbb{R}^{2\times 2} \end{tikzcd}$$

where $L_Q \,:\, \mathsf{S}^1(\mathbb{R}) \to \mathsf{S}^1(\mathbb{R}), L_Q(u) = Qu$ and $\widehat{L}_Q \,:\, \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}, \widehat{L}_Q(X) = QX$

▷ Note that both L_Q and $\widehat{L}_Q$ are smooth (why?). In fact they are diffeomorphisms because their inverse maps correspond to $L_{Q^\top}$ and $\widehat{L}_{Q^\top}$, respectively, which are smooth

▷ Now, choose Q such that $L_Q(p) = q$. The previous diagram induces the next one, expressed in terms of push-forwards:

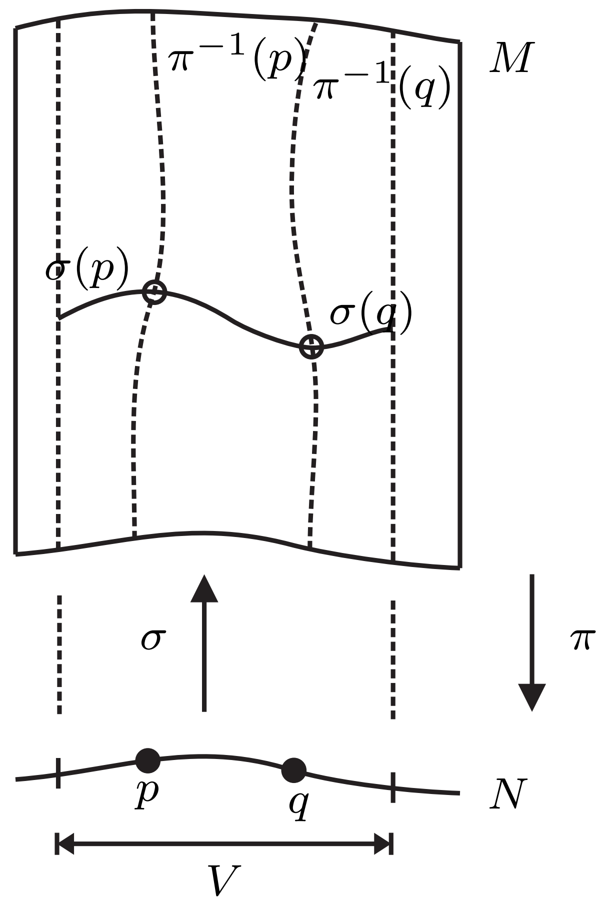
$$\begin{tikzcd}
T_p\mathsf{S}^1(\mathbb{R}) \arrow[r, "L_{Q*}"] \arrow[d, "F_{*p}"'] & T_q\mathsf{S}^1(\mathbb{R}) \arrow[d, "F_{*q}"] \\
T_{F(p)}\mathbb{R}^{2\times 2} \arrow[r, "\widehat{L}_{Q*}"'] & T_{F(q)}\mathbb{R}^{2\times 2}
\end{tikzcd}$$

Equivalently: $\widehat{L}_{Q*} \circ F_{*p} = F_{*q} \circ L_{Q*}$. Since $\widehat{L}_{Q*}$ and L_{Q*} are isomorphisms,

$$\operatorname{rank}\left(\widehat{L}_{Q*} \circ F_{*p}\right) = \operatorname{rank}\left(F_{*p}\right) \quad \text{and} \quad \operatorname{rank}\left(F_{*q} \circ L_{Q*}\right) = \operatorname{rank}\left(F_{*q}\right).$$

The conclusion is $\operatorname{rank}(F_{*p}) = \operatorname{rank}(F_{*q})$

□ **Definition [Local section]** Let $\pi : M \to N$ be a smooth map between smooth manifolds. A smooth local section of π is a pair (V, σ) where $V \subset N$ is open and $\sigma : V \to M$ is a smooth map satisfying $\pi \circ \sigma = \mathrm{id}_V$.



$*$ *Intuition: σ is a smooth choice of a representative in each fiber of π*

☐ **Lemma [Properties of submersions: part I]** Let $\pi : M \to N$ be a smooth map between smooth manifolds. Suppose π is a submersion. Then, π is an open map. Moreover, for every $p \in M$, there exists a local section (V, σ) of π such that $p \in \sigma(V)$

☐ **Lemma [Properties of submersions: part II]** Let M, N, P be smooth manifolds and $\pi : M \to N$ be a surjective submersion. Then, a map $F : N \to P$ is smooth if and only if $\widehat{F} = F \circ \pi$ is smooth

$$
\begin{tikzcd}
M \arrow[d, "\pi"'] \arrow[dr, "\widehat{F}"] & \\
N \arrow[r, "F"'] & P
\end{tikzcd}
$$

$*$ *Intuition: smoothness of the "hard" map F can be investigated via the easier $\widehat{F}$*

☐ **Example (an immersion of the unit-sphere):** consider the map

$$F : \mathsf{S}^{n-1}(\mathbb{R}) \to \mathbb{R}^{n \times n} \qquad F(u) = uu^\top.$$

We already know that F is smooth. Here is an alternative proof of smoothness of F:

▷

$$\pi : \mathbb{R}^n - \{0\} \to \mathsf{S}^{n-1}(\mathbb{R}) \qquad \pi(x) = \frac{x}{\|x\|}$$

is a surjective submersion

▷

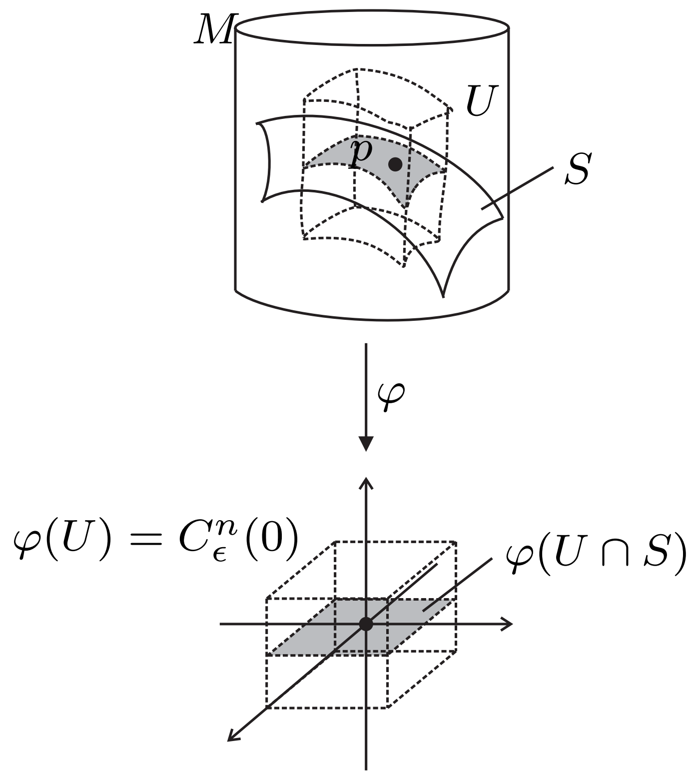
$$\widehat{F} : \mathbb{R}^n - \{0\} \to \mathbb{R}^{n \times n} \qquad \widehat{F}(x) = \frac{xx^\top}{\|x\|^2}$$

is <u>clearly</u> smooth

▷ $\widehat{F} = F \circ \pi$

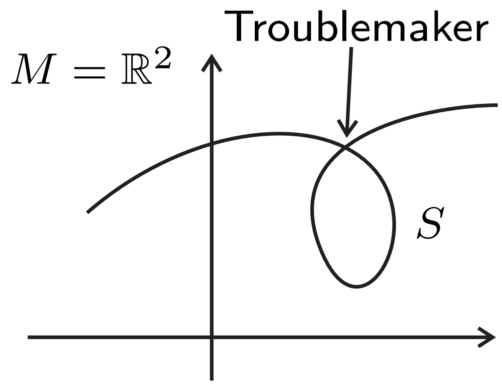
□ **Definition [Embedded submanifold]** Let M be an n-dimensional smooth manifold. A subset $S \subset M$ is called an embedded k-submanifold of M if, for each point $p \in S$, there is a smooth chart (U, φ) centered at p with $\varphi(U) = C^n_\epsilon(0)$ and

$$\varphi(U \cap S) = \{(x^1, x^2, \ldots, x^k, x^{k+1}, \ldots, x^n) : x^{k+1} = x^{k+2} = \cdots = x^n = 0\}.$$



$*$ *Intuition: the subset $S \subset M$ can be flattened (locally)*

□ **Example:**



S is not an embedded submanifold of $\mathbb{R}^2$

□ **Example (linear subspaces):** if $S \subset \mathbb{R}^n$ is a linear subspace with (linear) dimension k then, S is an embedded k-submanifold of $\mathbb{R}^n$.

▷ the linear subspace of symmetric matrices

$$\mathsf{S}(n, \mathbb{R}) = \{X \in \mathbb{R}^{n \times n} \, : \, X = X^\top\}$$

is an embedded $n(n+1)/2$-submanifold of $\mathbb{R}^{n \times n}$

▷ same holds for the linear subspace of skew-symmetric matrices

$$\mathsf{K}(n, \mathbb{R}) = \{X \in \mathbb{R}^{n \times n} \, : \, X = -X^\top\}$$

☐ **Example (unit-sphere):** $\mathsf{S}^{n-1}(\mathbb{R}) = \{x \in \mathbb{R}^n \,:\, \|x\| = 1\}$ is an embedded $(n-1)$-submanifold of $\mathbb{R}^n$

☐ **Lemma [Embedding submanifolds are local constructions]** Let M be a smooth manifold. The subset $S \subset M$ is a embedded submanifold of M if and only if each $p \in S$ has a neighborhood $U \subset M$ such that $S \cap U$ is an embedded submanifold of U

☐ **Lemma [Open subsets are embedded submanifolds]** Let $U \subset M$ be an open subset of the n-dimensional smooth manifold M. Then, U is an embedded n-submanifold of M

☐ **Example (positive definite matrices):** the set of positive definite matrices

$$\mathsf{P}(n, \mathbb{R}) = \{X \in \mathsf{S}(n, \mathbb{R}) \,:\, X \succ 0\}$$

is an embedded $n(n+1)/2$-submanifold of $\mathsf{S}(n, \mathbb{R})$

☐ **Definition [Embedding]** A smooth map $F : N \to M$ between smooth manifolds is said to be an embedding if it is an immersion and a topological embedding (a homeomorphism of N onto its image $\widetilde{N} = F(N)$, viewed as a subspace of M).

☐ **Lemma [Useful criterion for detecting embeddings]** Let the smooth map $F : M \to N$ be an injective immersion. If M is compact, F is an embedding.

☐ **Example (an embedding of the unit-circle):** consider the map

$$F : \mathsf{S}^1(\mathbb{R}) \to \mathbb{R}^{2\times 2} \qquad F(u) = \begin{bmatrix} u & Ju \end{bmatrix}$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We already know that F is a smooth immersion. Since $\mathsf{S}^1(\mathbb{R})$ is compact, F is an embedding

□ **Theorem [Embedded submanifolds are smooth manifolds]** Let the subset $S \subset M$ be an embedded k-dimensional submanifold of M, where $\dim M = n$.

Then, as a subspace of M, S is a topological manifold of dimension k and it has an unique smooth structure such that the inclusion map $\iota : S \to M$ is a smooth embedding.

With this smooth structure on S, let (U, φ) be a smooth chart in M with $\varphi(U) = C^n_\epsilon(0)$ and

$$\varphi(U \cap S) = \{(x^1, x^2, \ldots, x^k, x^{k+1}, \ldots, x^n) : x^{k+1} = x^{k+2} = \cdots = x^n = 0\}.$$

Then, $(S \cap U, \widehat{\pi} \circ \varphi)$ is a smooth chart in S, where

$$\widehat{\pi}(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n) = (x^1, \ldots, x^k).$$

□ **Theorem [Smooth embeddings provide embedded submanifolds]** The image of a smooth embedding is an embedded submanifold

☐ **Example ($\mathsf{SO}(2)$ is an embedded submanifold of $\mathbb{R}^{2\times 2}$):** the subset

$$\mathsf{SO}(2) = \left\{ \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \right\}$$

is an embedded submanifold of $\mathbb{R}^{2\times 2}$ because $\mathsf{SO}(2) = F(\mathsf{S}^1(\mathbb{R}))$ where F is the embedding

$$F : \mathsf{S}^1(\mathbb{R}) \to \mathbb{R}^{2\times 2} \qquad F(u) = \begin{bmatrix} u & Ju \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

☐ **Lemma [Composition of embeddings]** The composition of embeddings is an embedding.

☐ **Theorem [Constant-rank level set theorem]** Let $F : M \to N$ be a smooth map with constant rank k. Then, for each $c \in \mathrm{Im}\, F$, the level set

$$F^{-1}(c) = \{p \in M \,:\, F(p) = c\}$$

is a closed, embedded submanifold of dimension $\dim M - k$

☐ **Example (Stiefel):** let

$$\mathsf{O}(n,m) = \{X \in \mathbb{R}^{n\times m} \;:\; X^\top X = I_m\}$$

be the set of $n \times m$ orthonormal frames in $\mathbb{R}^n$.

Then $\mathsf{O}(n,m)$ is an embedded submanifold of $\mathbb{R}^{n\times m}$ and

$$\dim \mathsf{O}(n,m) = nm - \frac{m(m+1)}{2}.$$

The manifold $\mathsf{O}(n,m)$ is known as the Stiefel manifold

☐ **Example (special orthogonal group $\mathsf{SO}(n)$):** since $\mathsf{SO}(n)$ is an open subset of the smooth manifold $\mathsf{O}(n)$, it is an embedded submanifold of $\mathsf{O}(n)$ and

$$\dim \mathsf{SO}(n) = \dim \mathsf{O}(n) = \frac{n(n-1)}{2}.$$

Since $\mathsf{O}(n)$ is embedded in $\mathbb{R}^{n\times}$, $\mathsf{SO}(n)$ is an embedded submanifold of $\mathbb{R}^{n\times n}$

□ **Example (matrices with fixed rank):** let

$$\mathsf{Rank}_{=k}(n, m, \mathbb{R}) = \{X \in \mathbb{R}^{n \times m} \ : \ \mathsf{rank}\, X = k\}$$

be the set of $n \times m$ matrices with rank k.

Then $\mathsf{Rank}_{=k}(n, m, \mathbb{R})$ is an embedded submanifold of $\mathbb{R}^{n \times m}$ and

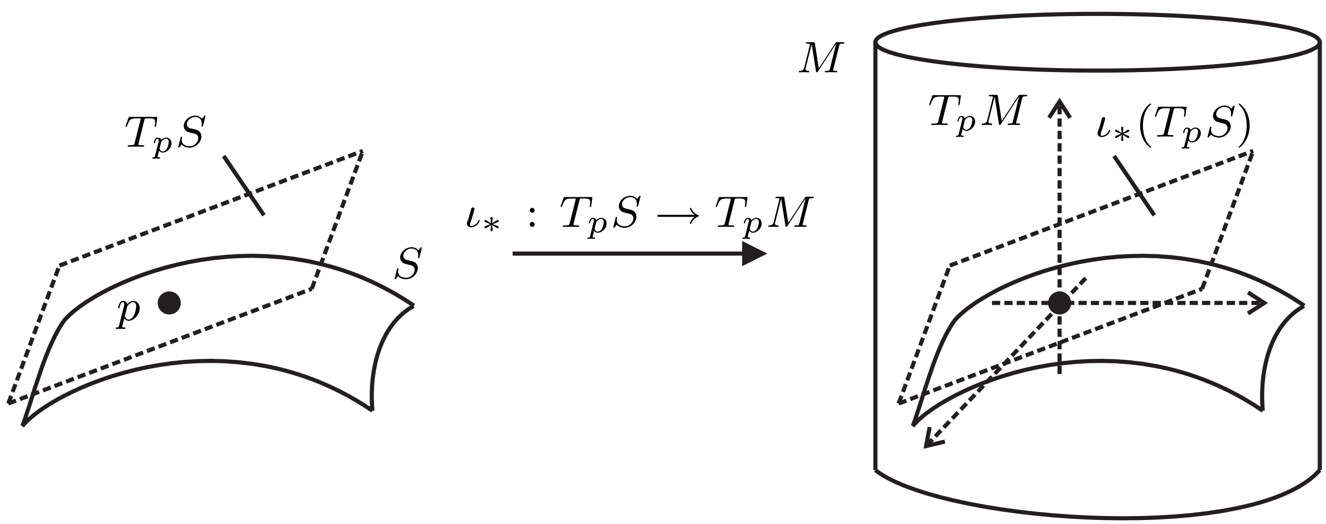
$$\mathsf{dim}\, \mathsf{Rank}_{=k}(n, m, \mathbb{R}) = (m + n - k)k$$

$\square$ **Lemma [Identifications for tangent spaces]** Let $F : M \to N$ be a smooth map with constant rank k. Let $c \in \operatorname{Im} F$. Thus, the level set $S = F^{-1}(c)$ is an embedded submanifold of M and the dimension of S is $d = \dim M - k$.

Since the inclusion $\iota : S \to M$ is an embedding (in particular, an immersion), it follows that, for any $p \in S$, the push-forward $\iota_* : T_pS \to T_PM$ is injective and

$$\iota_*(T_pS) \subset T_pM$$

is a d-dimensional subspace of T_pM. We usually make the identification $T_pS \simeq \iota_*(T_pS)$. Further, in our case, $\iota_*\,(T_pS) = \operatorname{Ker} F_{*p}$. Thus, $T_pS \simeq \operatorname{Ker} F_{*p}$.



$\triangleright$ *T_pS after being push-forwarded by ι_* appears as a subspace of T_pM*

☐ **Example (unit-sphere):**

▷ $\mathsf{S}^{n-1}(\mathbb{R})$ is a level set of the constant-rank map

$$F : \mathbb{R}^n - \{0\} \to \mathbb{R} \qquad F(x) = x^\top x$$

▷ thus, we have

$$T_p \mathsf{S}^{n-1}(\mathbb{R}) \simeq \mathsf{Ker}\, F_{*p}$$

for any $p \in \mathsf{S}^{n-1}(\mathbb{R})$

▷ using the identifications

$$T_p \mathbb{R}^n - \{0\} \simeq \mathbb{R}^n \quad \text{and} \quad T_{F(p)} \mathbb{R} \simeq \mathbb{R}$$

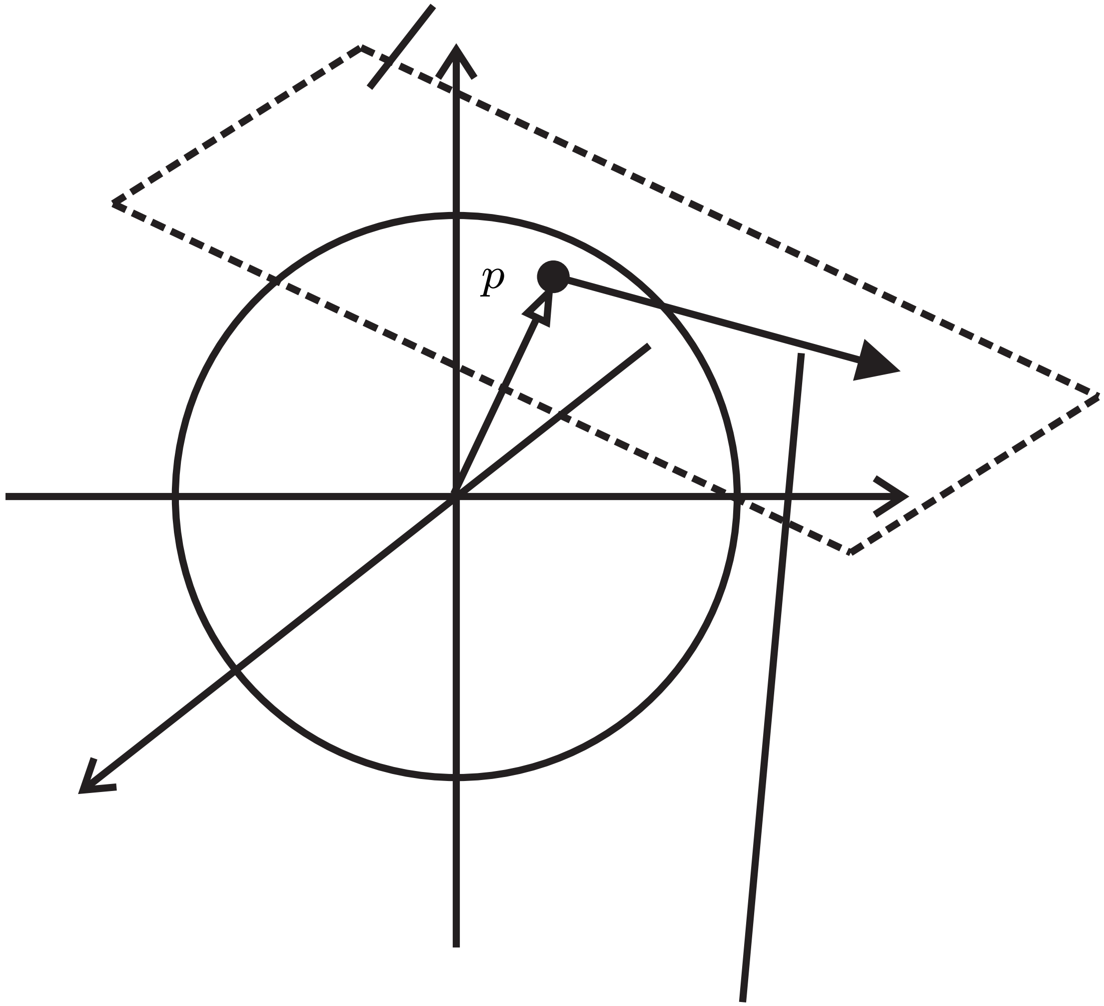
the push-forward $F_{*p} : T_p \mathbb{R}^n - \{0\} \to T_{F(p)} \mathbb{R}$ is represented by the linear map

$$F_{*p} : \mathbb{R}^n \to \mathbb{R} \qquad F_{*p}(\delta) = \delta^\top p + p^\top \delta$$

▷ hence,

$$T_p \mathsf{S}^{n-1}(\mathbb{R}) \simeq \mathsf{Ker}\, F_{*p} = \{\delta \in \mathbb{R}^n : p^\top \delta = 0\}$$

$$T_p\mathsf{S}^2(\mathbb{R}) \simeq \iota_*\left(T_p\mathsf{S}^2(\mathbb{R})\right) \subset T_pM$$



$$\delta = (\delta^1, \ldots, \delta^n) \in \mathbb{R}^n \simeq \delta^1 \frac{\partial}{\partial x^1}\bigg|_p + \cdots + \delta^n \frac{\partial}{\partial x^n}\bigg|_p \in T_pM$$

□ **Example (orthogonal group):**

▷ $\mathsf{O}(n)$ is a level set of the constant-rank map

$$F : \mathsf{GL}(n,\mathbb{R}) \to \mathbb{R}^{n\times n} \qquad F(X) = X^\top X$$

▷ thus, we have

$$T_Q\mathsf{O}(n) \simeq \mathsf{Ker}\, F_{*Q}$$

for any $Q \in \mathsf{O}(n)$

▷ using the identifications

$$T_Q\mathsf{GL}(n,\mathbb{R}) \simeq \mathbb{R}^{n\times n} \quad \text{and} \quad T_{F(Q)}\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n\times n}$$

the push-forward $F_{*Q} : T_Q\mathsf{GL}(n,\mathbb{R}) \to T_{F(Q)}\mathbb{R}^{n\times n}$ is represented by the linear map

$$F_{*Q} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n} \qquad F_{*Q}(\Delta) = \Delta^\top Q + Q^\top \Delta$$

▷ hence,

$$T_Q\mathsf{O}(n) \simeq \mathsf{Ker}\, F_{*Q} = \{QK : K \in \mathsf{K}(n,\mathbb{R})\} = Q\mathsf{K}(n,\mathbb{R})$$

☐ **Proposition [Restricting the domain and/or range of smooth maps]** Let $F : M \to N$ be a smooth map.

(a) If A is an embedded submanifold of M, then the map

$$F|_A : A \to N \qquad F|_A(p) = F(p)$$

is smooth.

$$\begin{tikzcd}
M \arrow[r, "F"] & N \\
A \arrow[u, "\iota_A"] \arrow[ur, "F|_A"'] &
\end{tikzcd}$$

(b) If B is an embedded submanifold of N and $F(M) \subset B$, then the map

$$F|^B : M \to B \qquad F|^B(p) = F(p)$$

is smooth.

$$\begin{tikzcd}
M \arrow[d, "F|^B"'] \arrow[dr, "F"] & \\
B \arrow[r, "\iota_B"'] & N
\end{tikzcd}$$

□ **Example (an immersion of the unit-sphere):** consider the map

$$F : \mathsf{S}^{n-1}(\mathbb{R}) \to \mathbb{R}^{n\times n} \qquad F(u) = uu^\top.$$

The map F is smooth because

Step 1:

$$\widehat{F} : \mathbb{R}^n \to \mathbb{R}^{n\times n} \qquad \widehat{F}(x) = xx^\top$$

is <u>clearly</u> smooth

Step 2: $\mathsf{S}^{n-1}(\mathbb{R})$ is an embedded submanifold of $\mathbb{R}^n$

Step 3: $F = \widehat{F}|_{\mathsf{S}^{n-1}(\mathbb{R})}$

☐ **Example (a submersion onto the unit-sphere):** consider the map

$$F : \mathbb{R}^n - \{0\} \to \mathsf{S}^{n-1}(\mathbb{R}) \qquad F(x) = \frac{x}{\|x\|}.$$

The map F is smooth because

Step 1:

$$\widehat{F} : \mathbb{R}^n - \{0\} \to \mathbb{R}^n \qquad \widehat{F}(x) = \frac{x}{\|x\|}$$

is clearly smooth

Step 2: $\mathsf{S}^{n-1}(\mathbb{R})$ is an embedded submanifold of $\mathbb{R}^n$

Step 3: $F = \widehat{F}|^{\mathsf{S}^{n-1}(\mathbb{R})}$

□ **Example (concatenating the techniques):** consider the map

$$F \,:\, \mathsf{O}(n) \to \mathsf{S}^{n-1}(\mathbb{R}) \qquad F(X) = F([\,x_1\, x_2\, \cdots\, x_n\,]) = x_1.$$

The map F is smooth because

Step 1:

$$\widehat{F} \,:\, \mathbb{R}^{n\times n} \to \mathbb{R}^n \qquad \widehat{F}(X) = F([\,x_1\, x_2\, \cdots\, x_n\,]) = x_1$$

is <u>clearly</u> smooth

Step 2: $\mathsf{O}(n)$ is an embedded submanifold of $\mathbb{R}^{n\times n}$, hence,

$$\widehat{F}|_{\mathsf{O}(n)} \,:\, \mathsf{O}(n) \to \mathbb{R}^n, \qquad \widehat{F}|_{\mathsf{O}(n)}(X) = x_1$$

is smooth

$$\begin{tikzcd}
\mathbb{R}^{n\times n} \arrow[r, "\widehat{F}"] & \mathbb{R}^n \\
\mathsf{O}(n) \arrow[u, "\iota_{\mathsf{O}(n)}"] \arrow[ur, "\widehat{F}|_{\mathsf{O}(n)}"'] &
\end{tikzcd}$$

Step 3: $\mathsf{S}^{n-1}(\mathbb{R})$ is an embedded submanifold of $\mathbb{R}^n$ and $\widehat{F}|_{\mathsf{O}(n)}(\mathsf{O}(n)) \subset \mathsf{S}^{n-1}(\mathbb{R})$; hence,

$$F = \widehat{F}|_{\mathsf{O}(n)}^{\mathsf{S}^{n-1}(\mathbb{R})}$$

is smooth

$$\begin{tikzcd}
 & \mathbb{R}^n \\
\mathsf{O}(n) \arrow[ur, "\widehat{F}|_{\mathsf{O}(n)}"] \arrow[r, "F"'] & \mathsf{S}^{n-1}(\mathbb{R}) \arrow[u, "\iota_{\mathsf{S}^{n-1}(\mathbb{R})}"']
\end{tikzcd}$$

☐ **Example (using identifications for computations):** let $F : A \to B$ be a smooth map between smooth manifolds. Assume that A and B are embedded in M and N, respectively. Suppose that there exists a smooth map $\widehat{F} : M \to N$ such that the following diagram commutes (i.e., $\iota_B \circ F = \widehat{F} \circ \iota_A$)

$$\begin{tikzcd} M \arrow[r, "\widehat{F}"] & N \\ A \arrow[u, "\iota_A"] \arrow[r, "F"'] & B \arrow[u, "\iota_B"'] \end{tikzcd}$$

For any $p \in A$, we have the corresponding diagram in terms of the push-forwards

$$\begin{tikzcd}
\iota_{A*}(T_pA) \subset T_pM \arrow[r, "\widehat{F}_*"] & \iota_{B*}(T_{F(p)}B) \subset T_{F(p)}N \\
T_pA \arrow[u, "\iota_{A*}"] \arrow[r, "F_*"'] & T_{F(p)}B \arrow[u, "\iota_{B*}"']
\end{tikzcd}$$

This means that we can represent the push-forward map $F_* : T_pA \to T_{F(p)}B$ by the push-forward map

$$\widehat{F}_* : \iota_{A*}(T_pA) \to \iota_{B*}(T_{F(p)}B).$$

□ **Example (an immersion of the unit-sphere):** consider the map

$$F : \mathsf{S}^{n-1}(\mathbb{R}) \to \mathbb{R}^{n\times n} \qquad F(x) = xx^\top.$$

▷ We have the commutative diagram

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\widehat{F}"] & \mathbb{R}^{n\times n} \\ \mathsf{S}^{n-1}(\mathbb{R}) \arrow[u, "\iota_{\mathsf{S}^{n-1}(\mathbb{R})}"] \arrow[ur, "F"'] & \end{tikzcd}$$

▷ It is easy to obtain the push-forward of $\widehat{F}$ at any point $p \in \mathbb{R}^n$:

$$\widehat{F}_* : T_p\mathbb{R}^n \simeq \mathbb{R}^n \to T_{\widehat{F}(p)}\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n\times n} \qquad \widehat{F}(\delta) = \delta p^\top + p\delta^\top$$

▷ On the other hand,

$$T_p\mathsf{S}^{n-1}(\mathbb{R}) \simeq \{\delta \in \mathbb{R}^n : p^\top\delta = 0\}$$

▷ In conclusion, we can represent the push-forward

$$F_* \, : \, T_p\mathsf{S}^{n-1}(\mathbb{R}) \longrightarrow T_{F(p)}\mathbb{R}^{n\times n}$$

by the linear map

$$F_* \, : \, \{\delta \, : \, p^\top \delta = 0\} \longrightarrow \mathbb{R}^{n\times n} \qquad F_*(\delta) = \delta p^\top + p\delta^\top$$

▷ As an example, we can exploit the representation above to prove that the smooth map F is an immersion, that is, F_* is injective (its kernel is zero-dimensional):

$$\begin{aligned} F_*(\delta) = 0 \quad &\Rightarrow \quad \delta p^\top + p\delta^\top = 0 \\ &\Rightarrow \quad p^\top \left(\delta p^\top + p\delta^\top\right) = 0 \\ &\Rightarrow \quad \delta^\top = 0. \end{aligned}$$

We used the facts that $p^\top \delta = 0$ and $p^\top p = 1$.

□ **Example (a submersion onto the unit-sphere):** consider the map

$$F : \mathbb{R}^n - \{0\} \to \mathsf{S}^{n-1}(\mathbb{R}) \qquad F(x) = \frac{x}{\|x\|}.$$

▷ We have the commutative diagram

$$\begin{tikzcd} \mathbb{R}^n - \{0\} \arrow[r, "\widehat{F}"] \arrow[dr, "F"'] & \mathbb{R}^n \\ & \mathsf{S}^{n-1}(\mathbb{R}) \arrow[u, "\iota_{\mathsf{S}^{n-1}(\mathbb{R})}"'] \end{tikzcd}$$

▷ It is easy to obtain the push-forward of $\widehat{F}$ at any point $p \in \mathbb{R}^n - \{0\}$:

$$\widehat{F}_* : T_p\mathbb{R}^n - \{0\} \simeq \mathbb{R}^n \to T_{\widehat{F}(p)}\mathbb{R}^n \simeq \mathbb{R}^n$$

$$\widehat{F}_*(\delta) = \frac{1}{\|p\|}\left(I_n - \frac{pp^\top}{\|p\|^2}\right)\delta = \frac{1}{\|p\|}\left(I_n - F(p)F(p)^\top\right)\delta$$

▷ On the other hand,

$$T_{F(p)}\mathsf{S}^{n-1}(\mathbb{R}) \simeq \{\gamma \in \mathbb{R}^n \,:\, F(p)^\top \gamma = 0\}$$

▷ In conclusion, we can represent the push-forward

$$F_* \,:\, \mathbb{R}^n - \{0\} \to T_{F(p)}\mathsf{S}^{n-1}(\mathbb{R})$$

by the linear map

$$F_* \,:\, \mathbb{R}^n - \{0\} \to \{\gamma \,:\, F(p)^\top \gamma = 0\} \qquad F_*(\delta) = \frac{1}{\|p\|}\left(I_n - F(p)F(p)^\top\right)\delta.$$

▷ As an example, we can exploit the representation above to prove that the smooth map F is a submersion, that is, F_* is surjective: choose γ such that $F(p)^\top\gamma = 0$. Letting $\delta = \|p\|\,\gamma$, we have $F_*(\delta) = \gamma$

□ **Example (another submersion onto the unit-sphere):** consider the map

$$F : \mathsf{O}(n) \to \mathsf{S}^{n-1}(\mathbb{R}) \qquad F(X) = Xe_1,$$

where $e_1 = (1, 0, \ldots, 0)^\top$.

▷ The map F is smooth and we have the commutative diagram

$$\begin{tikzcd} \mathbb{R}^{n\times n} \arrow[r, "\widehat{F}"] & \mathbb{R}^n \\ \mathsf{O}(n) \arrow[u, "\iota_{\mathsf{O}(n)}"] \arrow[r, "F"'] & \mathsf{S}^{n-1}(\mathbb{R}) \arrow[u, "\iota_{\mathsf{S}^{n-1}(\mathbb{R})}"'] \end{tikzcd}$$

where $\widehat{F}(X) = Xe_1$

▷ The push-forward of $\widehat{F}$ at any point $X_0 \in \mathbb{R}^{n\times n}$ is easily obtained:

$$\widehat{F}_* : T_{X_0}\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n\times n} \to T_{\widehat{F}(X_0)}\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n\times n}$$

$$\widehat{F}_*(\Delta) = \Delta e_1$$

▷ Now, for $Q_0 = [\, q_1 \, q_2 \, \cdots \, q_n \,] \in \mathsf{O}(n)$,

$$\begin{aligned} T_{Q_0}\mathsf{O}(n) &\simeq Q_0\mathsf{K}(n,\mathbb{R}) \\ T_{F(Q_0)}\mathsf{S}^{n-1}(\mathbb{R}) &\simeq \{\delta \in \mathbb{R}^n \;\; q_1^\top \delta = 0\} \end{aligned}$$

▷ Thus, we can represent the push-forward

$$F_* \,:\, T_{Q_0}\mathsf{O}(n) \to T_{F(Q_0)}\mathsf{S}^{n-1}(\mathbb{R})$$

by the linear map

$$F_* \,:\, Q_0\mathsf{K}(n,\mathbb{R}) \to \{\delta \,:\, q_1^\top \delta = 0\} \qquad F_*(Q_0 K) = Q_0 K e_1$$

▷ Exploiting the representation above, it is straightforward to show that F_* is surjective and, therefore, F is a submersion

☐ **Lemma [Embedded submanifolds of product manifolds]** If A is embedded in M and B is embedded in N, then $A \times B$ is embedded in $M \times N$

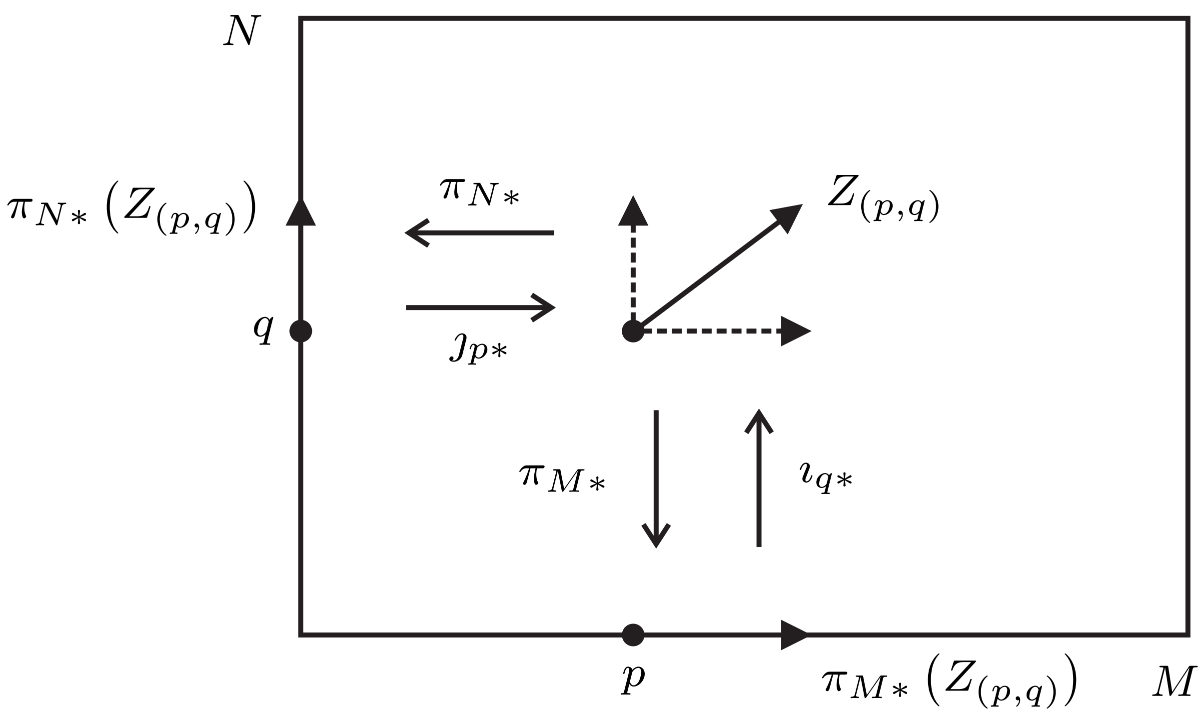
☐ **Example (tangent space identifications for product manifolds):** let $(p, q) \in M \times N$. We have the identification

$$T_{(p,q)} M \times N \simeq T_p M \oplus T_q N$$

due to the isomorphism

$$\pi_{M*} \times \pi_{N*} \, : T_{(p,q)} M \times N \to T_p M \oplus T_q N$$

$$\pi_{M*} \times \pi_{N*} \left(Z_{(p,q)} \right) = \left(\pi_{M*}(Z_{(p,q)}), \pi_{N*}(Z_{(p,q)}) \right)$$



The inverse map is given by

$$\imath_{q*} \oplus \jmath_{p*} \,:\, T_pM \oplus T_qN \to T_{(p,q)}M \times N \qquad \imath_{q*} \oplus \jmath_{p*}(X_p, Y_q) = \imath_{q*}(X_p) + \jmath_{q*}(Y_q),$$

where

$$\begin{aligned} \imath_q \,&:\, M \to M \times N \qquad & x \mapsto (x, q) \\ \jmath_p \,&:\, N \to M \times N \qquad & y \mapsto (p, y) \end{aligned}$$

The discussed identification $T_{(p,q)}M \times N \simeq T_pM \oplus T_qN$ can be used as follows. Suppose we have a smooth map $F : M \times N \to P$. We want to compute the push-forward of F at the point (p,q), that is, the linear map

$$F_* : T_{(p,q)}M \times N \to T_{F(p,q)}P \qquad Z_{(p,q)} \mapsto F_*(Z_{(p,q)}).$$

Since $T_{(p,q)}M \times N \simeq T_pM \oplus T_qN$, we know that it can be represented by a linear map

$$F_* : T_pM \oplus T_qN \to T_{F(p,q)}P \qquad (X_p, Y_q) \mapsto F_*(X_p, Y_q).$$

The next diagram illustrates the idea:

$$\begin{tikzcd}
T_pM \oplus T_qN \arrow[r, "\imath_{q*} \oplus \jmath_{p*}"'] \arrow[rr, bend right, "F_* \text{ (the identification)}"'] & T_{(p,q)}M \times N \arrow[r, "F_*"'] & T_{F(p,q)}P
\end{tikzcd}$$

To find out how to represent F_* by this latter map, we reason as follows:

$$\begin{aligned} F_*(X_p, Y_q) &\simeq F_* \circ \imath_{q*} \oplus \jmath_{p*}(X_p, Y_q) \\ &= F_*(\imath_{q*} X_p + \jmath_{p*} Y_q) \\ &= (F \circ \imath_q)_* X_p + (F \circ \jmath_p)_* Y_q \\ &= F_{q*} X_p + F_{p*} Y_q, \end{aligned}$$

where

$$F_q = F \circ \imath_q \,:\, M \to P \qquad x \mapsto F(x, q)$$

and

$$F_p = F \circ \jmath_p \,:\, M \to P \qquad y \mapsto F(p, y).$$

That is, F_q and F_p correspond to F when we hold fixed the 2nd and 1st argument at q and p, respectively.

For a specific example, let $M = N = P = \mathbb{R}^{n \times n}$ and consider the smooth map

$$F \,:\, M \times N \to P \qquad F(X, Y) = XY.$$

The push-forward of F at the point (X_0, Y_0) can be represented by the linear map

$$F_* \,:\, \mathbb{R}^{n \times n} \oplus \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n} \qquad F_*(\Delta, \Omega) = \Delta Y_0 + X_0 \Omega$$

☐ **Example (embedded submanifolds of product manifolds):** let A, B and C be embedded submanifolds of M, N and P, respectively. Let $F : A \times B \to C$ be a smooth map. Suppose that there exists a smooth map $\widehat{F} : M \times N \to P$ such that the following diagram commutes

$$\begin{tikzcd} M \times N \arrow[r, "\widehat{F}"] & P \\ A \times B \arrow[u, "\iota_{A\times B}"] \arrow[r, "F"'] & C \arrow[u, "\iota_C"'] \end{tikzcd}$$

Note that $A \times B$ is embedded in $M \times N$. Thus, for any given $(a,b) \in A \times B$, we already know that we have the following diagram

$$\begin{tikzcd}
\iota_{A\times B*}\left(T_{(a,b)}A \times B\right) \arrow[r, "\widehat{F}_*"] & \iota_{C*}\left(T_{F(a,b)}C\right) \\
T_{(a,b)}A \times B \arrow[u, "\iota_{A\times B*}"] \arrow[r, "F_*"'] & T_{F(a,b)}C \arrow[u, "\iota_{C*}"']
\end{tikzcd}$$

which allows us to represent the "hard" linear map

$$F_* \; : \; T_{(a,b)}A \times B \to C$$

by the "easier" one

$$\widehat{F}_* \; : \; \iota_{A\times B*}\left(T_{(a,b)}A \times B\right) \subset T_{(a,b)}M \times N \to \iota_{C*}\left(T_{F(a,b)}C\right) \subset T_{F(a,b)}P$$

Our goal here is to exploit the tangent space identifications discussed in the previous example to find out another representation for F_*.

We start by noting that we have the following two diagrams

$$\begin{tikzcd} M \arrow[r, "\widehat{\imath}_b"] & M \times N \\ A \arrow[u, "\iota_A"] \arrow[r, "\imath_b"'] & A \times B \arrow[u, "\iota_{A\times B}"'] \end{tikzcd} \qquad \begin{tikzcd} N \arrow[r, "\widehat{\jmath}_a"] & M \times N \\ B \arrow[u, "\iota_B"] \arrow[r, "\jmath_a"'] & A \times B \arrow[u, "\iota_{A\times B}"'] \end{tikzcd}$$

where

$$\begin{aligned} \imath_b &: A \to A \times B && x \mapsto (x, b) \\ \widehat{\imath}_b &: M \to M \times N && x \mapsto (x, b) \\ \jmath_a &: B \to A \times B && y \mapsto (a, y) \\ \widehat{\jmath}_a &: N \to M \times N && y \mapsto (a, y) \end{aligned}$$

From this, it follows that

$$\begin{tikzcd}
\iota_{A*}(T_aA) \oplus \iota_{B*}(T_bB) \arrow[r, "\widehat{\imath}_{b*} \oplus \widehat{\jmath}_{a*}"] & \iota_{A\times B*}\left(T_{(a,b)}A \times B\right) \\
T_aA \oplus T_bB \arrow[u, "\iota_{A*} \times \iota_{B*}"] \arrow[r, "\imath_{b*} \oplus \jmath_{a*}"'] & T_{(a,b)}A \times B \arrow[u, "\iota_{A\times B*}"']
\end{tikzcd}$$

where

$$\begin{aligned}
\imath_{b*} \oplus \jmath_{a*} &: T_aA \oplus T_bB \to T_{(a,b)}A \times B & (X_a, Y_b) &\mapsto \imath_{b*}(X_a) + \jmath_{a*}(Y_b) \\
\widehat{\imath}_{b*} \oplus \widehat{\jmath}_{a*} &: T_aM \oplus T_bN \to T_{(a,b)}M \times N & (X_a, Y_b) &\mapsto \widehat{\imath}_{b*}(X_a) + \widehat{\jmath}_{a*}(Y_b) \\
\iota_{A*} \times \iota_{B*} &: T_aA \oplus T_bB \to T_aM \oplus T_bN & (X_a, Y_b) &\mapsto (\iota_{A*}(X_a), \iota_{B*}(Y_b))
\end{aligned}$$

The equality

$$\iota_{A\times B*} \circ \imath_{b*} \oplus \jmath_{a*} = \widehat{\imath}_{b*} \oplus \widehat{\jmath}_{a*} \circ \iota_{A*} \times \iota_{B*}$$

expressed in the last diagram can be proved as follows:

$$\begin{aligned}
\iota_{A\times B*} \circ \imath_{b*} \oplus \jmath_{a*}(X_a, Y_b) &= \iota_{A\times B*}\left(\imath_{b*}(X_a) + \jmath_{a*}(Y_b)\right) \\
&= \iota_{A\times B*} \circ \imath_{b*}(X_a) + \iota_{A\times B*} \circ \jmath_{a*}(Y_b) \\
&\overset{(a)}{=} \left(\iota_{A\times B} \circ \imath_b\right)_* (X_a) + \left(\iota_{A\times B} \circ \jmath_a\right)_* (Y_b) \\
&\overset{(b)}{=} \left(\widehat{\imath}_b \circ \iota_A\right)_* (X_a) + \left(\widehat{\jmath}_a \circ \iota_B\right)_* (Y_b) \\
&= \widehat{\imath}_{b*}\left(\iota_{A*}(X_a)\right) + \widehat{\jmath}_{a*}\left(\iota_{B*}(Y_b)\right) \\
&= \widehat{\imath}_{b*} \oplus \widehat{\jmath}_{a*}\left(\iota_{A*}(X_a), \iota_{B*}(Y_b)\right) \\
&= \widehat{\imath}_{b*} \oplus \widehat{\jmath}_{a*} \circ \iota_{A*} \times \iota_{B*}(X_a, Y_b).
\end{aligned}$$

In (a), the chain rule for push-forwards was used. In (b), we used the two commutative diagrams in page 64.

Now, taking the last diagram in page 64 and plugging it on the left of the last diagram in page 63 yields

$$
\begin{tikzcd}
\iota_{A*}\left(T_a A\right) \oplus \iota_{B*}\left(T_b B\right) \arrow[r, "\simeq"] \arrow[rr, bend left, "F_* \text{ (the identification)}"] & \iota_{A\times B*}\left(T_{(a,b)} A \times B\right) \arrow[r, "\widehat{F}_*"] & \iota_{C*}\left(T_{F(a,b)} P\right) \\
T_a A \oplus T_b B \arrow[u, "\iota_{A*} \times \iota_{B*}"] \arrow[r, "\simeq"'] & T_{(a,b)} A \times B \arrow[u, "\iota_{A\times B*}"] \arrow[r, "F_*"'] & T_{F(a,b)} C \arrow[u, "\iota_{C*}"']
\end{tikzcd}
$$

Since the arrows marked with $\simeq$ denote isomorphisms, this shows that F_* can be represented by the linear map

$$F_* \,:\, \iota_{A*}\left(T_a A\right) \oplus \iota_{B*}\left(T_b B\right) \to \iota_{C*}\left(T_{F(a,b)} C\right)$$

given by

$$\begin{aligned} \left(\iota_{A*}(X_a), \iota_{B*}(Y_b)\right) &\mapsto \widehat{F}_*\left(\widehat{\imath}_{b*}\left(\iota_{A*}(X_a)\right) + \widehat{\jmath}_{a*}\left(\iota_{B*}(Y_b)\right)\right) \\ &= \widehat{F}_{b*}\left(\iota_{A*}(X_a)\right) + \widehat{F}_{a*}\left(\iota_{B*}(Y_b)\right), \end{aligned}$$

where

$$\begin{aligned} \widehat{F}_b \,:\, M \to P \qquad & x \mapsto \widehat{F}(x,b) \\ \widehat{F}_a \,:\, N \to P \qquad & y \mapsto \widehat{F}(a,y) \end{aligned}$$

□ **Example: (Polar decomposition is a diffeomorphism):** consider the map

$$F : \mathsf{P}(n,\mathbb{R}) \times \mathsf{O}(n) \to \mathsf{GL}(n,\mathbb{R}) \qquad F(P,Q) = PQ.$$

The map F is smooth (why?).

We have the following commutative diagram

$$\begin{tikzcd}
\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \arrow[r, "\widehat{F}"] & \mathbb{R}^{n\times n} \\
\mathsf{P}(n,\mathbb{R}) \times \mathsf{O}(n) \arrow[u, "\iota_{\mathsf{P}(n,\mathbb{R})\times \mathsf{O}(n)}"] \arrow[r, "F"'] & \mathsf{GL}(n,\mathbb{R}) \arrow[u, "\iota_{\mathsf{GL}(n,\mathbb{R})}"']
\end{tikzcd}$$

where

$$\widehat{F} : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n} \qquad \widehat{F}(X,Y) = XY$$

Note that $\mathsf{P}(n,\mathbb{R})$, $\mathsf{O}(n)$ and $\mathsf{GL}(n,\mathbb{R})$ are embedded submanifolds of $\mathbb{R}^{n\times n}$.

Also:

$$\begin{aligned}
\iota_{\mathsf{P}(n,\mathbb{R})*}\big(T_{P_0}\mathsf{P}(n,\mathbb{R})\big) &\simeq \mathsf{S}(n,\mathbb{R}) \\
\iota_{\mathsf{O}(n)*}\big(T_{Q_0}\mathsf{O}(n)\big) &\simeq Q_0\mathsf{K}(n,\mathbb{R}) \\
\iota_{\mathsf{GL}(n,\mathbb{R}))*}\big(T_{X_0}\mathsf{GL}(n,\mathbb{R})\big) &\simeq \mathbb{R}^{n\times n}.
\end{aligned}$$

Thus, the push-forward

$$F_* \,:\, T_{(P_0,Q_0)}\mathsf{P}(n,\mathbb{R}) \times \mathsf{O}(n) \to T_{F(P_0,Q_0)}\mathsf{GL}(n,\mathbb{R})$$

can be represented by the linear map

$$F_* \,:\, \mathsf{S}(n,\mathbb{R}) \oplus Q_0\mathsf{K}(n,\mathbb{R}) \to \mathbb{R}^{n\times n} \qquad F_*(\Delta,\Psi) = \Delta Q_0 + P_0\Psi.$$

Since F_* is an injective linear map for any (P_0,Q_0), we conclude that F is a diffeomorphism.