Nonlinear Signal Processing
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Smooth manifolds

(Ch.1, "Introduction to Smooth Manifolds", J. Lee, Springer-Verlag)

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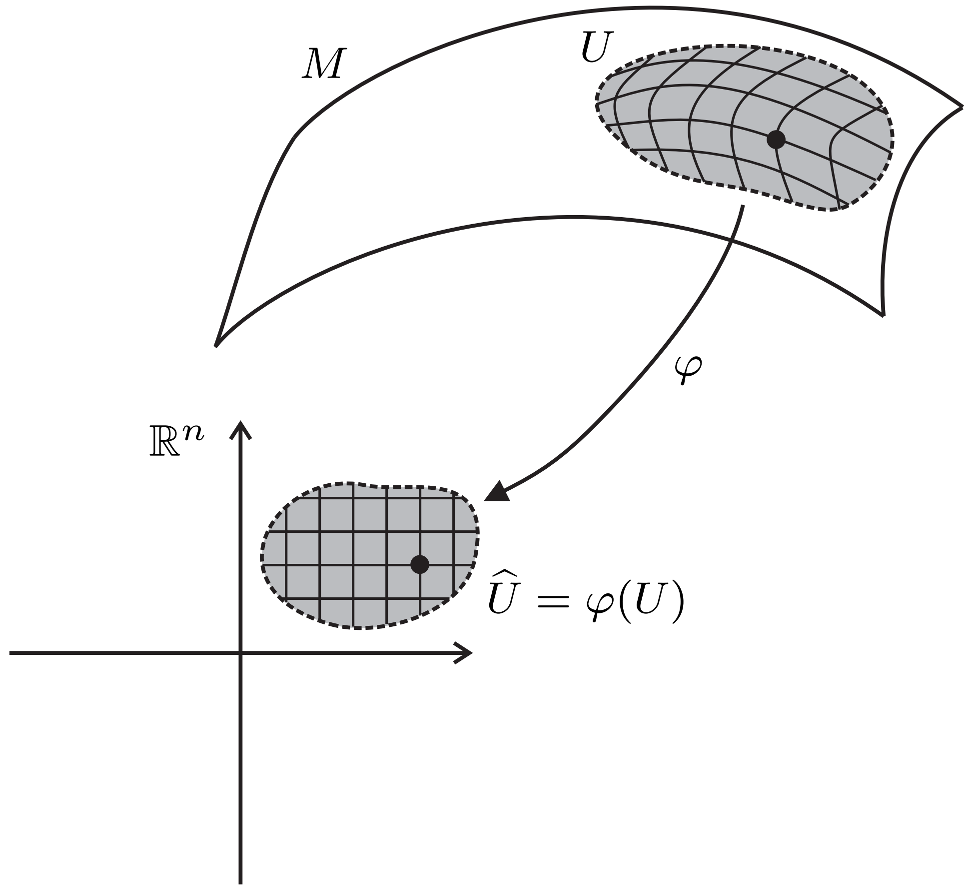
Lecture's key-points

☐ A smooth manifold M has (many) local coordinates around each point

☐ These local coordinates overlap smoothly

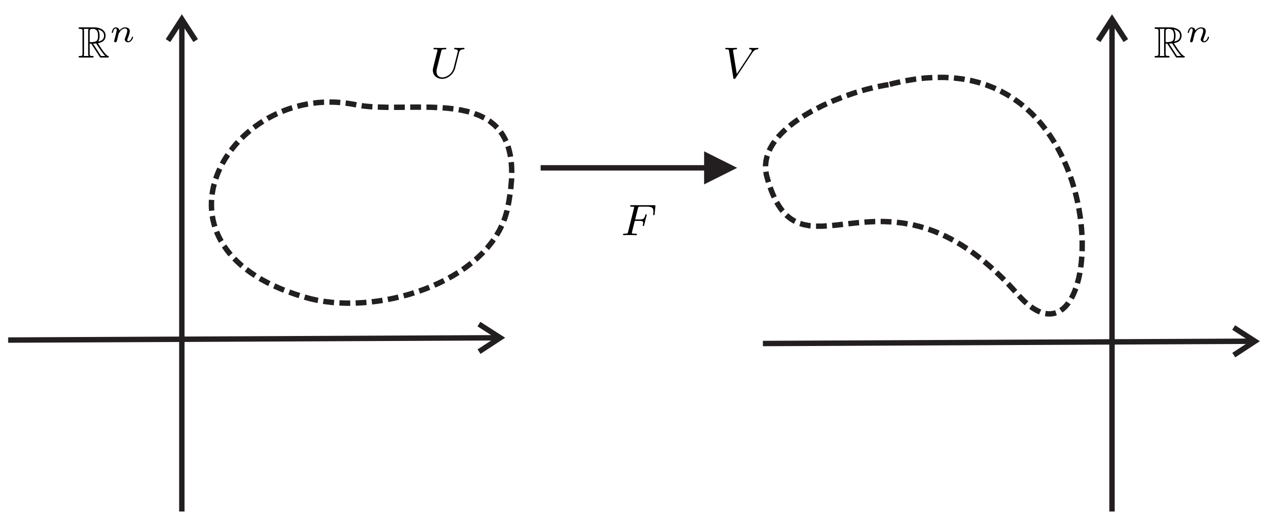
☐ **Definition [Coordinate chart]** Let M be a topological n-manifold. A coordinate chart on M is a pair (U, φ) where U is an open subset of M and $\varphi : U \to \widehat{U}$ is a homeomorphism from U to an open subset $\widehat{U} = \varphi(U) \subset \mathbb{R}^n$



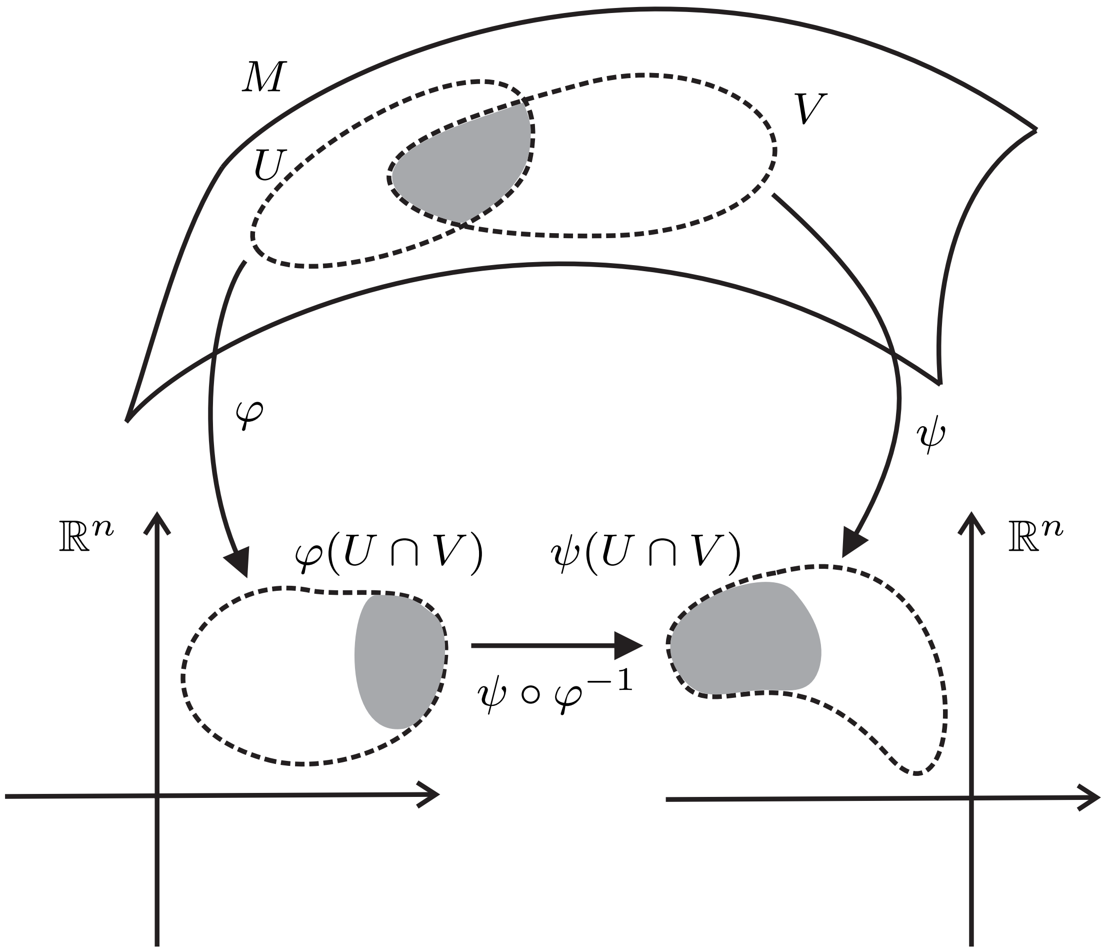
☐ U=coordinate domain, φ=coordinate map

☐ **Definition [Smooth maps in Euclidean spaces]** Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a map defined on an open subset $U \subset \mathbb{R}^n$. The map F is said to be smooth if all the component functions of $F = (F^1, F^2, \ldots, F^m)$ have continuous partial derivatives of all orders

☐ **Definition [Diffeomorphisms in Euclidean spaces]** Let $F : U \to V$ be a map defined on open subsets U, V of $\mathbb{R}^n$. The map F is said to be a diffeomorphism if it is bijective, smooth and its inverse map $F^{-1} : V \to U$ is smooth

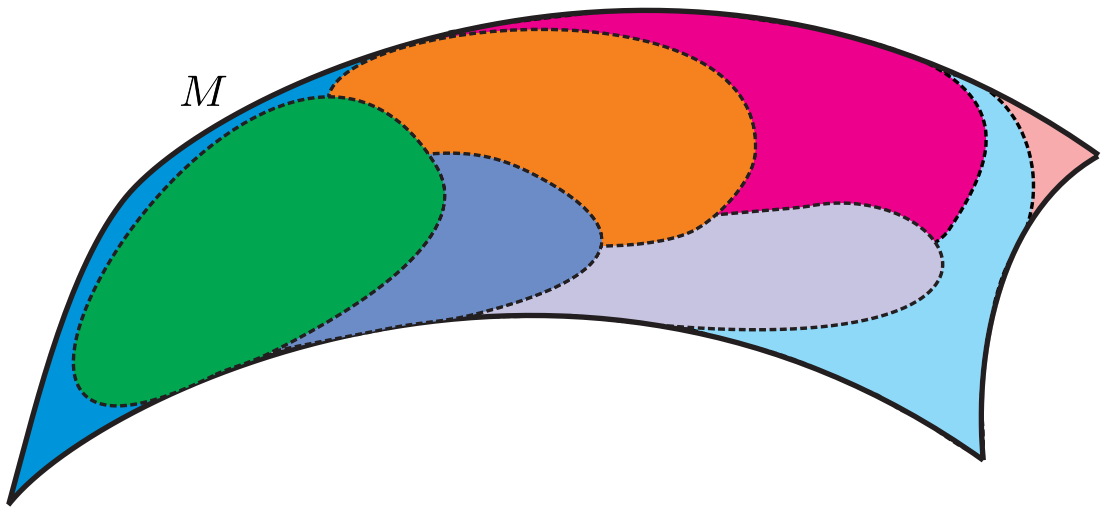


□ **Definition [Smooth compatibility of charts]** Let M be a topological manifold. Two charts (U, φ) and (V, ψ) on M are said to be smoothly compatible if either $U \cap V = \emptyset$ or $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism



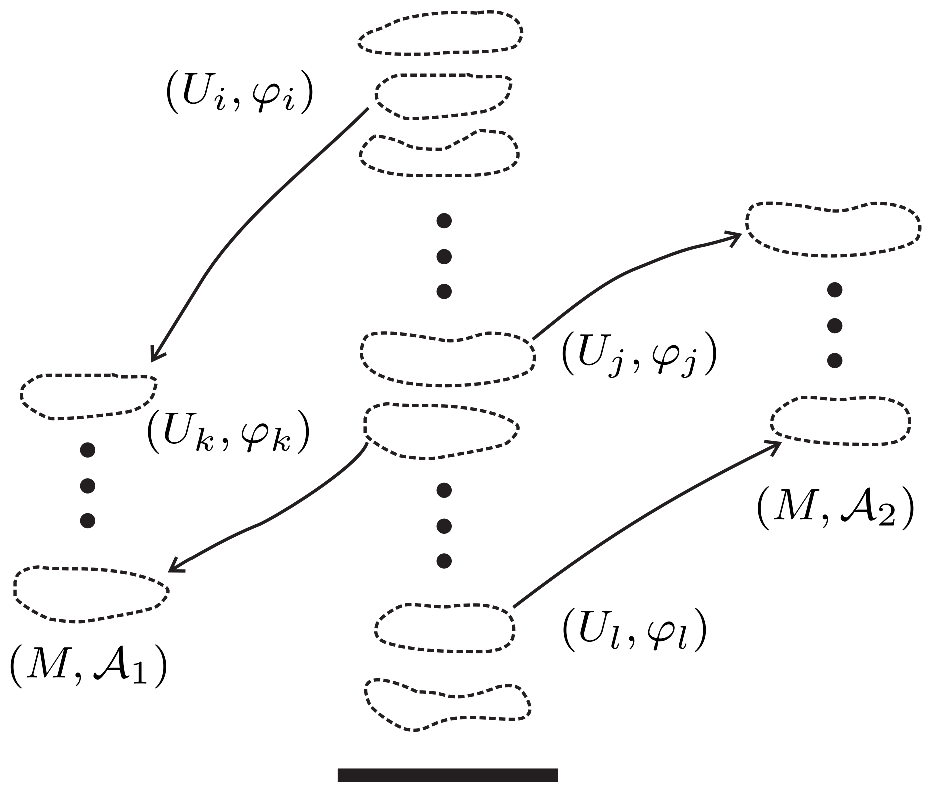
□ **Definition [Smooth atlas, maximal smooth atlas]** Let M be a topological manifold. A smooth atlas $\mathcal{A}$ on M is a collection of smoothly compatible charts whose domains cover M.

A smooth atlas $\mathcal{A}$ is said to be maximal if it does not exist a smooth atlas $\widetilde{\mathcal{A}}$ strictly larger that $\mathcal{A}$ ($\mathcal{A} \subset \widetilde{\mathcal{A}}$ and $\mathcal{A} \neq \widetilde{\mathcal{A}}$)



□ **Definition [Smooth manifold]** A smooth manifold is a pair $(M, \mathcal{A})$ where M is a topological manifold and $\mathcal{A}$ is a maximal smooth atlas on M

∗ *Intuition: a smooth manifold is a topological manifold which can be covered by coordinate charts which fit nicely together*



Stack of all coordinate charts of M

□ **Theorem [Maximal smooth atlas]** Let M be a topological manifold. If $\mathcal{A}$ is a smooth atlas on M, then there exists an unique maximal smooth atlas on M which contains $\mathcal{A}$

$*$ *Intuition: a straight-line through the origin can be specified by a single non-zero point (you don't need to furnish <u>all</u> points)*

☐ **Example:** $\mathbb{R}^n$ is a smooth manifold of dimension n

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☐ **Example (vector space):** a real vector space of dimension n is a smooth manifold of dimension n.

▷ $\mathsf{S}(n, \mathbb{R}) = \{X \in \mathbb{R}^{n \times n} \, : \, X = X^\top\}$

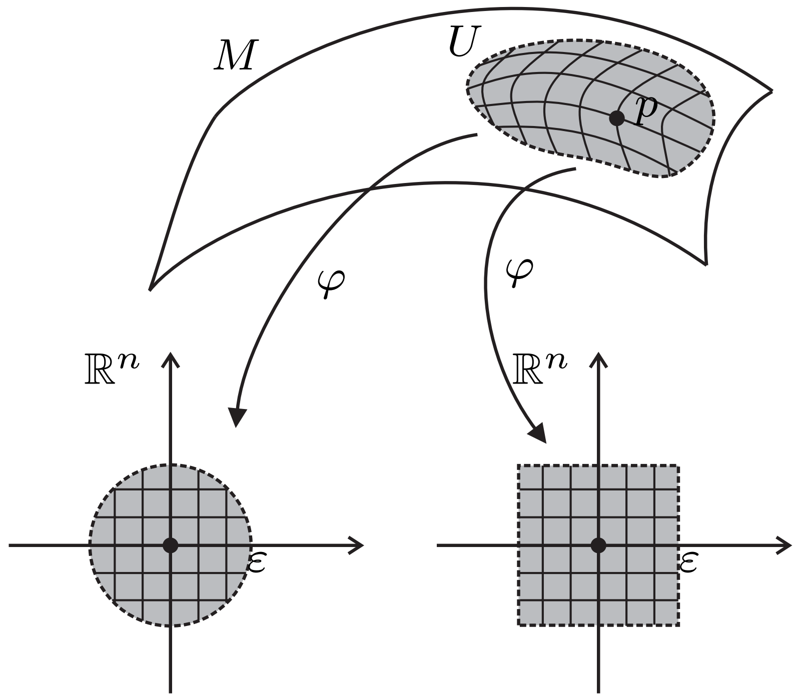
☐ **Example (unit-sphere):** the unit-sphere

$$\mathsf{S}^{n-1}(\mathbb{R}) = \{x \in \mathbb{R}^n \, : \, \|x\| = 1\}$$

is a smooth manifold of dimension $n - 1$

☐ **Definition [Smooth chart]** Let $(M, \mathcal{A})$ be a smooth manifold. Any coordinate chart (U, φ) contained in the given maximal smooth atlas $\mathcal{A}$ is called a smooth chart

□ **Lemma [Existence of special smooth charts]** Let $(M, \mathcal{A})$ be a smooth manifold of dimension m. If (U, φ) is a smooth chart and $V \subset U$ is an open subset, then $(V, \varphi|_V)$ is a smooth chart. For any $p \in M$ and $\varepsilon > 0$ there exists a smooth chart (U, φ) containing p such that $\varphi(U) = B^m_\varepsilon(0)$ or $\varphi(U) = C^m_\varepsilon(0)$, where
$B^m_\varepsilon(0) = \{x \in \mathbb{R}^m : \|x\|_2 < \varepsilon\} \quad C^m_\varepsilon(0) = \{x \in \mathbb{R}^m : \|x\|_\infty < \varepsilon\}$



∗ *Intuition: A smooth manifold contains a lot of smooth charts, many of them nice*

☐ **Lemma [Open submanifold]** Let $(M, \mathcal{A})$ be a smooth n-manifold and $W \subset M$ an open subset. Then, $\mathcal{A}_W = \{(U, \varphi) \in \mathcal{A} : U \subset W\}$ is a smooth atlas on the topological n-manifold W (W is given the subspace topology)

☐ **Theorem [Product manifolds]** Let $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ be smooth manifolds of dimension m and n. Then,

$$\mathcal{A}_{M\times N} = \{(U \times V, \varphi \times \psi) : (U, \varphi) \in \mathcal{A}_M, (V, \psi) \in \mathcal{A}_N\}$$

is a smooth atlas on $M \times N$ and it is called the smooth (canonical) product atlas on the topological $(m+n)$-manifold $M \times N$