Nonlinear Signal Processing
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Connectedness and compactness

(Ch.4, "Introduction to Topological Manifolds", J. Lee, Springer-Verlag)

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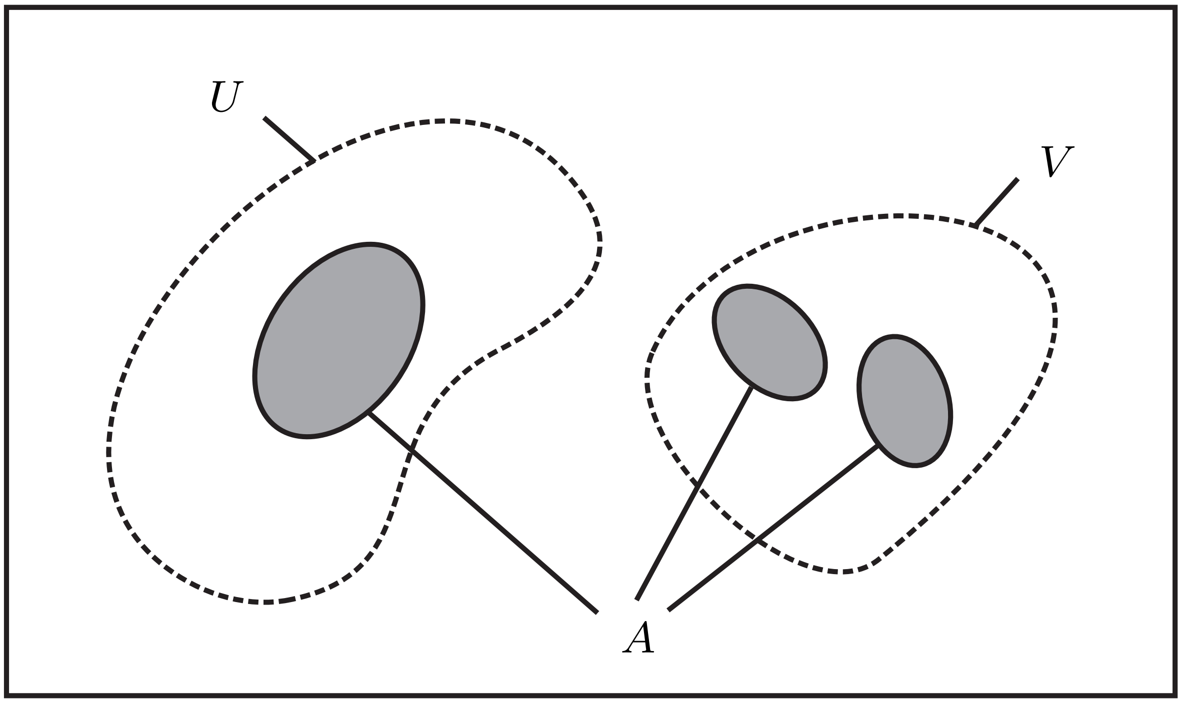
Lecture's key-points

- ☐ A connected space is made of one piece

- ☐ A compact space behaves muck like a "finite space"

- ☐ Continuous maps preserve connectedness and compactness

☐ **Definition [Connected space]** Let X be a topological space. A separation of X is a pair of nonempty, disjoint, open subsets $U, V \subset X$ such that $X = U \cup V$. X is said to be disconnected if there exists a separation of X, and connected otherwise



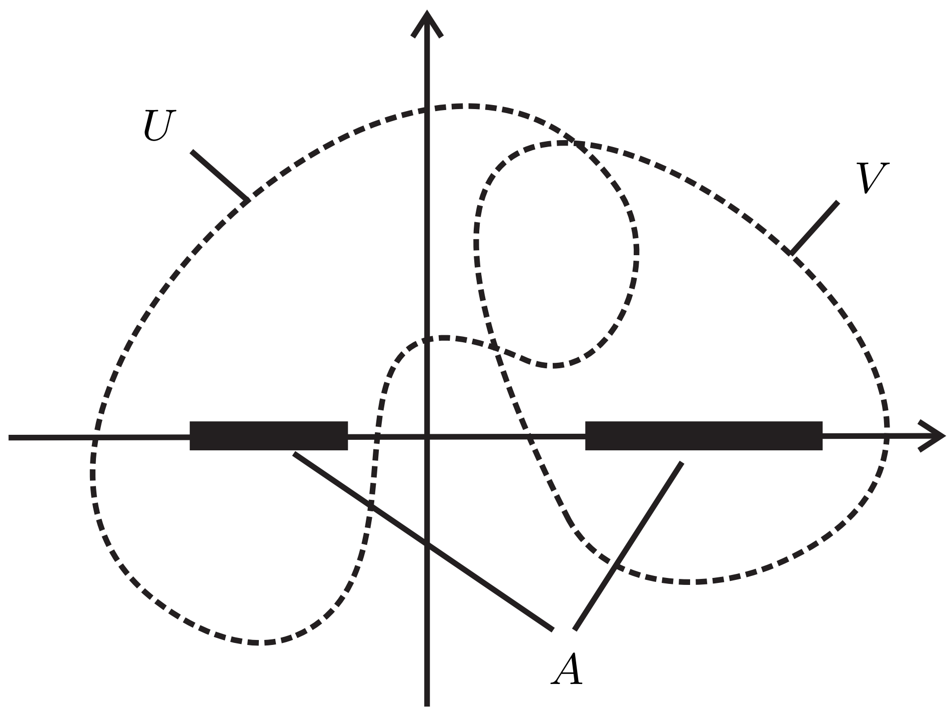
☐ **Definition [Connected subset]** Let X be a topological space. A subset $A \subset X$ is said to be connected if the subspace A is connected: there do not exist open sets U, V in X such that $A \cap U \neq \emptyset, \quad A \cap V \neq \emptyset, \quad (A \cap U) \cap (A \cap V) = \emptyset, \quad A \subset U \cup V$

☐ **Example:** $\mathbb{R}^n$ is connected

☐ **Example (simple disconnected subset):** the subset

$$A = \{(x, y) \in \mathbb{R}^2 \,:\, x \in [-3, 1[\cup]2, 5],\ y = 0\}$$

of $\mathbb{R}^2$ is disconnected. Equivalently, the topological space A (endowed with the subspace topology) is disconnected



□ **Example (more interesting disconnected subset):** the subset

$$\mathsf{O}(n) = \{X \in \mathbb{R}^{n \times n} \, : \, X^\top X = I_n\}$$

of $\mathbb{R}^{n \times n}$ is disconnected. Equivalently, the topological space $\mathsf{O}(n)$ (endowed with the subspace topology) is disconnected.

The open sets

▷ $U = \{X \in \mathbb{R}^{n \times n} \, : \, \det X < 0\}$

▷ $V = \{X \in \mathbb{R}^{n \times n} \, : \, \det X > 0\}$

provide a separation of $\mathsf{O}(n)$

(note that $\mathsf{O}(n) \cap U \neq \emptyset$ and $\mathsf{O}(n) \cap V \neq \emptyset$; why ?)

☐ **Proposition [Characterization of connectedness]** A topological space X is connected if and only if the only subsets of X that are both open and closed are $\emptyset$ and X

☐ **Example:** want to prove that all points in a connected space X have property P

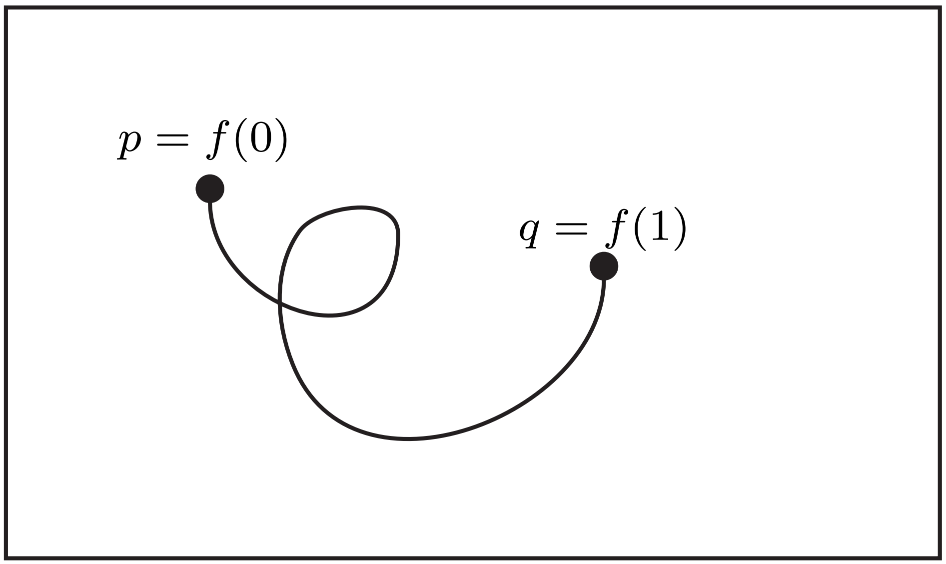
- ▷ define $S = \{x \in X \, : \, x \text{ has property } P\}$
- ▷ show S is non-empty
- ▷ show S is closed
- ▷ show S is open

Conclude that $S = X$

☐ **Example:** let X be a connected space and $A : X \to \mathsf{S}(n, \mathbb{R})$ a continuous map. Suppose that the eigenvalues of $A(x)$ belong to $\{0, 1\}$ for any $x \in X$. Then, $\operatorname{rank} A(x)$ is constant over $x \in X$

☐ **Proposition [Characterization of connected subsets of $\mathbb{R}$]** A nonempty subset of $\mathbb{R}$ is connected if and only if it is an interval

☐ **Definition [Path connected space]** Let X be a topological space and $p, q \in X$. A path in X from p to q is a continuous map $f : [0, 1] \to X$, $f(0) = p$ and $f(1) = q$. We say that X is path connected if for any $p, q \in X$ there is a path in X from p to q.



☐ **Theorem [Easy sufficient criterion for connectedness]** If X is a path connected topological space, then X is connected

☐ **Example:** convex sets are connected

▹ $\mathsf{S}(n, \mathbb{R}) = \{X \in \mathbb{R}^{n \times n} \,:\, X = X^\top\}$

▹ $\mathsf{U}^+(n, \mathbb{R}) = \{X \in \mathbb{R}^{n \times n} : X$ upper-triangular and $X_{ii} > 0\}$

☐ **Example (special orthogonal matrices):** $\mathsf{SO}(n) = \{X \in \mathsf{O}(n) \,:\, \det(X) = 1\}$ is connected because there is a path in $\mathsf{SO}(n)$ from I_n to any $X \in \mathsf{SO}(n)$

☐ **Example (non-singular matrices with positive determinant):**
$\mathsf{GL}^+(n, \mathbb{R}) = \{X \in \mathbb{R}^{n \times n} \,:\, \det(X) > 0\}$ is connected because there is a path in $\mathsf{GL}^+(n, \mathbb{R})$ from I_n to any $X \in \mathsf{GL}^+(n, \mathbb{R})$

☐ **Example (special Euclidean group):**

$$\mathsf{SE}(n) = \left\{ \begin{bmatrix} Q & \delta \\ 0 & 1 \end{bmatrix} \,:\, Q \in \mathsf{SO}(n), \delta \in \mathbb{R}^n \right\}$$

is connected because there is a path in $\mathsf{SE}(n)$ from I_{n+1} to any $X \in \mathsf{SE}(n)$

☐ **Theorem [Main theorem on connectedness]** Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map. If X is connected, then $f(X)$ (as a subspace of Y) is connected

☐ **Example (unit-sphere):** $\mathsf{S}^{n-1}(\mathbb{R}) = \{x \in \mathbb{R}^n : \|x\| = 1\}$ is connected, because it is the image of the connected space $\mathbb{R}^{n+1} - \{0\}$ through the continuous map

$$f : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^n \qquad f(x) = \frac{x}{\|x\|}$$

☐ **Example (ellipsoid):** any non-flat ellipsoid in $\mathbb{R}^n$ can be described as

$$\mathsf{E} = \left\{ Au + x_0 : u \in \mathsf{S}^{n-1}(\mathbb{R}) \right\}$$

where $x_0 \in \mathbb{R}^n$ is the center of the ellipsoid and $A \in \mathsf{GL}(n, \mathbb{R})$ defines the shape and spatial orientation of E.

Thus E is connected because it is the image of the connected space $\mathsf{S}^{n-1}(\mathbb{R})$ through the continuous map

$$f : \mathsf{S}^{n-1}(\mathbb{R}) \to \mathbb{R}^n \qquad f(x) = Ax + x_0.$$

☐ **Example (projective space $\mathbb{RP}^n$):** $\mathbb{RP}^n$ is connected because it is the image of the connected space $\mathbb{R}^{n+1} - \{0\}$ through the continuous projection map

$$\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n \qquad \pi(x) = [x]$$

☐ **Proposition [Properties of connected spaces]**

(a) Suppose X is a topological space and U, V are disjoint open subsets of X. If A is a connected subset of X contained in $U \cup V$, then either $A \subset U$ or $A \subset V$

(b) Suppose X is a topological space and $A \subset X$ is connected. Then $\overline{A}$ is connected

(c) Let X be a topological space, and let $\{A_i\}$ be a collection of connected subsets with a point in common. Then $\bigcup_i A_i$ is connected

(d) The Cartesian product of finitely many connected topological spaces is connected

(e) Any quotient space of a connected topological space is connected

☐ **Theorem [Intermediate value theorem]** Let X be a connected topological space and f is a continuous real-valued function on X. If $p, q \in X$ then f takes on all values between $f(p)$ and $f(q)$

☐ **Example (antipodal points at the same temperature)**: let

$$T \,:\, \mathsf{S}^1(\mathbb{R}) \subset \mathbb{R}^2 \to \mathbb{R}$$

be a continuous map on the unit-circle in $\mathbb{R}^2$. Then, there exist a point $p \in \mathsf{S}^1(\mathbb{R})$ such that $T(p) = T(-p)$.

Consequence: there are two antipodal points in the Earth's equator line at the same temperature

☐ **Definition [Components]** Let X be a topological space. A component of X is a maximally connected subset of X, that is, a connected set that is not contained in any larger connected set.

$*$ *Intuition: X consists of a union of disjoint "islands"/components*

□ **Example (orthogonal group):** the orthogonal group

$$\mathsf{O}(n) = \{X \in \mathsf{M}(n, \mathbb{R}) \,:\, X^T X = I_n\}$$

has two components:

$$\mathsf{SO}(n) = \{X \in \mathsf{O}(n, \mathbb{R}) \,:\, \det X = 1\}$$
$$\mathsf{O}^-(n) = \{X \in \mathsf{O}(n, \mathbb{R}) \,:\, \det X = -1\}$$

□ **Proposition [Properties of components]** Let X be any topological space.

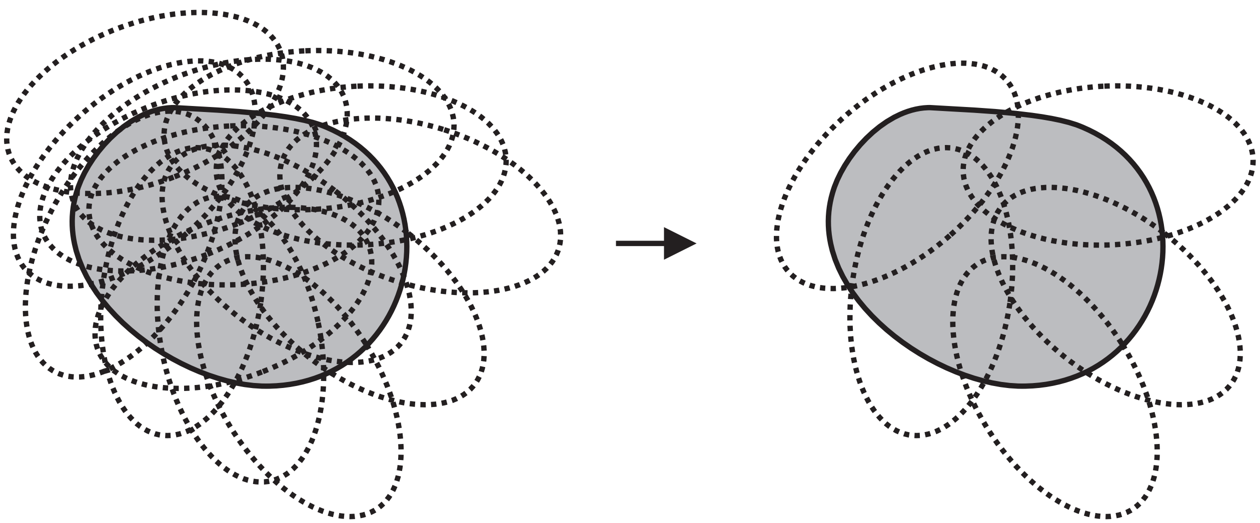
(a) Each component of X is closed in X

(b) Any connected subset of X is contained in a single component

□ **Definition [Compact space]** A topological space X is said to be compact if every open cover of X has a finite subcover. That is, if $\mathcal{U}$ is any given open cover of X, then there are finitely many sets $U_1, \ldots, U_k \in \mathcal{U}$ such that $X = U_1 \cup \cdots \cup U_k$

□ **Definition [Compact subset]** Let X be a topological space. A subset $A \subset X$ is said to be compact if the subspace A is compact.

In equivalent terms, the subset A is compact if and only if given any collection of open subsets of X covering A, there is a finite subcover



□ **Example:** the interval $A =]0, 1] \subset \mathbb{R}$ is not compact

☐ **Proposition [Characterization of compact sets in $\mathbb{R}^n$]** A subset X in $\mathbb{R}^n$ is compact if and only if X is closed and bounded

☐ **Example (Stiefel manifold):** the set

$$\mathsf{O}(n,m) = \{X \in \mathbb{R}^{n\times m} \,:\, X^\top X = I_m\}$$

is compact because it is closed and bounded.

▷ closed because $\mathsf{O}(n,m) = f^{-1}(\{I_m\})$ and

$$f \,:\, \mathbb{R}^{n\times m} \to \mathbb{R}^{m\times m} \qquad f(X) = X^\top X$$

is continuous

▷ bounded because if $X \in \mathsf{O}(n,m)$ then $\|X\|^2 = \mathsf{tr}(X^\top X) = \mathsf{tr}(I_m) = m$

Note that $\mathsf{O}(n,1) = \mathsf{S}^{n-1}(\mathbb{R})$ and $\mathsf{O}(n,n) = \mathsf{O}(n)$

☐ **Theorem [Main theorem on compactness]** Let X, Y be topological spaces and let $f \,:\, X \to Y$ be a continuous map. If X is compact, then $f(X)$ (as a subspace of Y) is compact

☐ **Example (projective space $\mathbb{RP}^n$):** the projective space $\mathbb{RP}^n$ is compact because it is the image of the compact set $\mathsf{S}^n(\mathbb{R})$ through the continuous projection map $\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n$

☐ **Proposition [Properties of compact spaces]**

(a) Every closed subset of a compact space is compact

(b) In a Hausdorff space X, compact sets can be separated by open sets. That is, if $A, B \subset X$ are disjoint compact subsets, there exist disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$

(c) Every compact subset of a Hausdorff space is closed

(d) The Cartesian product of finitely many compact topological spaces is compact

(e) Any quotient space of a compact topological space is compact

☐ **Example (special orthogonal matrices):**

$$\mathsf{SO}(n) = \{X \in \mathsf{O}(n) \,:\, \det X = 1\}$$

is compact because it is a closed subset of the compact space $\mathsf{O}(n)$.

It is closed because $\mathsf{SO}(n) = f^{-1}(\{1\})$ and

$$f \,:\, \mathsf{O}(n) \to \mathbb{R} \qquad f(X) = \det X$$

is continuous

☐ **Theorem [Extreme value theorem]** If X is a compact space and $f \,:\, X \to \mathbb{R}$ is continuous, then f attains its maximum and minimum values on X

☐ **Proposition [Characterization of compactness in 2nd countable Hausdorff spaces]** Let X be a 2nd countable Hausdorff space. The following are equivalent:

(a) X is compact

(b) Every sequence in X has a subsequence that converges to a point in X

☐ **Example:** continuity of singular values

▷ **Lemma** Let X, Y be 2nd countable Hausdorff spaces. Furthermore, let Y be compact. Let $F : X \times Y \to \mathbb{R}$ be a continuous function. For each $x \in X$, we define the function $F_x : Y \to \mathbb{R}$, $F_x(y) = F(x, y)$. The function

$$f : X \to \mathbb{R} \quad f(x) = \max_{y \in Y} F_x(y)$$

is continuous

▷ The function $\lambda_{\mathsf{max}} : \mathsf{S}(n, \mathbb{R}) \to \mathbb{R}$, $X \mapsto \lambda_{\max}(X)$ is continuous

▷ For $A \in \mathsf{S}(n, \mathbb{R})$, order its eigenvalues

$$\underbrace{\lambda_n(A)}_{\lambda_{\mathsf{min}}(A)} \leq \lambda_{n-1}(A) \leq \cdots \leq \lambda_2(A) \leq \underbrace{\lambda_1(A)}_{\lambda_{\mathsf{max}}(A)}$$

The function $\lambda_k : \mathsf{S}(n, \mathbb{R}) \to \mathbb{R}$, $X \mapsto \lambda_k(X)$ is continuous

▷ For $A \in \mathbb{R}^{n \times m}$, order its singular values

$$\sigma_p(A) \leq \cdots \leq \sigma_2(A) \leq \underbrace{\sigma_1(A)}_{\sigma_{\mathsf{max}}(A)} \quad (p = \min\{n, m\})$$

The function $\sigma_k : \mathbb{R}^{n \times m} \to \mathbb{R}$, $X \mapsto \sigma_k(X)$ is continuous

□ **Example:** Principal Component Analysis (PCA) is a continuous map

▷ **Lemma** Let X, Y be 2nd countable Hausdorff spaces. Furthermore, let Y be compact. Let $F : X \times Y \to \mathbb{R}$ be a continuous function. For each $x \in X$, we define the function $F_x : Y \to \mathbb{R}$, $F_x(y) = F(x, y)$. Suppose that, for each $x \in X$, there exists only one global minimizer in Y of the function F_x. Let $\phi : X \to Y$ be the map which, given $x \in X$, returns the (unique) global minimizer in Y of the function F_x. The map ϕ is continuous.

▷ Let $P = [\, p_1 \, p_2 \, \ldots \, p_k] \in \mathbb{R}^{n \times k}$ denote a constellation of k points in $\mathbb{R}^n$. A one-dimensional principal component analysis (PCA) of P consists in extracting the "dominant" straight line in P, i.e., the straight line spanned by a vector

$$
\begin{aligned}
\widehat{x}(P) \quad & \in \quad \underset{x \in \mathbb{R}^n - \{0\}}{\arg\min} \; \sum_{j=1}^{k} \left\| p_j - \frac{x x^\top}{\|x\|^2} p_j \right\|^2 \\
& = \quad \underset{x \in \mathbb{R}^n - \{0\}}{\arg\max} \; \frac{x^\top P P^\top x}{\|x\|^2}
\end{aligned}
$$

▷ The straight line is unique if $\lambda_{\mathsf{max}}(PP^\top)$ is simple: order the eigenvalues

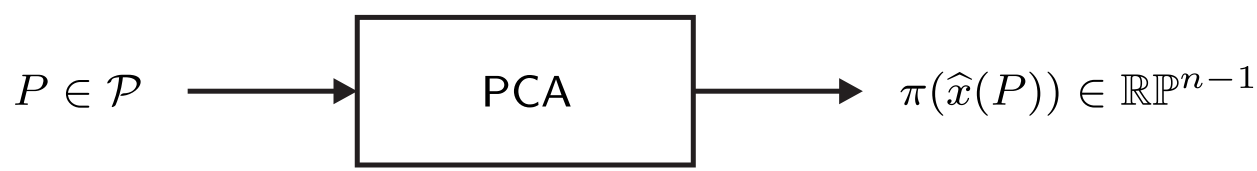
$$\underbrace{\lambda_n(PP^\top)}_{\lambda_{\mathsf{min}}(PP^\top)} \le \lambda_{n-1}(PP^\top) \le \cdots \le \lambda_2(PP^\top) \le \underbrace{\lambda_1(PP^\top)}_{\lambda_{\mathsf{max}}(PP^\top)}$$

The dominant straight line is unique for those constellations P belonging to

$$\mathcal{P} = \left\{ P \in \mathbb{R}^{n \times k} \;:\; \lambda_1(PP^\top) > \lambda_2(PP^\top) \right\}$$

Note that the set $\mathcal{P}$ is open in $\mathbb{R}^{n \times k}$

▷ We have a map PCA : $\mathcal{P} \to \mathbb{RP}^{n-1}$



▷ The map PCA is continuous, because

Step 1: The map

$$F : \mathbb{RP}^{n-1} \times \mathbb{R}^{n \times k} \to \mathbb{R} \qquad F([x], P) = \sum_{j=1}^{k} \left\| p_j - \frac{xx^T}{\|x\|^2} p_j \right\|^2$$

is continuous (as we have already seen in a previous example)

Step 2: Its restriction to the subspace $\mathbb{RP}^{n-1} \times \mathcal{P} \subset \mathbb{RP}^{n-1} \times \mathbb{R}^{n \times k}$ is also continuous (for brevity of notation, we keep the same symbol F):

$$F : \mathbb{RP}^{n-1} \times \mathcal{P} \to \mathbb{R} \qquad F([x], P) = \sum_{j=1}^{k} \left\| p_j - \frac{xx^T}{\|x\|^2} p_j \right\|^2 .$$

Step 3: PCA : $\mathcal{P} \to \mathbb{RP}^{n-1}$ extracts, for each $P \in \mathcal{P}$, the (unique) minimizer of F_P in $\mathbb{RP}^{n-1}$. Since $\mathbb{RP}^{n-1}$ is compact, the last lemma shows that PCA is continuous