Nonlinear Signal Processing
2007-2008

New Spaces from Old

(Ch.3, "Introduction to Topological Manifolds", J. Lee, Springer-Verlag)

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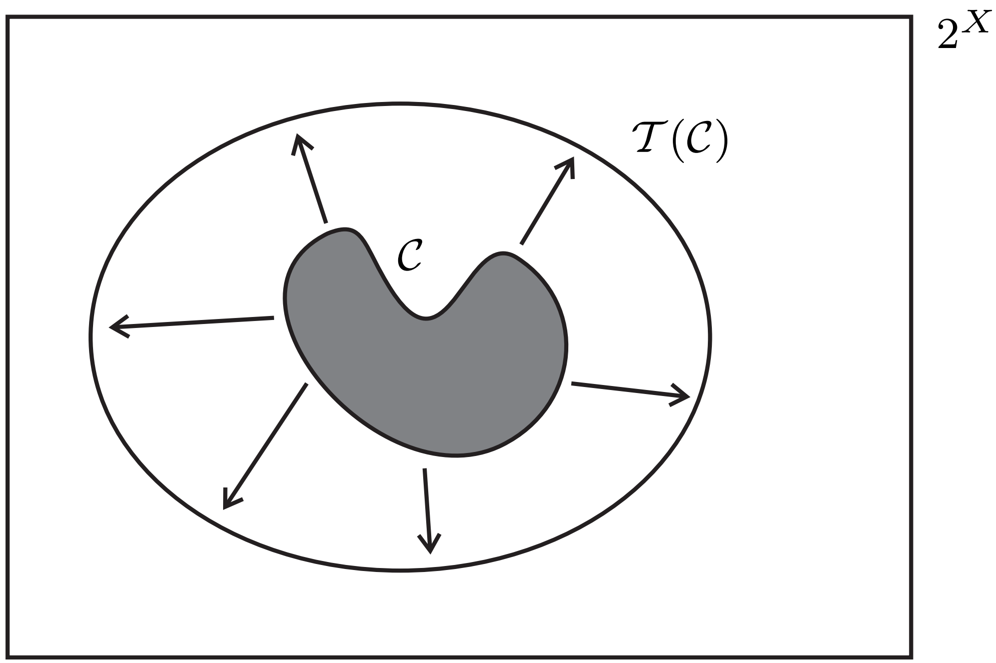
Lecture's key-points

☐ 3 canonical mechanisms to construct new topological spaces from old ones:

- ▷ subspaces
- ▷ Cartesian products
- ▷ quotients

☐ Questions about continuity in the new spaces can be answered in the old ones

☐ **Definition [Topology generated by a class of subsets]** Let X be a nonempty set and $\mathcal{C}$ a class of subsets of X. The topology generated by $\mathcal{C}$, written $\mathcal{T}(\mathcal{C})$, is defined as the smallest topology containing the class $\mathcal{C}$



☐ **Example:** if $X = \{a, b, c\}$, $\mathcal{C} = \{\{a\}, \{c\}\}$ then $\mathcal{T}(\mathcal{C}) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$

□ **Lemma [Characterization of generated topologies]** Let X be a nonempty set and $\mathcal{C}$ a class of subsets of X. Then $\mathcal{T}(\mathcal{C})$ is the class of all unions of finite intersections of sets in $\mathcal{C}$.

That is, $U \in \mathcal{T}(\mathcal{C})$ if and only if

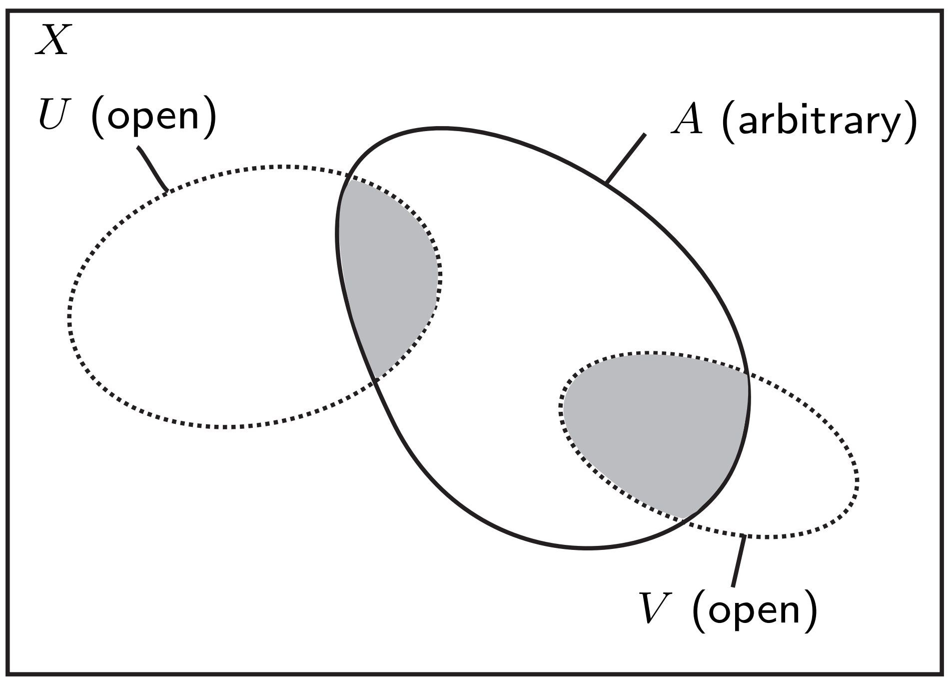
$$U = \bigcup_{\alpha \in A} U_\alpha, \qquad U_\alpha = C_\alpha^1 \cap C_\alpha^2 \cap \cdots C_\alpha^n, \qquad\qquad C_\alpha^i \in \mathcal{C}.$$

Also, the collection $\{U_\alpha\}$ is a basis for $\mathcal{T}(\mathcal{C})$.

☐ **Definition [Subspace topology]** Let X be a topological space and $A \subset X$ be any subset. The subspace topology $\mathcal{T}_A$ on A is defined as

$$\mathcal{T}_A = \{A \cap U \; : \; U \text{ open in } X\}\,.$$

Let $A \subset X$ be any subset. By the subspace A of X we mean the topological space $(A, \mathcal{T}_A)$ where $\mathcal{T}_A$ is the subspace topology on A.



☐ **Example:** consider the subspace $A = [0,2[$ of $X = \mathbb{R}$

▷ The set $[0,1[$ is not open in X

▷ The set $[0,1[$ is open in A: it can be written as

$$[0,1[= \underbrace{[0,2[}_{A} \cap \underbrace{]-1,1[}_{U\text{: open in } X}$$

☐ **Example (unit sphere):** consider the subspace

$$\mathsf{S}^{n-1}(\mathbb{R}) = \{x \in \mathbb{R}^n \,:\, \|x\| = 1\}$$

of $\mathbb{R}^n$.

▷ The set $U_i^+ = \left\{x \in \mathsf{S}^{n-1}(\mathbb{R}) \,:\, x_i > 0\right\}$ is open in $\mathsf{S}^{n-1}(\mathbb{R})$:

$$U_i^+ = \mathsf{S}^{n-1}(\mathbb{R}) \cap \underbrace{\{x \in \mathbb{R}^n \,:\, x_i > 0\}}_{\text{open in } \mathbb{R}^n}$$

☐ **Example (orthogonal matrices):** consider the subspace

$$\mathsf{O}(n) = \{X \in \mathbb{R}^{n \times n} \; : \; X^\top X = I_n\}$$

of $\mathbb{R}^{n \times n}$.

▷ The set $\mathsf{SO}(n) = \{X \in \mathsf{O}(n) \; : \; \det(X) > 0\}$ is open in $\mathsf{O}(n)$:

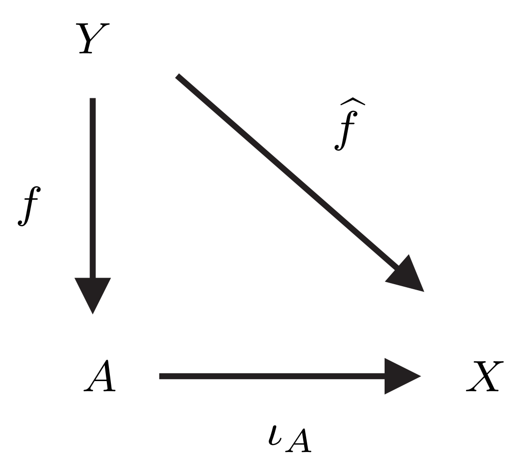
$$\mathsf{SO}(n) = \mathsf{O}(n) \; \cap \; \underbrace{\left\{X \in \mathbb{R}^{n \times n} \; : \; \det(X) > 0\right\}}_{\text{open in } \mathbb{R}^{n \times n}}$$

☐ **Trivial but important fact:** if A is a subspace of X, then

$$\iota_A : A \to X \qquad \iota_A(x) = x$$

is a continuous map

☐ **Theorem [Characteristic property of subspace topologies]** Let X, Y be topological spaces. Let A be a subspace of X. Then, a map $f : Y \to A$ is continuous if and only if $\widehat{f} = \iota_A \circ f$ is continuous

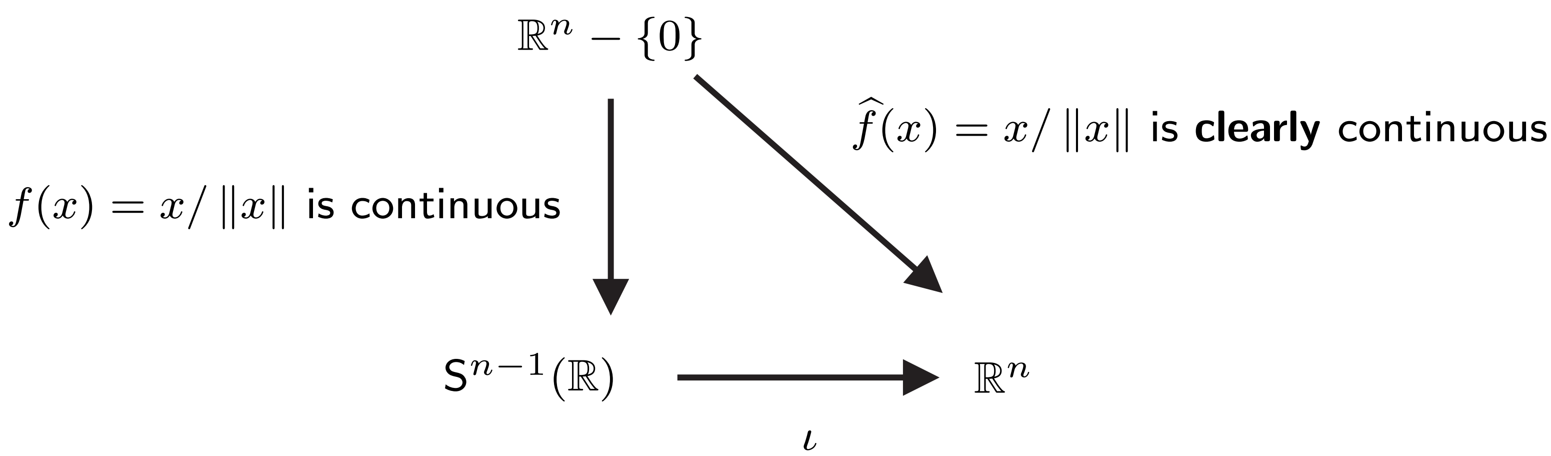


∗ *Intuition: continuity of the "hard" map f can be investigated through the easier $\widehat{f}$*

☐ **Example (map into the unit sphere):** the map

$$f : \mathbb{R}^n - \{0\} \to \mathsf{S}^{n-1}(\mathbb{R}) \qquad f(x) = \frac{x}{\|x\|}$$

is continuous



☐ **Definition [Topological embedding]** Let X and Y be topological spaces. An injective, continuous map $f : X \to Y$ is said to be a topological embedding if it is a homeomorphism onto its image $f(X)$ (endowed with the subspace topology)

$*$ *Intuition: we can interpret $X \simeq f(X)$ as a subspace of Y (X is simply another label for a subspace of Y)*

☐ **Example:** consider

▷ $X = \{1, 2, 3\}$ with the discrete topology $\mathcal{T}_X = 2^X$

▷ $Y = \{1, 2, 3, 4, \ldots, 10\}$ with the trivial topology $\mathcal{T}_Y = \{\emptyset, Y\}$

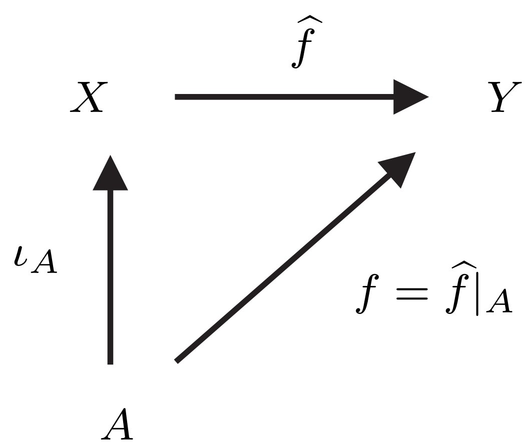
▷ the map $f : X \to Y$, $f(x) = x$ is injective and continuous

▷ the map f is **not** a homeomorphism onto the subspace $f(X) = \{1, 2, 3\}$ of Y

☐ **Lemma [Other properties of the subspace topology]** Suppose A is a subspace of the topological space X.

(a) The inclusion map $\iota_A : A \to X$ is continuous (in fact, a topological embedding)

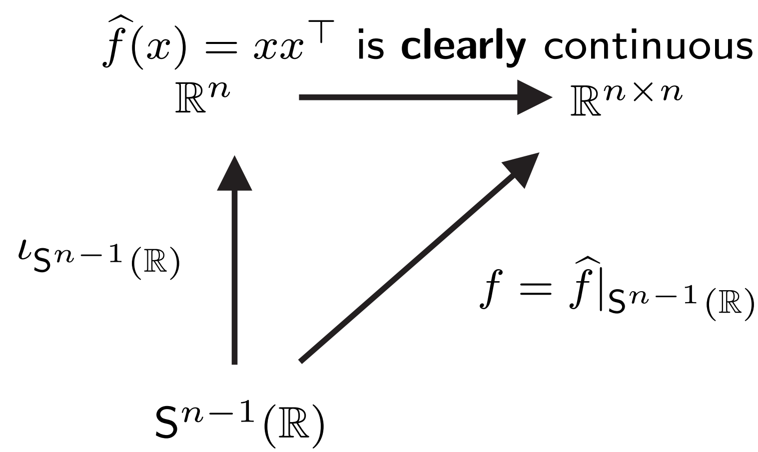
(b) If $\widehat{f} : X \to Y$ is continuous then $f = \widehat{f}|_A : A \to Y$ is continuous



☐ **Example (map out of the unit sphere):** the map

$$f\ :\ \mathsf{S}^{n-1}(\mathbb{R}) \to \mathbb{R}^{n\times n} \qquad\qquad f(x) = xx^\top$$

is continuous



☐ **Example (concatenating the techniques...):** the map

$$f \,:\, \mathsf{O}(n) \to \mathsf{S}^{n-1}(\mathbb{R}) \qquad\qquad f(X) = f([\,x_1\, x_2\, \cdots\, x_n\,]) = x_1$$

is continuous because

Step 1:

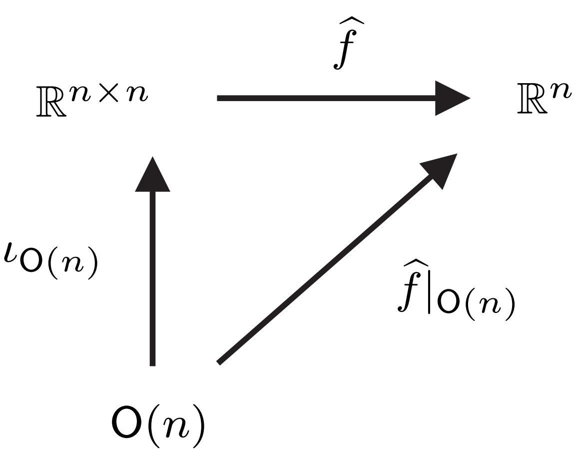
$$\widehat{f} \,:\, \mathbb{R}^{n\times n} \to \mathbb{R}^n \qquad\qquad f(X) = x_1$$

is **clearly** continuous

Step 2:

$$\widehat{f}|_{\mathsf{O}(n)} \,:\, \mathsf{O}(n) \to \mathbb{R}^n \qquad\qquad \widehat{f}|_{\mathsf{O}(n)}(X) = x_1$$

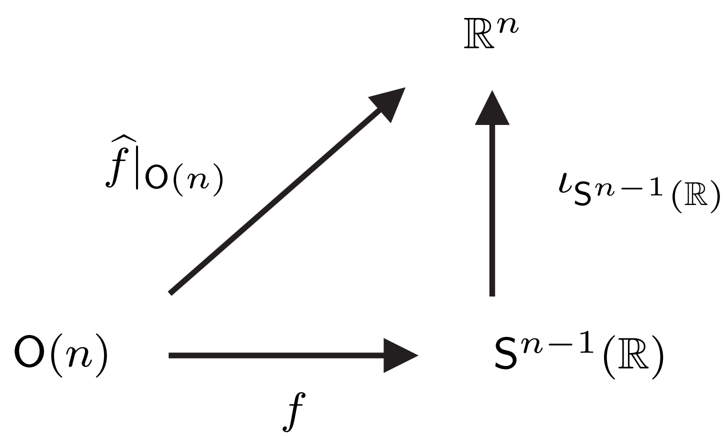
is continuous due to



Step 3:

$$f\ :\ \mathsf{O}(n) \to \mathsf{S}^{n-1}(\mathbb{R}) \qquad\qquad f(X) = x_1$$

is continuous due to



□ **Lemma [Other properties of the subspace topology (cont.)]**

(c) If $B \subset A$ is a subspace of A, then B is a subspace of X; in other words, the subspace topologies that B inherits from A and from X agree

(d) If $\mathcal{B}$ is a basis for the topology of X, then

$$\mathcal{B}_A = \{B \cap A \,:\, B \in \mathcal{B}\}$$

is a basis for the topology of A

(e) If X is Hausdorff and 2nd countable then A is Hausdorff and 2nd countable

□ **Example (a simple topological manifold):** $\mathsf{S}^{n-1}(\mathbb{R})$ is a topological manifold of dimension $n-1$

☐ **Example (another topological manifold?):** the set of 2×2 special orthogonal matrices

$$\mathsf{SO}(2) = \left\{ X \in \mathbb{R}^{2\times 2} \,:\, X^\top X = I_2,\, \det(X) = 1 \right\}$$

is a topological manifold of dimension 1 because the map

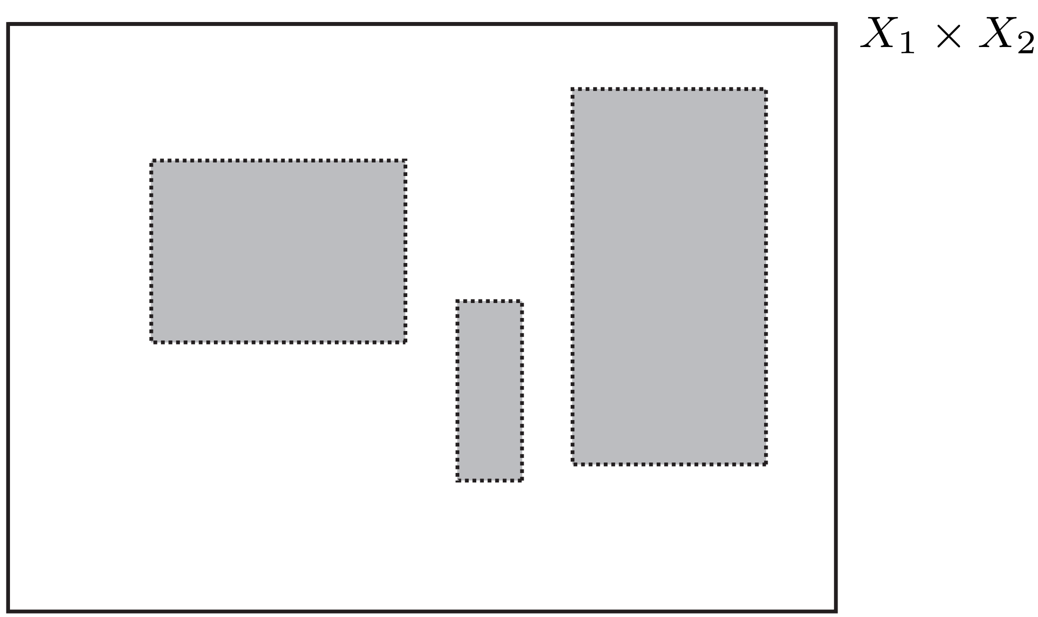
$$f \,:\, \mathsf{S}^1(\mathbb{R}) \to \mathsf{SO}(2) \qquad\qquad f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

is a homeomorphism

☐ **Definition [Product topology]** Let $X_1, X_2, \ldots, X_n$ be topological spaces. The product topology on the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ is the topology generated by the collection of rectangles

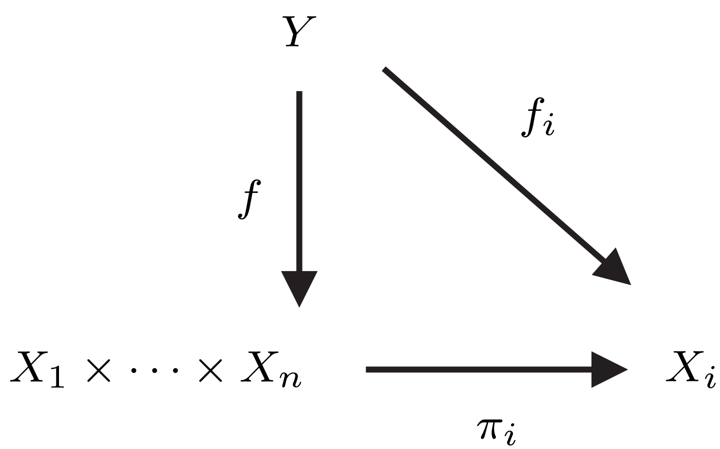
$$\mathcal{C} = \{U_1 \times U_2 \times \cdots U_n \ : \ U_i \text{ is open in } X_i\}.$$

The set $X_1 \times \cdots \times X_n$ equipped with the product topology is called a product space.



☐ Note that $\mathcal{C}$ is a basis for the product topology

☐ **Theorem [Characteristic property of product topologies]** Let $X_1 \times \cdots \times X_n$ be a product space and let Y be a topological space. Then, the map $f : Y \to X_1 \times \cdots \times X_n$ is continuous if and only if each map $f_i : Y \to X_i$, $f_i = \pi_i \circ f$ is continuous



☐ $\pi_i : X_1 \times X_2 \times \cdots \times X_n \to X_i$, $\pi_i(x_1, x_2, \ldots, x_n) = x_i$ denotes the projection map onto the ith factor X_i

☐ **Example (decomposing a vector in amplitude and direction):** the map

$$f : \mathbb{R}^n - \{0\} \to \mathbb{R}^+ \times \mathsf{S}^{n-1}(\mathbb{R}) \qquad f(x) = \left(\|x\| \, , \frac{x}{\|x\|} \right)$$

is continuous because

- ▷ $f_1 : \mathbb{R}^n - \{0\} \to \mathbb{R}^+$, $f_1(x) = \|x\|$ is continuous
- ▷ $f_2 : \mathbb{R}^n - \{0\} \to \mathsf{S}^{n-1}(\mathbb{R})$, $f_2(x) = \frac{x}{\|x\|}$ is continuous

☐ **Lemma [Other properties of the product topology]** Let $X_1, \ldots, X_n$ be topological spaces.

(a) The projection maps $\pi_i : X_1 \times \cdots \times X_n \to X_i$ are continuous and open

(b) Let $x_j \in X_j$ be fixed for $j \neq i$. The map

$$f : X_i \to X_1 \times \cdots \times X_n, \qquad f(x) = (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)$$

is a topological embedding

☐ **Lemma [Other properties of the product topology (cont.)]**

(c) If $\mathcal{B}_i$ is a basis for the topology of X_i, then the class

$$\mathcal{B} = \{B_1 \times \cdots \times B_n \,:\, B_i \in \mathcal{B}_i\}$$

is a basis for the topology of the product space $X_1 \times \cdots \times X_n$

(d) If A_i is a subspace of X_i, for $i = 1, \ldots, n$, the product topology and the subspace topology on $A_1 \times \cdots \times A_n \subset X_1 \times \cdots \times X_n$ are identical

(e) If each X_i is Hausdorff and second countable then the product space $X_1 \times \cdots \times X_n$ is also Hausdorff and second countable

☐ **Definition [Product map]** If $f_i \,:\, X_i \to Y_i$ are maps for $i = 1, \ldots, n$, their product map, written $f_1 \times \cdots \times f_n$, is defined as

$$f_1 \times \cdots \times f_n \,:\, X_1 \times \cdots \times X_n \to Y_1 \times \cdots \times Y_n,$$

$$(f_1 \times \cdots \times f_n)(x_1, \ldots, x_n) = (f_1(x_1), \ldots, f_n(x_n)) \,.$$

☐ **Proposition [Product map]** A product of continuous maps is continuous, and a product of homeomorphisms is a homeomorphism

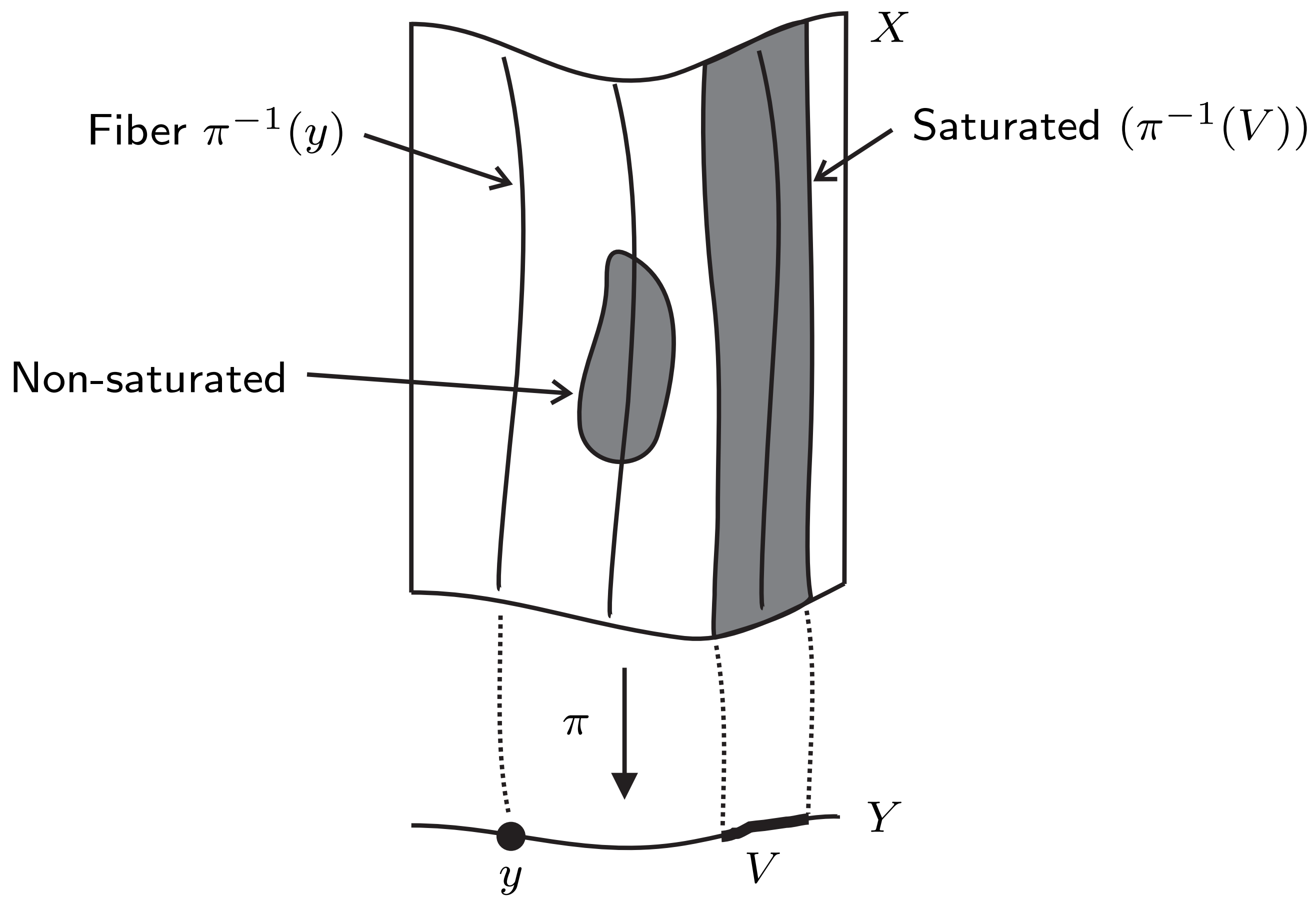
☐ **Proposition [Product manifolds]** If $M_1, \ldots, M_k$ are topological manifolds of dimensions $n_1, \ldots, n_k$, respectively, the product space $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$

▷ *Intuition: if each M_i has n_i "degrees of freedom", then $M_1 \times \cdots \times M_k$ has $n_1 + \cdots + n_k$ "degrees of freedom"*

☐ **Definition [Saturated sets, fibers]** Let X and Y be sets and $\pi : X \to Y$ be a surjective map.

A subset $\pi^{-1}(y) \subset X$ for $y \in Y$ is called a fiber of π.

A subset $U \subset X$ is saturated if $U = \pi^{-1}(V)$ for some $V \subset Y$ (U=union of fibers)



☐ **Example:** consider the surjective map

$$\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}_0^+ \qquad \pi(x) = \|x\|$$

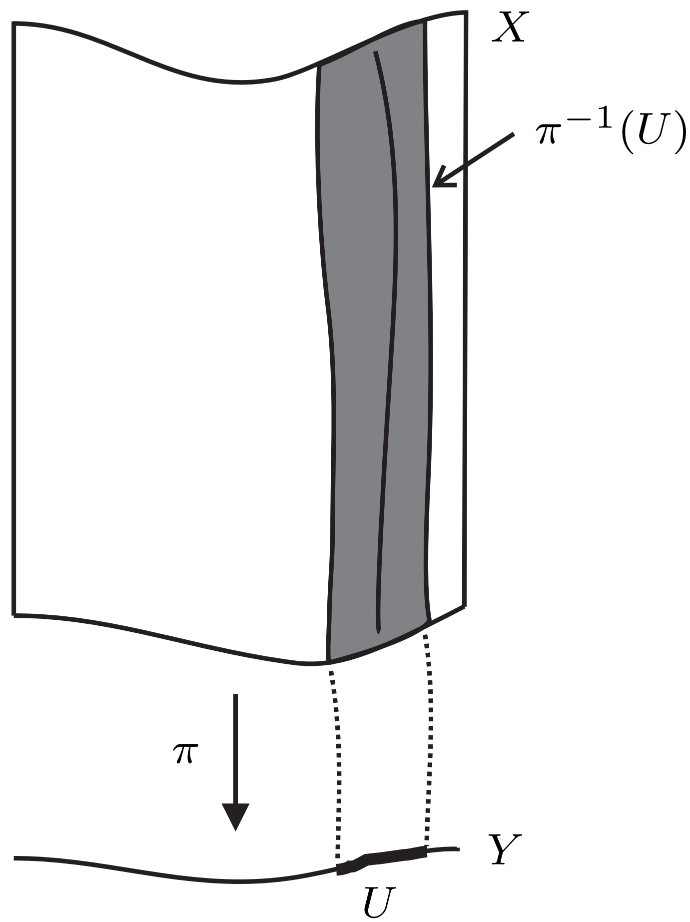
▷ the fibers of π are the circles centered at the origin and the origin itself

▷ the annulus $U = \{x \in \mathbb{R}^2 : 1 < \|x\| \leq 2\}$ is a saturated set

▷ each coordinate axis of $\mathbb{R}^2$ is non-saturated

☐ **Definition [Quotient topology]** Let X be a topological space, Y be any set, and $\pi : X \to Y$ be a surjective map. The quotient topology on Y induced by the map π is defined as

$$\mathcal{T}_\pi = \{U \subset Y \,:\, \pi^{-1}(U) \text{ is open in } X\}$$



☐ **Example (real projective space $\mathbb{RP}^n$):** introduce the equivalence relation $\sim$ in $X = \mathbb{R}^{n+1} - \{0\}$

$$x \sim y \qquad \text{if and only if} \qquad x = \lambda y \text{ for some } \lambda \neq 0$$

- ▷ let $\mathbb{RP}^n = X/\sim$ denote the set of equivalence classes
- ▷ the map $\pi : X \to \mathbb{RP}^n$, $x \mapsto \pi(x) = [x]$ is surjective
- ▷ $\mathbb{RP}^n$ becomes a topological space by letting π induce the quotient topology
- ▷ the fibers of π are straight lines in $\mathbb{R}^{n+1} - \{0\}$

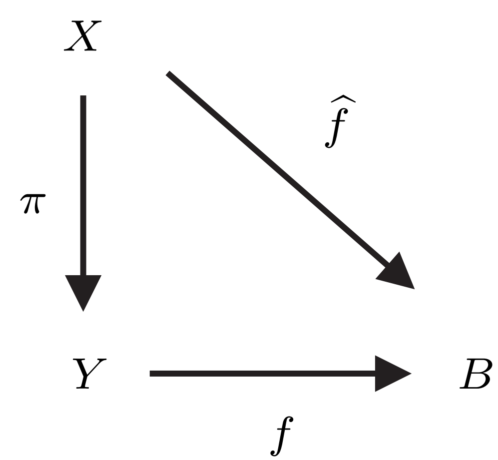
☐ **Definition [Quotient map]** Let X and Y be topological spaces. A surjective map $f : X \to Y$ is called a quotient map if the topology of Y coincides with $\mathcal{T}_f$ (the quotient topology induced by f). This is equivalent to saying that U is open in Y if and only if $f^{-1}(U)$ is open in X

☐ **Lemma [Characterization of quotient maps]** Let X and Y be topological spaces. A continuous surjective map $f : X \to Y$ is a quotient map if and only if it takes saturated open sets to open sets, or saturated closed sets to closed sets

☐ **Lemma [Easy sufficient conditions for quotient maps]** If $f : X \to Y$ is a surjective continuous map that is also an open or closed map, then it is a quotient map

☐ **Lemma [Composition property of quotient maps]** Suppose $\pi_1 : X \to Y$ and $\pi_2 : Y \to Z$ are quotient maps. Then their composition $\pi_2 \circ \pi_1 : X \to Z$ is also a quotient map

☐ **Theorem [Characteristic property of quotient topologies]** Let $\pi : X \to Y$ be a quotient map. For any topological space B, a map $f : Y \to B$ is continuous if and only if $\widehat{f} = f \circ \pi$ is continuous.



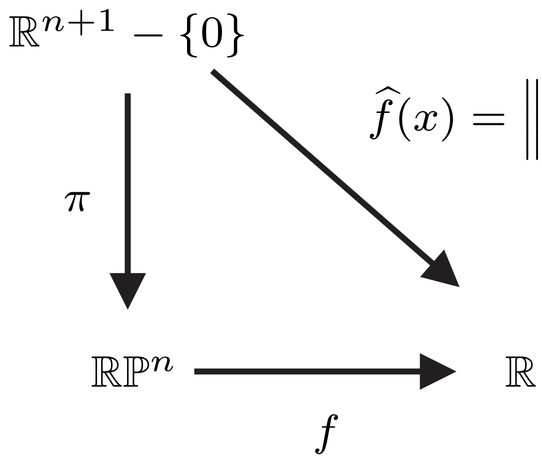
∗ *Intuition: continuity of the "hard" map f can be investigated through the easier $\widehat{f}$*

☐ **Example (real projective space $\mathbb{RP}^n$)**: for $[x] \in \mathbb{RP}^n$, let line$([x])$ be the straight line spanned by x and let $x_0 \in \mathbb{R}^{n+1}$ be fixed

The map

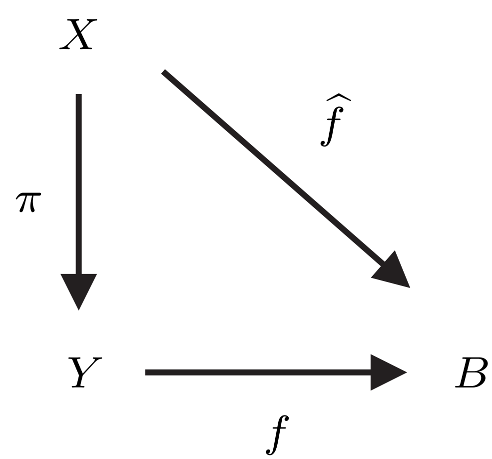
$$f : \mathbb{RP}^n \to \mathbb{R} \qquad f([x]) = \mathsf{dist}(x_0, \mathsf{line}([x]))$$

is continuous



$$\widehat{f}(x) = \left\| \left(I_n - \frac{xx^\top}{\|x\|^2} \right) x_0 \right\|$$ is **clearly** continuous

☐ **Corollary [Passing to the quotient]** Suppose $\pi : X \to Y$ is a quotient map, B is a topological space, and $\widehat{f} : X \to B$ is any continuous map that is constant on the fibers of π (that is, if $\pi(p) = \pi(q)$ then $\widehat{f}(p) = \widehat{f}(q)$). Then, there exists an unique continuous map $f : Y \to B$ such that $\widehat{f} = f \circ \pi$:



☐ **Example (elementary descent to $\mathbb{RP}^n$):** let $x_0 \in \mathbb{R}^{n+1} - \{0\}$ be fixed. The map

$$\widehat{f} : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R} \qquad \widehat{f}(x) = \text{arccos}\left(\frac{|x_0^T x|}{\|x_0\| \, \|x\|}\right)$$

is continuous and descends to a continuous map in $\mathbb{RP}^n$

□ **Definition [Group]** A group is an ordered pair $(G, *)$ consisting of a set G and a binary operation $* : G \times G \to G$ such that

(a) **(associativity)** for every $x, y, z \in G$ we have $(x * y) * z = x * (y * z)$

(b) **(identity)** there is $e \in G$ such that $e * x = x * e = x$ for all $x \in G$

(c) **(inverse)** for each $x \in G$ there is a $y \in G$ such that $x * y = y * x = e$

If understood from the context,

▷ $(G, *)$ is simply denoted by G

▷ we write xy instead of $x * y$

□ **Lemma [Elementary properties of groups]** Let $(G, *)$ be a group.

(a) The identity element is unique (and is usually denoted by e)

(b) The inverse is unique (and is usually denoted by x^{-1})

☐ **Example (general linear group):** $\mathsf{GL}(n, \mathbb{R}) = \{X \in \mathbb{R}^{n \times n} \,:\, \det X \neq 0\}$ is a group with matrix multiplication as the group operation

- ▷ the identity element of the group is I_n
- ▷ the inverse of A is A^{-1}

☐ **Example (group of orthogonal matrices):** $\mathsf{O}(n) = \{X \in \mathbb{R}^{n \times n} \,:\, X^\top X = I_n\}$ is a group with matrix multiplication as the group operation

☐ **Example (group of special orthogonal matrices):**

$$\mathsf{SO}(n) = \{X \in \mathsf{O}(n) \,:\, \det(X) = 1\}$$

is a group with matrix multiplication as the group operation

☐ **Example (upper triangular matrices with positive diagonal entries):**

$$U^+(n, \mathbb{R}) = \{X \in \mathbb{R}^{n \times n} : X \text{ is upper-triangular and } X_{ii} > 0 \text{ for all i } \}$$

is a group with matrix multiplication as the group operation

$\square$ **Example (group of rigid motions in $\mathbb{R}^n$):**

$$\mathsf{SE}(n) = \left\{ \begin{bmatrix} Q & \delta \\ 0 & 1 \end{bmatrix} : Q \in \mathsf{SO}(n), \delta \in \mathbb{R}^n \right\}$$

is a group with matrix multiplication as the group operation

$\triangleright$ the identity element of the group is

$$\begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}$$

$\triangleright$ the inverse of $\begin{bmatrix} Q & \delta \\ 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} Q^\top & -Q^\top \delta \\ 0 & 1 \end{bmatrix}$

☐ **Definition [Subgroup, left translation, right translation, homomorphism, kernel of a homomorphism]** Let $(G, *)$ be a group.

▷ a subgroup of G is a set $H \subset G$ such that $e \in H$, $x * y \in H$ whenever $x, y \in H$, and $x^{-1} \in H$ whenever $x \in H$

▷ for each $g \in G$, we define the left translation map $L_g : G \to G$, $L_g(x) = g * x$. Similarly, we have the right translation map $R_g : G \to G$, $R_g(x) = x * g$

▷ let $(H, \widetilde{*})$ denote a group with identity element $\widetilde{e}$. A map $F : G \to H$ is said to be a homomorphism if $F(x * y) = F(x) \widetilde{*} F(y)$ for all $x, y \in G$. The kernel of F is defined as

$$\mathsf{Ker}\, F = \{x \in G \,:\, F(x) = \widetilde{e}\}\,.$$

Note that Ker F is a subgroup of G.

☐ **Example (subgroups of the general linear group):** O(n), SO(n) and U$^+(n, \mathbb{R})$ are subgroups of GL$(n, \mathbb{R})$. SE(n) is a subgroup of GL$(n + 1, \mathbb{R})$

☐ **Example (homomorphism):** the map

$$F \,:\, \mathsf{GL}(n,\mathbb{R}) \to \mathsf{GL}(1,\mathbb{R}) \qquad\qquad F(X) = \det(X)$$

is a homomorphism. Its kernel is the subgroup

$$\mathsf{SL}(n,\mathbb{R}) = \{X \,:\, \det(X) = 1\}$$

☐ **Example (generalization of the previous result):** the map

$$F \,:\, \mathsf{GL}(n,\mathbb{R}) \to \mathsf{GL}\left(\begin{pmatrix} n \\ k \end{pmatrix}, \mathbb{R}\right) \qquad\qquad f(X) = X^{[k]}$$

is a homomorphism (Cauchy-Binet formula)

☐ **Definition [Topological group]** Let G be a group which is at the same time a topological space. Then, G is said to be a topological group if the maps

(a) $\iota : G \to G \quad \iota(x) = x^{-1}$

(b) $m : G \times G \to G \quad m(x,y) = xy$

are continuous

☐ **Examples:** GL$(n, \mathbb{R})$, O(n), SO(n), U$^+(n, \mathbb{R})$, SE(n) are topological groups

☐ **Definition [Group action]** Let G be a group and X be a set. A left action of G on X is a map $\theta : G \times X \to X$ such that

(a) $\theta(e, x) = x$ for all $x \in X$

(b) $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all $g, h \in G$ and $x \in X$

If the action θ is clear from the context, we use gx instead of $\theta(g, x)$

If G is a topological group and X is a topological space, the action is said to be continuous if θ is continuous.

☐ **Example ($\mathsf{GL}(n,\mathbb{R})$ acts on $\mathbb{R}^n$):** the map

$$\theta\,:\,\mathsf{GL}(n,\mathbb{R})\times\mathbb{R}^n\to\mathbb{R}^n \qquad \theta(A,x)=Ax$$

defines a continuous left action of $\mathsf{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$.

This is called the natural action of $\mathsf{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$.

☐ **Example ($\mathsf{O}(n)$ acts on $\mathsf{S}(n,\mathbb{R})$):** let

$$\mathsf{S}(n,\mathbb{R})=\{X\in\mathbb{R}^{n\times n}\,:\,X=X^\top\}$$

denote the set of $n\times n$ symmetric matrices with real entries.

The map

$$\theta\,:\,\mathsf{O}(n)\times\mathsf{S}(n,\mathbb{R})\to\mathsf{S}(n,\mathbb{R}) \qquad \theta(Q,S)=QSQ^\top$$

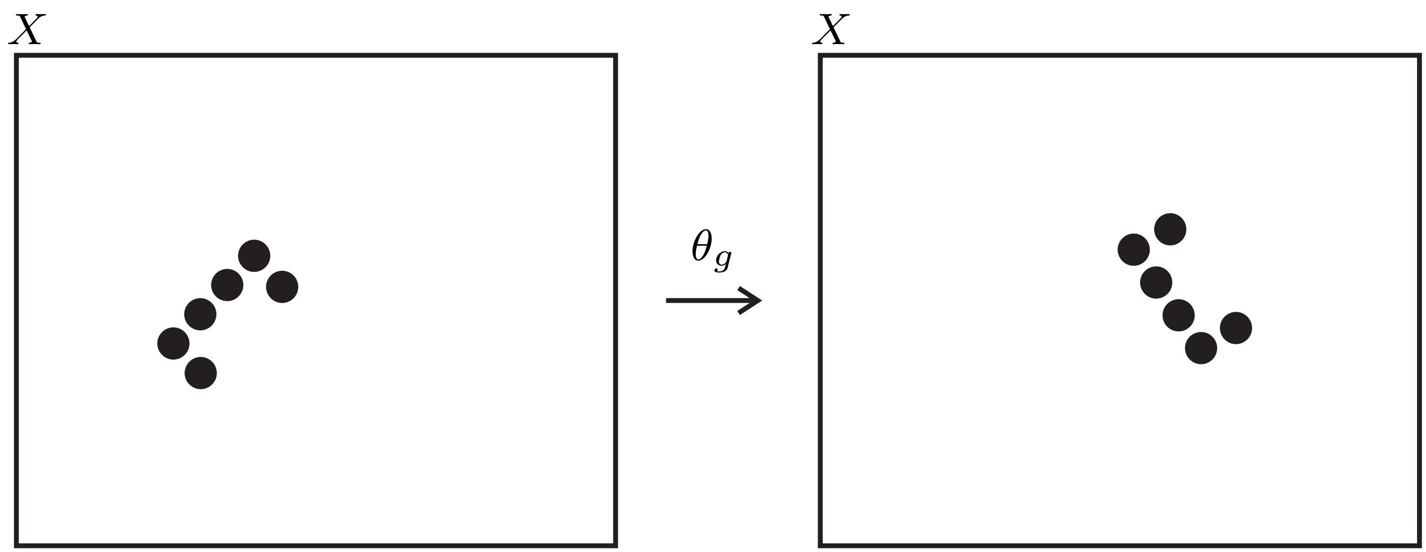
defines a continuous left action of $\mathsf{O}(n)$ on $\mathsf{S}(n,\mathbb{R})$

□ **Lemma [Continuous left actions]** Let $\theta : G \times X \to X$ be a continuous left action of G on X. For each $g \in G$, the map

$$\theta_g : X \to X \qquad \theta_g(x) = \theta(g,x) = gx$$

is a homeomorphism.

▷ *Proof: The map θ_g is bijective because the map $\theta_{g^{-1}}$ is a left and right inverse for it, that is, $\theta_g \circ \theta_{g^{-1}} = \theta_{g^{-1}} \circ \theta_g = \mathrm{id}_X$. The map θ_g is continuous because it is the composition of two continuous maps: $\theta_g = \theta \circ \iota_g$, where $\iota_g : G \to G \times X$, $\iota_g(x) = (g,x)$. It is a homeomorphism because its inverse is given by $\theta_{g^{-1}}$, which is continuous*□



☐ **Definition [Orbits,free/transitive actions,invariants,maximal invariants]** Let $\theta : G \times X \to X$ denote a left action of the group G on a set X.

▷ The orbit of $p \in X$ is the set $Gp = \{\theta(g, p) \, : \, g \in G\}$

▷ The action is said to be transitive if, for any given $p, q \in X$ there exists $g \in G$ such that $\theta(g, p) = q$

∗ *Intuition: there is only one orbit*

▷ The action is said to be free if $\theta(g, p) = p$ implies $g = e$

∗ *Intuition: each orbit is a "copy" of G*

▷ An invariant of the action is a map $\phi : X \to Y$ (where Y denotes a set) which is constant on orbits, that is, $x, y \in Gp$ imply $\phi(x) = \phi(y)$

A maximal invariant of the action is an invariant ϕ which differs from orbit to orbit, that is, $x \notin Gy$ implies $\phi(x) \neq \phi(y)$

∗ *Intuition: a maximal invariant permits to index the orbits*

☐ **Example:** the natural action of $\mathsf{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ is not transitive (it has two orbits, namely, $\{0\}$ and $\mathbb{R}^n - \{0\}$), it is not free and a maximal invariant is $\phi : \mathbb{R}^n \to \mathbb{R}$, $\phi(0) = 0$ and $\phi(x) = 1$ if $x \neq 0$

☐ **Example:** the action of $\mathsf{O}(n)$ on $\mathsf{S}(n, \mathbb{R})$ discussed above is not transitive, it is not free and a maximal invariant is $\phi : \mathsf{S}(n, \mathbb{R}) \to \mathbb{R}^n$,

$$\phi(S) = (\lambda_1(S), \lambda_2(S), \ldots, \lambda_n(S))^\top ,$$

where $\lambda_1(S) \geq \lambda_2(S) \geq \cdots \geq \lambda_n(S)$ denote the eigenvalues of S sorted in non-increasing order

☐ **Definition [Orbit space]** Let $\theta : G \times X \to X$ denote a continuous action of the topological group G on the topological space X.

Introduce an equivalence relation on X by declaring $x \sim y$ if they share the same orbit, that is, $x \sim y$ if and only if there exists $g \in G$ such that $y = \theta(g, x)$.

The set of equivalence classes is denoted by X/G and is called the orbit space of the action.

□ **Lemma [Orbit space]** Suppose the topological group G acts continuously on the left of the topological space X. Let X/G be given the quotient topology.

(a) The projection map $\pi : X \to X/G$ is open

(b) If X is second countable, then X/G is second countable

(c) X/G is Hausdorff if and only if the set

$$A = \{(p, q) \in X \times X \;:\; q = \theta(g, p) \text{ for some } g \in G\}$$

is closed in $X \times X$

▷ *Proof: (a) Let U be open in X. We must show that $\pi(U)$ is open in X/G, that is, $V = \pi^{-1}(\pi(U))$ is open in X. But*

$$V = \bigcup_{g \in G} \theta_g(U),$$

where $\theta_g : X \to X$, $\theta_g(x) = gx$. Since each θ_g is a homeomorphism, $\theta_g(U)$ is open in X. Thus, V is open in X. (b) If $\mathcal{B}$ is a countable basis for X, then $\pi(\mathcal{B}) = \{\pi(B) : B \in \mathcal{B}\}$ is a countable basis for X/G. (c) (⇒) Let $(x, y) \notin A$. Thus, x and y lie in distinct orbits, that is, $\pi(x) \neq \pi(y)$. Since X/G is Hausdorff, let U and V be disjoint neighborhoods of $\pi(x)$ and $\pi(y)$, respectively. Then,

$\pi^{-1}(U) \times \pi^{-1}(V)$ is open in $X \times X$, contains (x, y) and does not intersect A (why?). Thus, the complement of A in $X \times X$ is open. ($\Leftarrow$) Let $\pi(x)$ and $\pi(y)$ be two distinct points in X/G. Then, $(x, y) \notin A$. Let U and V be neighborhoods of x and y, respectively, such that $U \times V$ does not intersect A. Then, $\pi(U)$ and $\pi(V)$ are disjoint neighborhoods of $\pi(x)$ and $\pi(y)$, respectively (why?) $\square$

$\square$ **Example (projective space $\mathbb{RP}^n$):** let $G = \mathsf{GL}(1, \mathbb{R})$ act continuously on $X = \mathbb{R}^{n+1} - \{0\}$ as $\theta : G \times X \to X$, $\theta(\lambda, x) = \lambda x$. Then, $\mathbb{RP}^n = X/G$.

- ▷ $\mathbb{RP}^n$ is second countable
- ▷ $\mathbb{RP}^n$ is Hausdorff because

$$A = \{(x, y) \in X \times X \, : \, x \text{ and } y \text{ are in the same orbit}\}$$

is closed: it can be written as $A = f^{-1}(\{0\})$ where f if the continuous map

$$f \, : \, X \times X \to \mathbb{R} \qquad f(x, y) = (x^\top x)(y^\top y) - (x^\top y)^2$$

□ **Lemma [Product of open maps is open]** Let A, B, X, Y be topological spaces. Let $f : A \to X$ and $g : B \to Y$ be open maps. Then, the product map

$$f \times g : A \times B \to X \times Y \qquad (f \times g)(a, b) = (f(a), g(b))$$

is open.

▷ *Proof: Let W be an open set in $A \times B$. Then, W may be written as a union of rectangles*

$$W = \bigcup_i U_i \times V_i,$$

where each U_i in open in A and each V_i is open in B. We have

$$(f \times g)(W) = (f \times g)\left(\bigcup_i U_i \times V_i\right) = \bigcup_i (f \times g)(U_i \times V_i) = \bigcup_i f(U_i) \times g(V_i).$$

Since $f(U_i)$ is open in X and $g(V_i)$ is open in Y (by hypothesis), then $f(U_i) \times g(V_i)$ is open in $X \times Y$. Since W is an union of open sets, it is open□

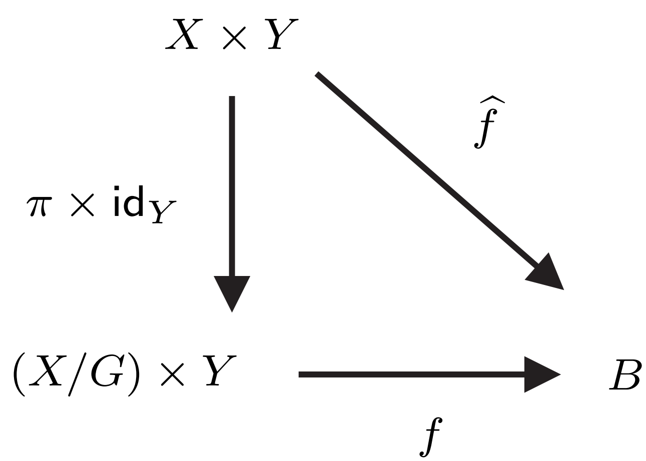
☐ **Lemma [Hybrid spaces]** Let the topological group G act continuously on the left of the topological space X. Let the orbit space X/G be given the quotient topology and let $\pi : X \to X/G$ be the corresponding projection map. Let Y be any topological space. Then, the map

$$\pi \times \mathrm{id}_Y : X \times Y \to (X/G) \times Y \qquad (\pi \times \mathrm{id}_Y)(x, y) = (\pi(x), y)$$

is a quotient map.

▷ *Proof: To abbreviate notation, let $f = \pi \times \mathrm{id}_Y$. The map f is clearly surjective and continuous. Thus, if we show that f is an open map, we are done. Now, both $\pi : X \to X/G$ and $\mathrm{id}_Y : Y \to Y$ are open maps. Since $f = \pi \times \mathrm{id}_Y$ is the product of open maps, it is itself open*☐

☐ **Corollary [Hybrid spaces]** Let the topological group G act continuously on the left of the topological space X. Let the orbit space X/G be given the quotient topology and let $\pi : X \to X/G$ be the corresponding projection map. Let Y and B be any topological spaces. Then, the map $f : (X/G) \times Y \to B$ is continuous if and only if the map $\widehat{f} : X \times Y \to B$, $\widehat{f} = f \circ (\pi \times \mathsf{id}_Y)$ is continuous.



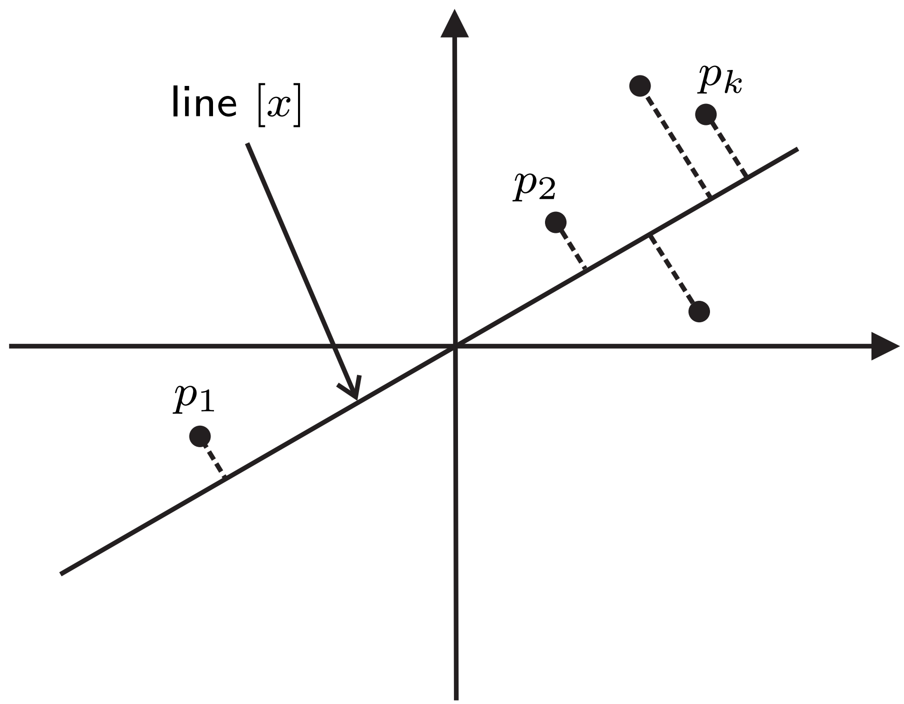
∗ *Intuition: continuity of the "hard" map f can be investigated through the easier $\widehat{f}$*

☐ **Example:** write a matrix $P \in \mathbb{R}^{n\times k}$ in columns $P = [\, p_1\, p_2\, \cdots\, p_k \,]$

Consider the map

$$f \,:\, \mathbb{RP}^{n-1} \times \mathbb{R}^{n\times k} \to \mathbb{R} \qquad f([x], P) = \sum_{j=1}^{k} \left\| p_j - \frac{xx^\top}{\|x\|^2} p_j \right\|^2 .$$

In geometric terms, the map f computes the total squared distance from the constellation of points $\{p_1, p_2, \ldots, p_k\}$ to the straight line $[x]$



The map f is continuous

$$\begin{array}{ccc} \mathbb{R}^n - \{0\} \times \mathbb{R}^{n \times k} & & \\ \Big\downarrow \pi \times \mathrm{id}_{\mathbb{R}^{n \times k}} & \searrow & \\ \mathbb{RP}^{n-1} \times \mathbb{R}^{n \times k} & \xrightarrow{\quad f \quad} & \mathbb{R} \end{array}$$

$\widehat{f}(x, P) = \sum_{j=1}^{k} \left\| p_j - \frac{xx^T}{\|x\|^2} p_j \right\|^2$ is **clearly** continuous