Nonlinear Signal Processing
2007-2008

Topological Spaces

(Ch.2, "Introduction to Topological Manifolds", J. Lee, Springer-Verlag)

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Lecture's key-points

- ☐ **Topology** = structure attached to a set to make sense of **continuity** issues:
  - ▷ convergence of sequences
  - ▷ continuity of maps
  - ▷ . . .
- ☐ Does **not** give access to **smoothness** issues:
  - ▷ differential of a functional
  - ▷ derivative of a map
  - ▷ . . .

☐ **Definition [Topology]** A topology on a set X is a collection $\mathcal{T}$ of subsets of X s.t.:

(a) $\emptyset$ and X are elements of $\mathcal{T}$

(b) $\mathcal{T}$ is closed under finite intersections:

$$U_1, \ldots, U_n \in \mathcal{T} \quad \Rightarrow \quad U_1 \cap \cdots \cap U_n \in \mathcal{T}$$

(c) $\mathcal{T}$ is closed under arbitrary unions:

$$\{U_\alpha\}_{\alpha \in A} \subset \mathcal{T} \quad \Rightarrow \quad \bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$$

where $\mathcal{A}$ is any index set (may be infinite, not countable)

☐ **Toy examples:** $X = \{a, b, c\}$

▷ $\mathcal{T} = \{\emptyset, \{b\}, \{a, b\}, X\}$ is a topology on X

▷ $\mathcal{T} = \{\emptyset, \{a, b\}, X\}$ is a topology on X

▷ $\mathcal{T} = \{\emptyset, \{a\}, \{c\}, X\}$ is **not** a topology on X

☐ **Two extreme examples:** for given X

▷ (trivial topology) $\mathcal{T} = \{\emptyset, X\}$

▷ (discrete topology) $\mathcal{T} = 2^X$=collection of all subsets of X

☐ A set X can accept many topologies

☐ **Definition [Topological space]** A pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X is called a topological space.

▷ The elements of $\mathcal{T}$ are called open sets by definition

▷ If $\mathcal{T}$ is understood from the context, we simply say X is a topological space

☐ **Important example:** $(\mathbb{R}^n, \mathcal{T})$ with $\mathcal{T}$ as the collection of all usual open sets in $\mathbb{R}^n$

☐ **Important example:** $\mathbb{R}^{m\times n} = \{X \,:\, X$ is a $m \times n$ matrix with real entries$\}$

- ▷ We use $\mathbb{R}^{m\times n} \simeq \mathbb{R}^{mn}$
- ▷ To illustrate: $\mathbb{R}^{3\times 2} \simeq \mathbb{R}^{6}$

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} \simeq \mathsf{vec}(X) = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{12} \\ x_{22} \\ x_{32} \end{bmatrix}$$

☐ **Important example:** $\mathbb{C}^n$

▷ We use $\mathbb{C}^n \simeq \mathbb{R}^{2n}$

▷ To illustrate: $\mathbb{C}^3 \simeq \mathbb{R}^6$

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \end{bmatrix} \simeq \iota(z) = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}$$

☐ **Important example:** $\mathbb{C}^{m\times n} = \{Z : Z$ is a $m \times n$ matrix with complex entries$\}$

- ▷ We use $\mathbb{C}^{m\times n} \simeq \mathbb{C}^{mn} \simeq \mathbb{R}^{2mn}$
- ▷ To illustrate: $\mathbb{C}^{3\times 2} \simeq \mathbb{C}^{6} \simeq \mathbb{R}^{12}$

$$Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \\ z_{31} & z_{32} \end{bmatrix} = \begin{bmatrix} x_{11}+iy_{11} & x_{12}+iy_{12} \\ x_{21}+iy_{21} & x_{22}+iy_{22} \\ x_{31}+iy_{31} & x_{32}+iy_{32} \end{bmatrix} \simeq \mathsf{vec}(Z) = \begin{bmatrix} z_{11} \\ z_{21} \\ z_{31} \\ z_{12} \\ z_{22} \\ z_{32} \end{bmatrix} \simeq \iota(\mathsf{vec}(Z)) = \begin{bmatrix} x_{11} \\ y_{11} \\ x_{21} \\ y_{21} \\ x_{31} \\ y_{31} \\ x_{12} \\ y_{12} \\ x_{22} \\ y_{22} \\ x_{32} \\ y_{32} \end{bmatrix}$$

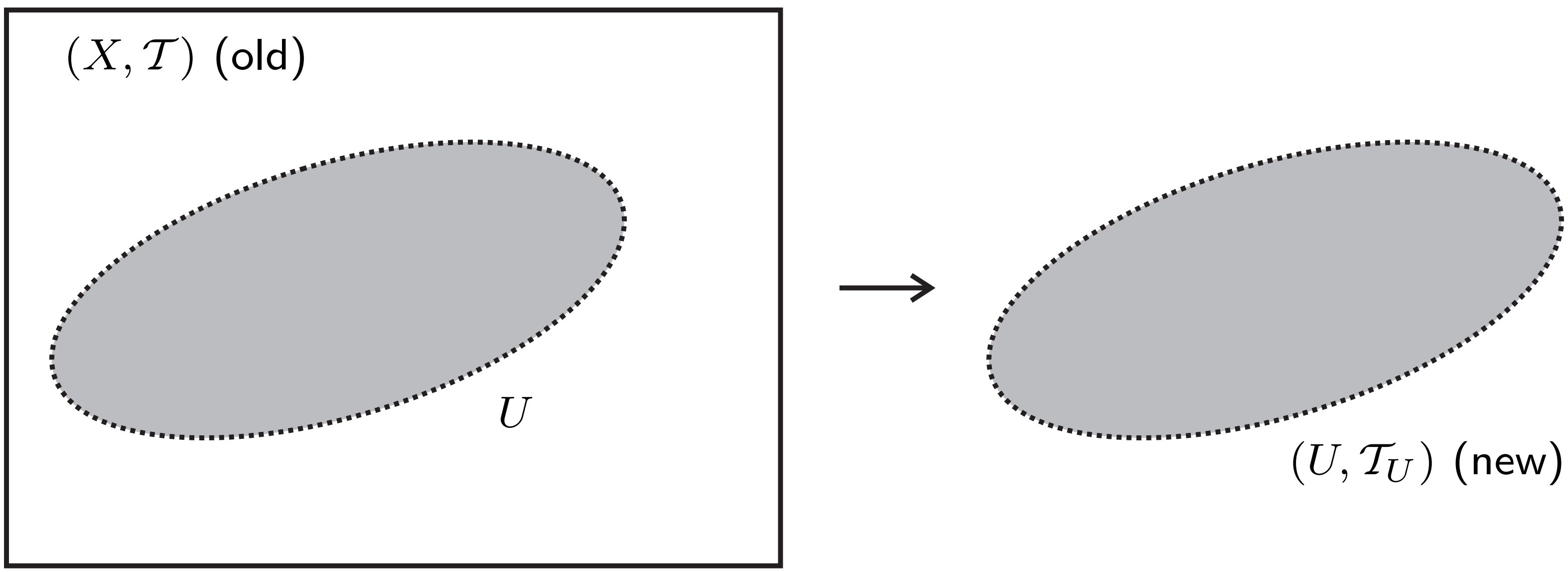
☐ **Example:** subspace topologies

▷ $(X, \mathcal{T})$ is a topological space

▷ U is an open set (that is, $U \in \mathcal{T}$)

▷ let $\mathcal{T}_U = \{V \in \mathcal{T} : V \subset U\} = \{W \cap U : W \in \mathcal{T}\}$

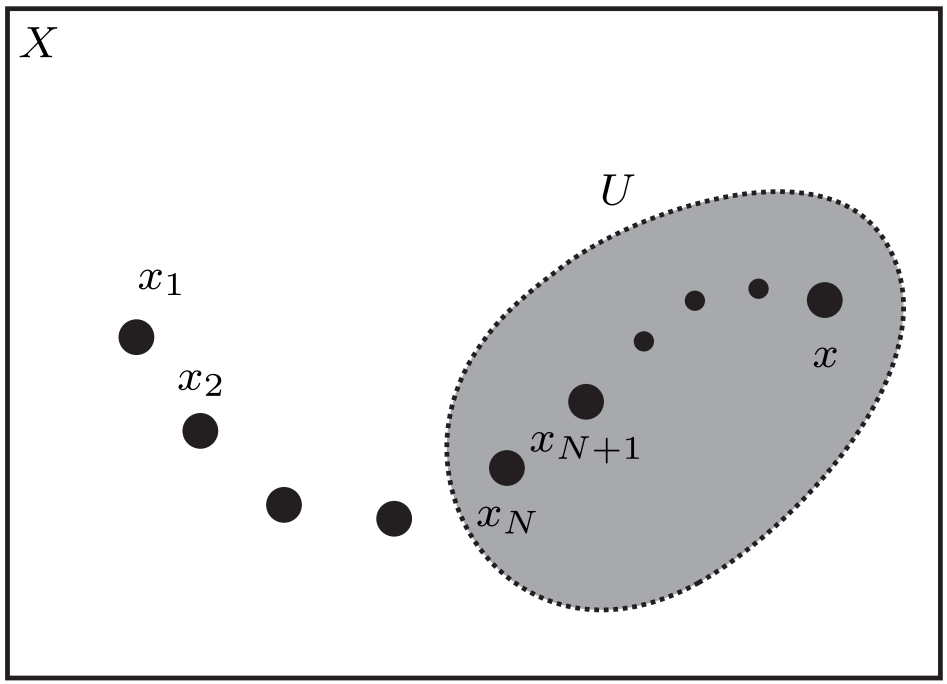
▷ then $(U, \mathcal{T}_U)$ is a topological space



☐ Hereafter, this construction is summarized by saying that "U is a subspace of X"

∗ *Intuition: the new topological space is "genetically" compatible with the old one*

☐ **Definition [Convergent sequence]** Let X be a topological space. A sequence $\{x_n : n = 1, 2, 3, \ldots\}$ of points in X is said to converge to $x \in X$ if for every open set U containing x there exists N such that $x_i \in U$ for all $i \geq N$



☐ $x_n \to x$ means any open neighborhood U of x contains the tail of $\{x_n\}$

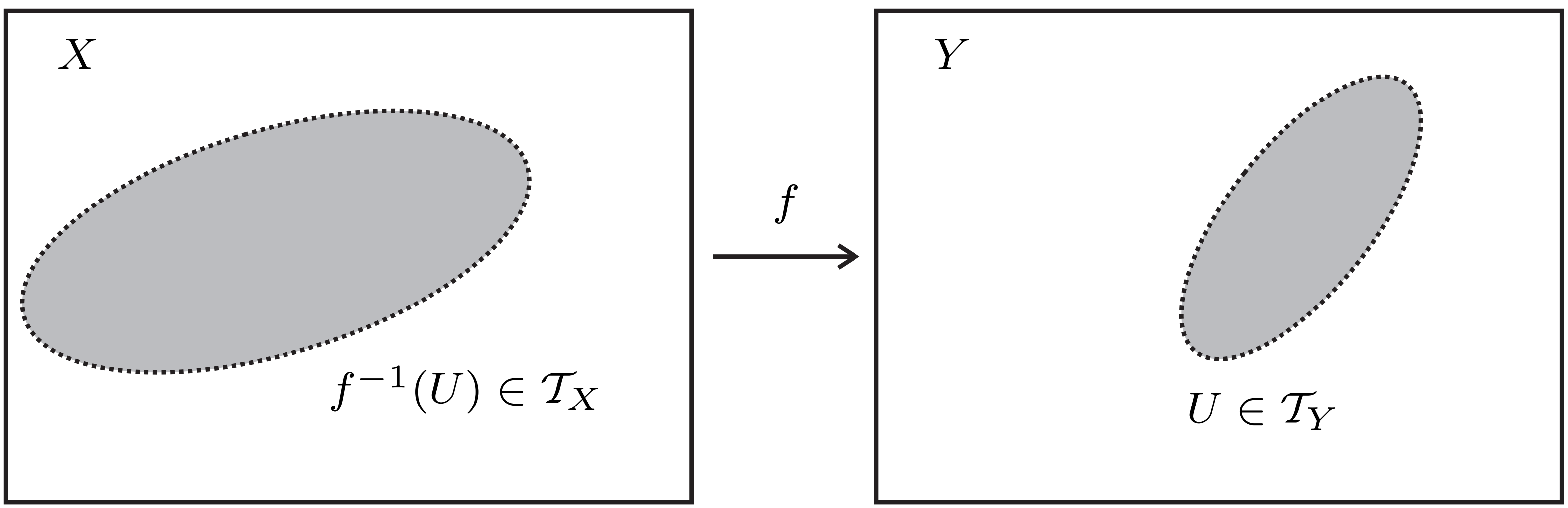
☐ **Important example:** in $\mathbb{R}^n$ (or $\mathbb{R}^{m\times n}, \mathbb{C}^n, \mathbb{C}^{m\times n}$), life proceeds as usual

☐ **Funny example:** in a trivial space, every sequence converges to every point

☐ **Definition [Continuous maps]** If X and Y are topological spaces, a map $f : X \to Y$ is said to be continuous if for every open set $U \subset Y$,

$$f^{-1}(U) = \{x \in X : f(x) \in U\}$$

is open in X



☐ f is continuous iff it pulls back open sets in Y to open sets in X

☐ **Important example:** in $\mathbb{R}^n$ (or $\mathbb{R}^{m\times n}, \mathbb{C}^n, \mathbb{C}^{m\times n}$), life proceeds as usual

- ▷ $f : \mathbb{R}^n \to \mathbb{R}$, $f(x,y,z) = 3x^2 z - 2y^3 + 5xy^2 z$
- ▷ $f : \mathbb{R}^{m\times n} \to \mathbb{R}^{m_k \times n_k}$, $f(X) = X^{[k]}$, $m_k = \begin{pmatrix} m \\ k \end{pmatrix}$, $n_k = \begin{pmatrix} n \\ k \end{pmatrix}$

To illustrate: for

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$A^{[2]} = \begin{bmatrix} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} & \det \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} & \det \begin{bmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{bmatrix} & \cdots & \det \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \\ \det \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix} & \det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} & \det \begin{bmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{bmatrix} & \cdots & \det \begin{bmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{bmatrix} \\ \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} & \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} & \det \begin{bmatrix} a_{21} & a_{24} \\ a_{31} & a_{34} \end{bmatrix} & \cdots & \det \begin{bmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{bmatrix} \end{bmatrix}$$

☐ **Application**: $\mathsf{Rank}_{\geq k}(m,n,\mathbb{R}) = \left\{X \in \mathbb{R}^{m\times n} \ : \ \mathsf{rank}(X) \geq k\right\}$ is open because

- ▷ $f \ : \ \mathbb{R}^{m\times n} \to \mathbb{R}^{m_k \times n_k}$ is continuous
- ▷ $U = \mathbb{R}^{m_k\times n_k} - \{0\}$ is open
- ▷ $\mathsf{Rank}_{\geq k}(m,n,\mathbb{R}) = f^{-1}(U)$

☐ Special cases:

- ▷ General Linear group

$$\mathsf{GL}(n,\mathbb{R}) = \{X \in \mathbb{R}^{n\times n} \ : \ \det(X) \neq 0\} = \mathsf{Rank}_{\geq n}(n,n,\mathbb{R})$$

- ▷ k-frames in $\mathbb{R}^n$

$$\mathsf{F}(n,k,\mathbb{R}) = \left\{X \in \mathbb{R}^{n\times k} \ : \ \mathsf{rank}(X) = k\right\} = \mathsf{Rank}_{\geq k}(n,k,\mathbb{R})$$

☐ **Important example:** if U is a subspace of X, then

$$\iota : U \to X, \quad \iota(x) = x$$

is continuous

∗ *Intuition: the "genetic" compatibility makes the natural link* $\iota : U \to X$ *continuous*

☐ **Example:** the topologies really matter

▷ $X = \{a, b, c\}$

▷ $\mathcal{T}_1 = \{\emptyset, \{b\}, \{a, b\}, X\}$

▷ $\mathcal{T}_2 = \{\emptyset, \{a, b\}, X\}$

Then,

▷ The map id : $(X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous

▷ The map id : $(X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is not continuous

☐ **Lemma [Elementary properties of continuous maps]** Let X, Y, and Z be topological spaces.

(a) Any constant map $f : X \to Y$ is continuous

(b) The identity map id $: X \to X$ is continuous

(c) If $f : X \to Y$ and $g : Y \to Z$ are continuous, so is $g \circ f : X \to Z$

☐ Application:

▷ $f : X \to Y$ is continuous

▷ U is a subspace of X

▷ Then, $f|_U : U \to Y$ is continuous

To illustrate: $f : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$, $f(X) = \mathrm{tr}(X)$ is continuous

☐ **Lemma [Local criterion for continuity]** A map $f : X \to Y$ between topological spaces is continuous if and only if each point of X has a neighborhood on which the restriction of f is continuous

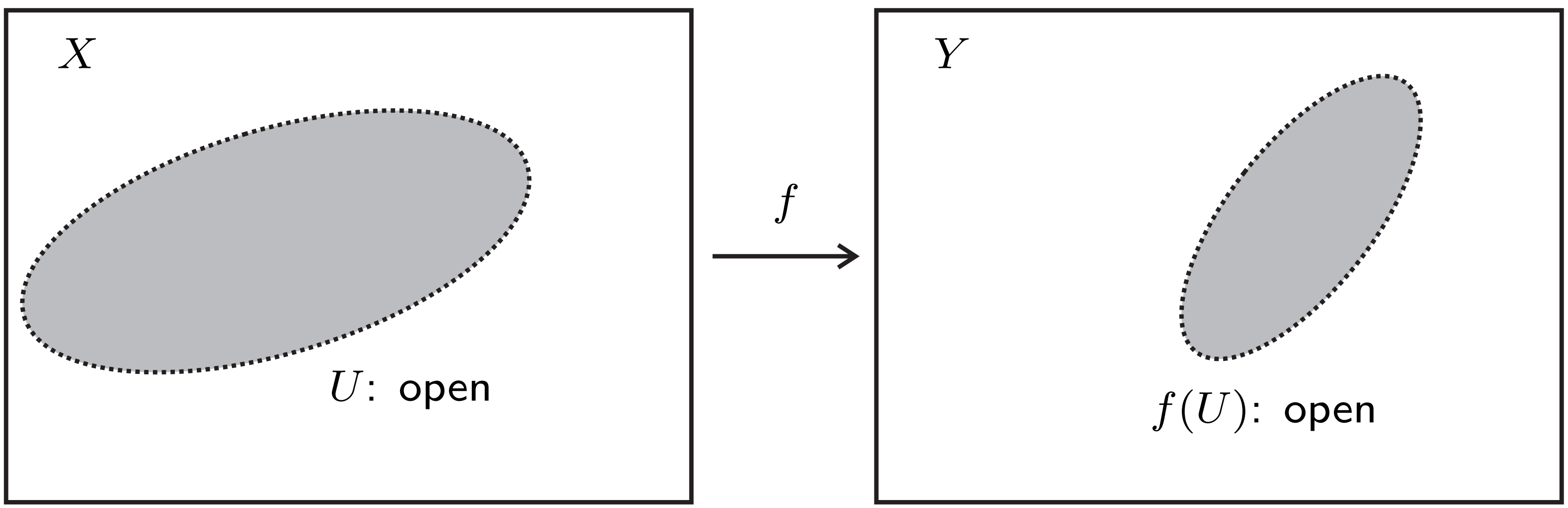
$*$ *Intuition: continuity is a local property*

☐ **Definition [Homeomorphism]** If X and Y are topological spaces, a homeomorphism from X to Y is a continuous bijective map $f : X \to Y$ with continuous inverse

$*$ *Intuition: X, Y are the "same" topological space (Y is simply another label for X)*

☐ Example: $X = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and $Y = \mathbb{R}$ are homeomorphic

☐ **Definition [Open map]** If X and Y are topological spaces, a map $f : X \to Y$ is said to be an open map if for any open set $U \subset X$, the image set $f(U)$ is open in Y



☐ f is open iff it pushes forward open sets in X to open sets in Y

☐ Examples:

▷ $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$, $f(x,y) = x$ is open

▷ $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$ is not open because $f(]-1,1[) = [0,1[$

☐ **Definition [Closed set]** A subset F of a topological space X is said to be closed if its complement $X - F$ is open

☐ Example:

$$\mathsf{Rank}_{\leq k}(m,n,\mathbb{R}) = \left\{X \in \mathbb{R}^{m \times n} \, : \, \mathsf{rank}(X) \leq k\right\}$$

is closed because

$$\mathsf{Rank}_{\leq k}(m,n,\mathbb{R}) = \mathbb{R}^{m \times n} - \underbrace{\mathsf{Rank}_{\geq k+1}(m,n,\mathbb{R})}_{\text{open}}$$

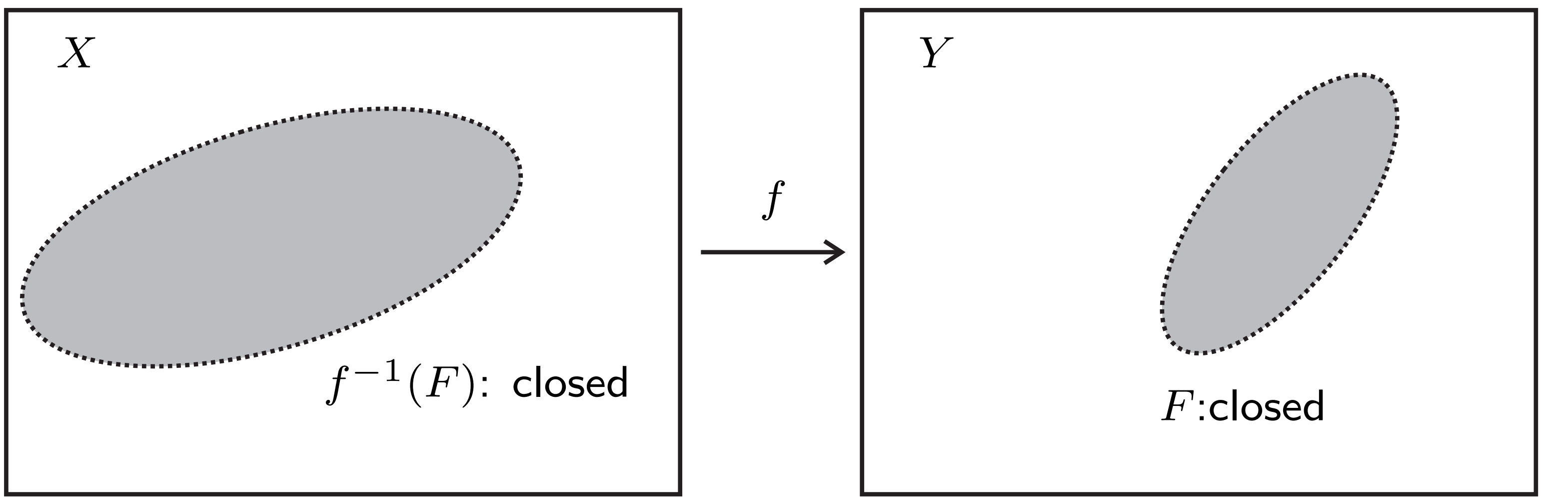
☐ **Lemma [Elementary properties of closed sets]** Let X be a topological space.

(a) X and $\emptyset$ are closed

(b) if $F_1, \ldots, F_n$ are closed, then $F_1 \cup \cdots \cup F_n$ is closed

(c) if $\{F_\alpha\}_{\alpha \in A}$ is any collection of closed sets, then $\bigcap_{\alpha \in A} F_\alpha$ is closed

☐ **Lemma [Characterization of continuous maps through closed sets]** A map between topological spaces is continuous if and only if the inverse image of every closed set is closed



☐ f is continuous iff it pulls back closed sets in Y to closed sets in X

☐ Example: $\text{Rank}_{\leq k}(m, n, \mathbb{R}) = f_{k+1}^{-1}(\{0\})$ is closed in $\mathbb{R}^{m\times n}$

☐ **Definition [Closed map]** If X and Y are topological spaces, a map $f : X \to Y$ is said to be a closed map if for any closed set $F \subset X$, the image set $f(F)$ is closed in Y

☐ Example: $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $f(x, y) = x$ is not closed

☐ **Definition [Elementary topological concepts]** Let X be a topological space and $A \subset X$.

▷ the closure of A in X, written $\overline{A}$ or cl(A), is

$$\overline{A} = \bigcap \{B : A \subset B \text{ and } B \text{ is closed in } X\}$$

∗ *Intuition:* $\overline{A}$ *is the "smallest" closed set containing* A

▷ the interior of A, written Int A, is

$$\text{Int}\, A = \bigcup \{C : C \subset A \text{ and } C \text{ is open in } X\}$$

∗ *Intuition:* Int(A) *is the "largest" open set contained in* A

▷ the exterior of A, written $\mathsf{Ext}\, A$, is

$$\mathsf{Ext}\, A = X - \overline{A}$$

▷ the boundary of A, written ∂A, is

$$\partial A = X - (\mathsf{Int}\, A \cup \mathsf{Ext}\, A)$$

□ **Example**: $X = \mathbb{R}^2$, $A =]-1,1] \times [-1,1]$

▷ $\overline{A} = [-1,1] \times [-1,1]$

▷ $\mathsf{Int}\, A =]-1,1[\times]-1,1[$

▷ $\mathsf{Ext}\, A = (\mathbb{R} - [-1,1]) \times \mathbb{R} \ \bigcup \ \mathbb{R} \times (\mathbb{R} - [-1,1])$

▷ $\partial A = [-1,1] \times \{\pm 1\} \ \bigcup \ \{\pm 1\} \times [-1,1]$

☐ **Lemma [Characterization of closure]** Let X be a topological space and $A \subset X$. A point $x \in \overline{A}$ if and only if every open set containing x intersects A

☐ Example: $X = \mathbb{R}^{n\times n}$, $A = \mathsf{GL}(n, \mathbb{R})$, $\overline{A} = \mathbb{R}^{n\times n}$

☐ **Lemma [Characterization of boundary]** Let X be a topological space and $A \subset X$. A point x is in the boundary of A if and only if every open set containing x contains both a point of A and a point of $X - A$

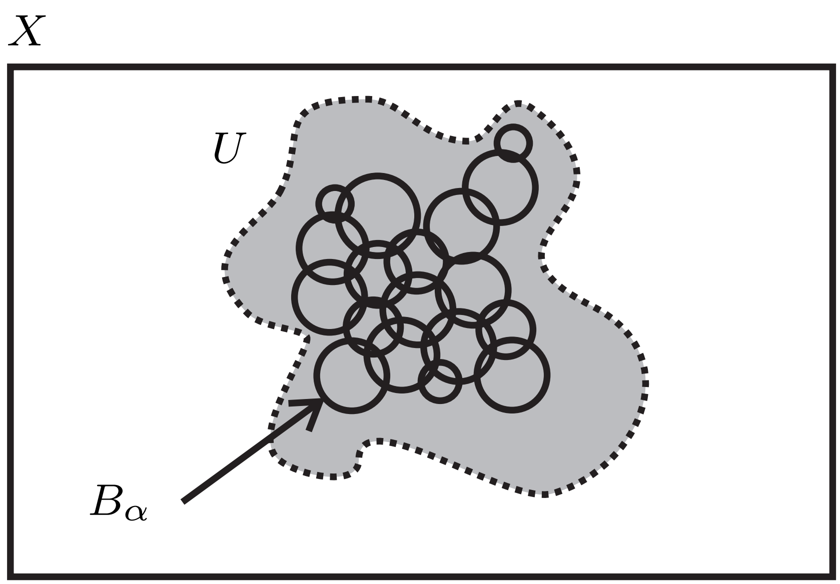
☐ Example: $X = \mathbb{R}^{m\times n}$, $A = \mathsf{Rank}_{=k}(m, n, \mathbb{R})$, $\partial A = \mathsf{Rank}_{\leq k}(m, n, \mathbb{R})$

☐ **Definition [Basis]** Let $(X, \mathcal{T})$ be a topological space. A basis for X is a class $\mathcal{B}$ of open sets (thus, $\mathcal{B} \subset \mathcal{T}$) with the property that every non-empty open set in X is a union of sets in the class $\mathcal{B}$. That is, if $U \subset X$ is open and non-empty, we can write

$$U = \bigcup_{\alpha \in A} B_\alpha \qquad \text{for some } B_\alpha \in \mathcal{B}.$$

The sets in a basis are called basic open sets.

∗ *Intuition: basis="DNA" of $\mathcal{T}$, basic open sets=building blocks of all open sets*



☐ The (strange) open set U is an union of basic (nice) sets B_α

□ **Examples:**

▷ $\mathcal{T}$ is a basis for itself (useless remark in practice)

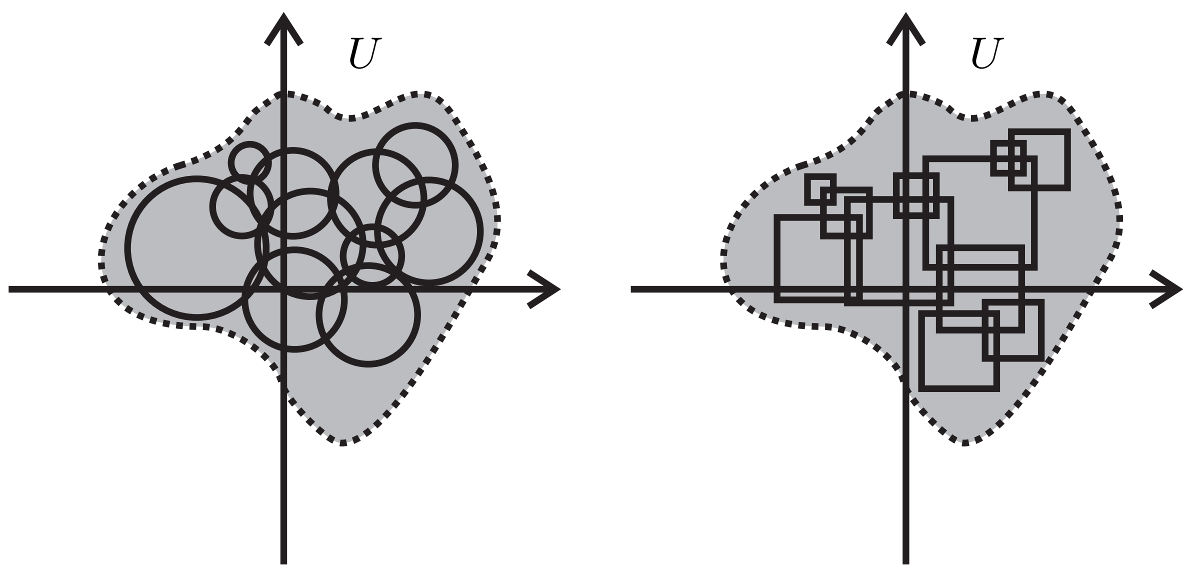
▷ $\mathcal{B} = \{\,]a, b[\subset \mathbb{R} \,:\, a < b\}$ is a basis for $\mathbb{R}$

▷ $\mathcal{B} = \{B^n_\epsilon(x_0) \subset \mathbb{R}^n \,:\, \epsilon > 0,\, x_0 \in \mathbb{R}^n\}$ is a basis for $\mathbb{R}^n$, where

$$B^n_\epsilon(x_0) = \{x \in \mathbb{R}^n \,:\, \|x - x_0\|_2 < \epsilon\}$$

▷ $\mathcal{B} = \{C^n_\epsilon(x_0) \subset \mathbb{R}^n \,:\, \epsilon > 0,\, x_0 \in \mathbb{R}^n\}$ is a basis for $\mathbb{R}^n$, where

$$C^n_\epsilon(x_0) = \{x \in \mathbb{R}^n \,:\, \|x - x_0\|_\infty < \epsilon\}$$



☐ **Lemma [Bases simplify the detection of continuous maps and open maps]** Let X and Y be topological spaces. Let $\mathcal{B}_X$ be a basis for X and $\mathcal{B}_Y$ be a basis for Y.

(a) A map $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open for every basic set $V \in \mathcal{B}_Y$

(b) A map $g : X \to Y$ is open if and only if the image $g(W)$ is open for every basic set $W \in \mathcal{B}_X$

☐ **Example (pointwise maximum of continuous functions is continuous):** let X be a topological space and $f_i : X \to \mathbb{R}$ be continuous for $i = 1, 2, \ldots, n$. Then,

$$f : X \to \mathbb{R} \qquad f(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

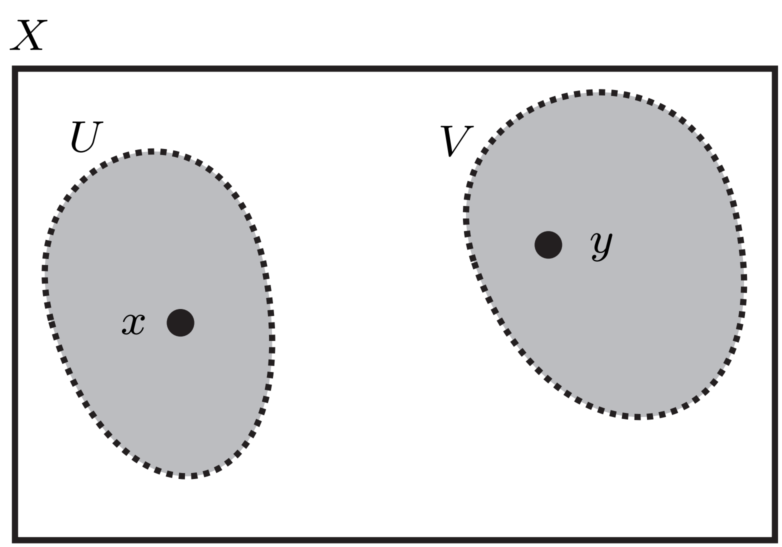
is continuous

☐ **Definition [Locally Euclidean]** A topological space X is said to be locally Euclidean of dimension n if every point of X has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$

$*$ *Intuition: around each point X looks like $\mathbb{R}^n$, but not globally*

☐ **Definition [Chart]** Let X be locally Euclidean of dimension n. A chart on X is a pair (U, φ) where $U \subset X$ is open and $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a homeomorphism

☐ **Definition [Hausdorff space]** A topological space X is said to be Hausdorff if for every $x \neq y$ there exist open neighborhoods U of x and V of y such that $U \cap V = \emptyset$



☐ **Examples:**

▷ $\mathbb{R}^n$ (or $\mathbb{R}^{m\times n}, \mathbb{C}^n, \mathbb{C}^{m\times n}$) are Hausdorff spaces

▷ $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, \{a, b\}, X\}$ is not Hausdorff (*Intuition: too few open sets*)

☐ **Lemma [Elementary properties of Hausdorff spaces]** Let X be a Hausdorff space.

(a) Every singleton set $\{x\}$ is closed in X

(b) The limits of convergent sequences in X are unique

☐ **Definition [Second countable space]** A topological space X is said to be second countable if it admits a countable basis

☐ **Important example:** $\mathbb{R}^n$ (or $\mathbb{R}^{m\times n}, \mathbb{C}^n, \mathbb{C}^{m\times n}$) are second countable. A countable basis for $\mathbb{R}^n$:

$$\mathcal{B} = \{B^n_\epsilon(x_0) \,:\, \epsilon \in \mathbb{Q}^+, \text{ coordinates of } x_0 \text{ in } \mathbb{Q}\}$$

□ **Definition [Cover/Subcover]** Let X be a topological space. A class $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of open sets is said to cover X if $X = \bigcup_{\alpha \in A} U_\alpha$. A subcover of $\mathcal{U}$ is a subclass $\mathcal{V} \subset \mathcal{U}$ which still covers X

□ **Lemma [Fundamental property of second countable spaces]** Let X be a second countable space. Then every open cover of X admits a countable subcover

□ **Lemma [Second countable spaces allow simple characterization of closures]** Let X be a second countable topological space. Let $A \subset X$. Then, $x_0 \in \overline{A}$ if and only if there exists a sequence $x_n \in A$ such that $x_n \to x_0$.

▷ *Proof: (⇒) Let $x_0 \in \overline{A}$ and $\mathcal{V} = \{V_1, V_2, V_3, \ldots\}$ the collection of basic open sets containing x_0. Define the shrinking sequence: $U_1 = V_1$, $U_2 = V_1 \cap V_2$, $U_3 = V_1 \cap V_2 \cap V_3$, ... Take a point $x_n \in A$ in each U_n (this can be done because each U_n is an open set containing $x_0 \in \overline{A}$). We have $x_n \to x_0$ (why?). (⇐) For the reverse direction, let U be an open set containing x_0. Since $x_n \to x_0$, there is a $x_N \in A$ in U. Thus, $A \cap U \neq \emptyset$* □

☐ **Lemma [Second countable spaces simplify detection of continuous maps]** Let $f : X \to Y$ be a map between topological spaces. Assume X is second countable. Then, f is continuous if and only if $x_n \to x_0$ implies $f(x_n) \to f(x_0)$.

▷ *Proof: (⇒) Let U be an open set containing $f(x_0)$. By hypothesis, $V = f^{-1}(U)$ is open. Since $x_n \to x_0$, the tail of x_n is in V. Thus, the tail of $f(x_n)$ is in $f(V) \subset U$. (⇐) Let F be a closed set. Suppose $A = f^{-1}(F)$ is not closed. Then there exists $x_0 \in \overline{A}$ such that $x_0 \notin A$. Let $x_n \in A$ with $x_n \to x_0$. We have $f(x_n) \in f(A) \subset F$ for all n and $f(x_0) \in Y - F$ (open set). Thus, $f(x_n) \not\to f(x_0)$ (contradiction)*☐

☐ **Definition [Topological manifold of dimension n]** A topological manifold of dimension n is a second countable Hausdorff space that is locally Euclidean of dimension n.

☐ **Important example:** $\mathbb{R}^n$ is an n-dimensional topological manifold

☐ **Lemma [Open subsets of manifolds are manifolds]** If U is an open subset of an n-dimensional topological manifold, then the subspace U is an n-dimensional topological manifold

☐ Example: $\mathsf{GL}(n, \mathbb{R})$ is an n^2-dimensional topological manifold

☐ **Lemma [Topological manifolds]** Let X and Y be homeomorphic topological spaces. Then, X is an n-dimensional topological manifold if and only if Y is an n-dimensional topological manifold