Nonlinear Signal Processing
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Geodesics and Distance

(Ch.6, "Riemannian Manifolds", J. Lee, Springer-Verlag)

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Lecture's key-points

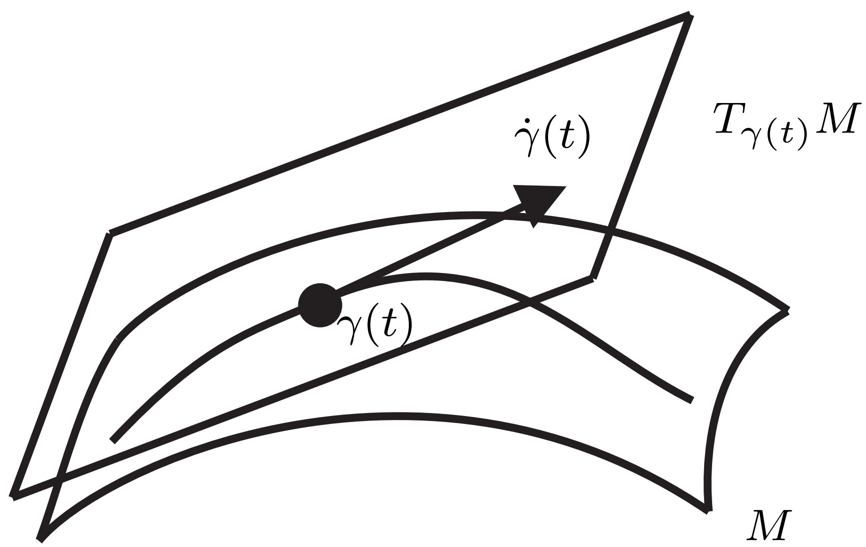
- A distance minimizing curve is a geodesic

- A geodesic is a locally minimizing curve

- Geodesically complete spaces coincide with complete metric spaces

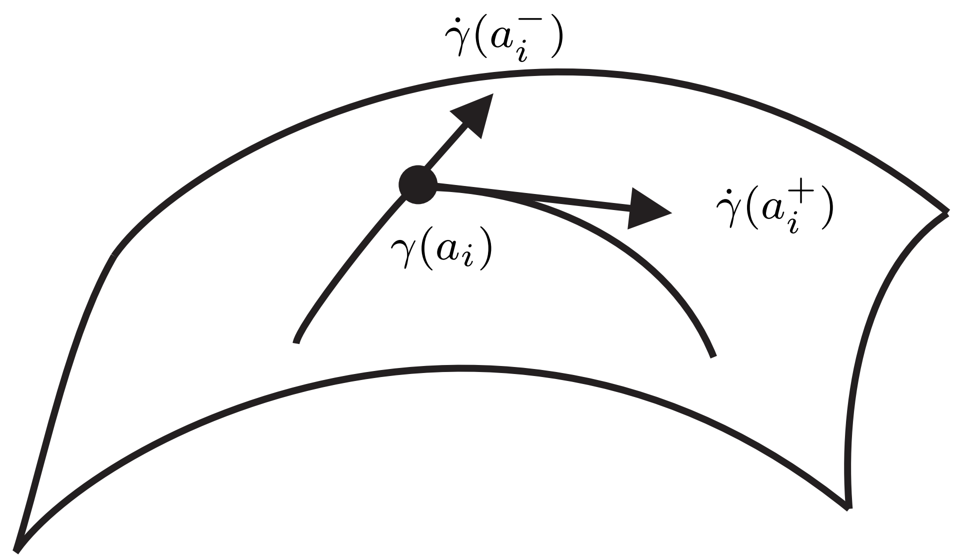
☐ **Definition [Length of a curve segment]** Let M be a Riemannian manifold and let $\gamma : [a,b] \to M$ be a smooth curve segment. The length of γ is given by

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)|\, dt = \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}\, dt$$

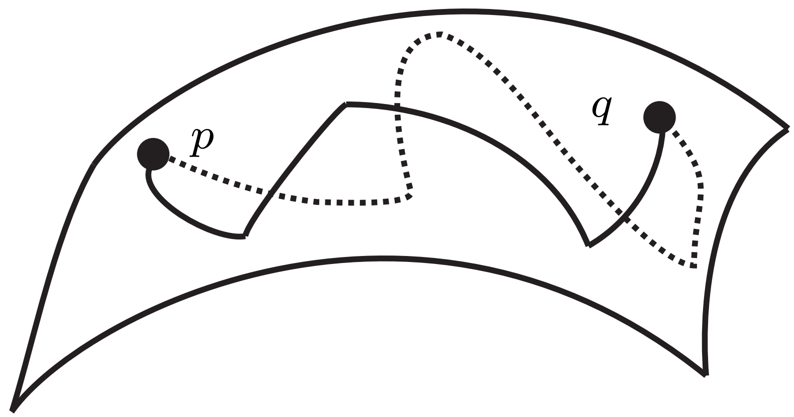


☐ **Definition [Regular curve]** Let M be a Riemannian manifold. A regular curve is a smooth curve $\gamma : I \subset \mathbb{R} \to M$ such that $\dot{\gamma}(t) \neq 0$ for all $t \in I$

☐ **Definition [Admissible curve]** Let M be a Riemannian manifold. A continuous map $\gamma : [a, b] \to M$ is called an admissible curve if is there exists a subdivision $a = a_0 < a_1 < \ldots < a_k = b$ such that $\gamma|_{[a_i, a_{i+1}]}$ is a regular curve



☐ **Definition [Riemannian distance function]** Let M be a connected Riemannian manifold. The Riemannian distance $d(p,q)$ between $p,q \in M$ is the infimum of the lengths of all admissible curves from p to q



☐ **Lemma [Riemannian distance function]** The topology induced by the Riemannian distance coincides with the manifold topology

☐ **Definition [Minimizing curve]** An admissible curve $\gamma : [a, b] \to M$ is said to be minimizing if

$$L(\gamma) \leq L(\widetilde{\gamma})$$

for any other admissible curve $\widetilde{\gamma}$ with the same endpoints

☐ **Theorem [Minimizing curves are geodesics]** Every minimizing curve is a geodesic when it is given a unit-speed parameterization

☐ **Example (a geodesic is not necessarily a minimizing curve):** the curve

$$\gamma : [0, \theta] \to \mathsf{S}^{n-1}(\mathbb{R}) \qquad \gamma(t) = (\cos t, \sin t, 0, \ldots, 0)$$

is a geodesic.

However, γ is not a minimizing curve when $\theta > \pi$

□ **Example (minimizing curves may not exist):** consider the Riemannian manifold $M = \mathbb{R}^2 - \{0\}$. There is not a minimizing curve from $p = (-1, 0)$ to $q = (1, 0)$

□ **Example (minimizing curve between points might not be unique)**: on the unit-sphere $\mathsf{S}^{n-1}(\mathbb{R})$, there are several minimizing curves from the North pole to the South pole

□ **Example (Riemannian distance on the unit-sphere):** if $\mathsf{S}^{n-1}(\mathbb{R})$ is viewed as a Riemannian submanifold of $\mathbb{R}^n$ then

$$d(p, q) = \mathsf{acos}\left(p^\top q\right)$$

for all $p, q \in \mathsf{S}^{n-1}(\mathbb{R})$

□ **Theorem [Riemannian geodesics are locally minimizing]** Let M be a Riemannian manifold and $\gamma : [a, b] \to M$ a geodesic. Then, for any $t_0 \in]a, b[$, there exists a $\epsilon > 0$ such that $\gamma|_{[t_0 - \epsilon, t_0 + \epsilon]}$ is minimizing

☐ **Definition [Geodesically complete manifolds]** A Riemannian manifold M is said to be geodesically complete if for all $X_p \in T_pM$ there exists a geodesic $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$, $\dot{\gamma}(0) = X_p$

☐ **Theorem [Hopf-Rinow]** A connected Riemannian manifold M is complete if and only if it is complete as a metric space (i.e., Cauchy sequences converge)

☐ **Corollary** If there exists one point $p \in M$ such that Exp is defined on all of T_pM, then M is complete

☐ **Corollary** M is complete if and only if any two points in M can be joined by a minimizing geodesic segment

☐ **Corollary** If M is compact, then M is complete

☐ **Example (geodesically complete manifolds):** the following manifolds are geodesically complete, because they are connected and compact

▷ the Stiefel manifold $\mathsf{O}(n,p)$ $(p < n)$ (includes the unit-sphere)

▷ the special orthogonal group $\mathsf{SO}(n)$

▷ the Grassmann manifold $\mathsf{G}(n,p)$