Nonlinear Signal Processing (2004/2005)

jxavier@isr.ist.utl.pt

Lie group actions

Definition [Left/right group action] A left action of a group G on a set M is a map

$$G \times M \to M \qquad (g, p) \mapsto g \cdot p$$

that satisfies

$$g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p \quad \text{and} \quad e \cdot p = p$$

for all $g_1, g_2 \in G$ and $p \in M$ (e denotes the identity element of G).

A right action of a group G on a set M is a map

$$M \times G \to M \qquad (p, g) \mapsto p \cdot g$$

that satisfies

$$(p \cdot g_1) \cdot g_2 = p \cdot (g_1 g_2) \quad \text{and} \quad p \cdot e = p$$

for all $g_1, g_2 \in G$ and $p \in M$.

If G is a Lie group and M is a smooth manifold, the left (right) action is said to be smooth if the map $G \times M \to M$ (resp. $M \times G \to M$) is smooth.

Proposition [Property of smooth actions] Let $\theta : G \times M \to M$ be a smooth action. Then, for each $g \in G$, the map

$$\theta_g : M \to M \qquad x \mapsto \theta(g, x) = g \cdot x$$

is a diffeomorphism.

Definition [Equivariant maps] Suppose G acts on the left of M and N as

$$\theta : G \times M \to M \qquad \varphi : G \times N \to N.$$

A map $F : M \to N$ is said to be equivariant with respect to the given actions if $F(\theta(g, p)) = \varphi(g, F(p))$, or, in a more compact notation

$$F(g \cdot p) = g \cdot F(p),$$

for all $g \in G$ and $p \in M$.

Equivalently, for any given $g \in G$, we have the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\;F\;} & N \\ \Big\downarrow{\scriptstyle \theta_g} & & \Big\downarrow{\scriptstyle \varphi_g} \\ M & \xrightarrow[\;F\;]{} & N \end{array}$$

Example 1 **[Sample mean and covariance]**

View $M = \mathsf{M}(n, k, \mathbb{R})$ as the set of all constellations of k points in $\mathbb{R}^n$.

Consider the map F which, given a constellation, extracts its sample mean and covariance

$$F : \mathsf{M}(n, k, \mathbb{R}) \to \mathbb{R}^n \times \mathsf{S}(n, \mathbb{R}) \qquad F(X) = \left(\frac{1}{k} Xe, \frac{1}{k-1} X \Pi X^T \right),$$

where $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^k$ and $\Pi = I_k - \frac{1}{k} ee^T$.

Consider the action of the special Euclidean group

$$\mathsf{SE}(n) = \left\{ \begin{bmatrix} Q & \delta \\ 0 & 1 \end{bmatrix} : Q \in \mathsf{SO}(n), \delta \in \mathbb{R}^n \right\}$$

on $\mathsf{M}(n, k, \mathbb{R})$ given by

$$\begin{bmatrix} Q & \delta \\ 0 & 1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} x_1 \, x_2 \, \cdots \, x_k \end{bmatrix}}_{X} = \begin{bmatrix} Qx_1 + \delta & Qx_2 + \delta & \cdots & Qx_k + \delta \end{bmatrix} = QX + \delta e^T.$$

Consider also the action of $\mathsf{SE}(n)$ on $\mathbb{R}^n \times \mathsf{S}(n, \mathbb{R})$ given by

$$\begin{bmatrix} Q & \delta \\ 0 & 1 \end{bmatrix} \cdot (\mu, S) = (Q\mu + \delta, QSQ^T).$$

Then, F is an equivariant map with respect to the given actions.

Theorem [Equivariant Rank theorem] Let $F : M \to N$ be a smooth equivariant map with respect to a transitive smooth action of a Lie group G on M and a smooth G-action on N. Then, F has constant rank.

Example 1 [Orthogonal group] We already know that $\mathsf{O}(n)$ is an embedded submanifold of $\mathsf{GL}(n, \mathbb{R})$ (hence, of $\mathsf{M}(n, \mathbb{R})$).

Here is another quick proof.

Consider the actions of the Lie group $G = \mathsf{GL}(n, \mathbb{R})$ on $M = \mathsf{GL}(n, \mathbb{R})$ and $N = \mathsf{M}(n, \mathbb{R})$ given by

$$\begin{array}{ll} \mathsf{GL}(n,\mathbb{R}) \times \mathsf{GL}(n,\mathbb{R}) \to \mathsf{GL}(n,\mathbb{R}) & A \cdot X = AX \\ \mathsf{GL}(n,\mathbb{R}) \times \mathsf{M}(n,\mathbb{R}) \to \mathsf{M}(n,\mathbb{R}) & A \cdot X = AXA^T. \end{array}$$

Note that the first action is transitive.

The map

$$F : \mathsf{GL}(n,\mathbb{R}) \to \mathsf{M}(n,\mathbb{R}) \qquad F(X) = XX^T$$

is equivariant with respect to the given actions. Thus, it has constant rank and

$$\mathsf{O}(n) = \{X \in \mathsf{GL}(n,\mathbb{R}) : F(X) = I_n\}$$

is an embedded submanifold of $\mathsf{GL}(n, \mathbb{R})$.

Example 2 [Special linear group] The map

$$\det : \mathsf{GL}(n,\mathbb{R}) \to \mathbb{R}$$

is equivariant with respect to the two actions of $\mathsf{GL}(n, \mathbb{R})$ on $\mathsf{GL}(n, \mathbb{R})$ and $\mathbb{R}$:

$$\begin{array}{ll} \mathsf{GL}(n,\mathbb{R}) \times \mathsf{GL}(n,\mathbb{R}) \to \mathsf{GL}(n,\mathbb{R}) & A \cdot X = AX \\ \mathsf{GL}(n,\mathbb{R}) \times \mathbb{R} \to \mathbb{R} & A \cdot x = \det(A)x. \end{array}$$

Thus, det has constant rank and, since the first action is transitive, the special linear group

$$\mathsf{SL}(n,\mathbb{R}) = \{X \in \mathsf{GL}(n,\mathbb{R}) : \det(X) = 1\}$$

is an embedded submanifold of $\mathsf{GL}(n, \mathbb{R})$, hence, of $\mathsf{M}(n, \mathbb{R})$.

Definition [Proper map] Let M and N be topological spaces. A map $F : M \to N$ is said to be proper if, given any compact set $K \subset N$, the inverse image $F^{-1}(K) \subset M$ is compact.

- ▷ *F is proper if it pull-backs compact sets to compact sets*
- ▷ *Usually, properness of F is a substitute for compactness of M*

Example 1 [A continuous map which is not proper] The map

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = 1$$

is continuous, but not proper:

$$f^{-1}\Big(\underbrace{[0,2]}_{\text{compact}}\Big) = \underbrace{\mathbb{R}}_{\text{not compact}}.$$

Example 2 [The domain and/or range matter] The map

$$f : (0, +\infty) \to \mathbb{R} \qquad f(x) = x^2$$

is not proper: $f^{-1}([0,4]) = (0,2]$.

The map

$$f : (0, +\infty) \to (0, +\infty) \qquad f(x) = x^2$$

is proper.

Definition [Proper action] Let the Lie group G act smoothly on the smooth manifold M. The action is said to be proper if the map

$$G \times M \to M \times M \qquad (g, p) \mapsto (g \cdot p, p)$$

is proper.

Proposition [Characterization of proper actions] Let the Lie group G act smoothly on the smooth manifold M. The action is proper if and only if for any convergent sequences $\{p_n\}$ and $\{g_n \cdot p_n\}$ in M the sequence $\{g_n\}$ has a convergent subsequence.

Corollary [Actions by compact groups are proper] Let the Lie group G act smoothly on the smooth manifold M. If G is compact, the action is proper.

Example 1 [A discrete Lie group] Consider the discrete (0-dimensional) Lie group of signed $n \times n$ diagonal matrices

$$\mathsf{SD}(n) = \left\{ \begin{bmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{bmatrix} \in \mathsf{M}(n, \mathbb{R}) \right\}.$$

The Lie group $\mathsf{SD}(n)$ acts smoothly on the smooth manifold $\mathsf{O}(n)$ as

$$\mathsf{SD}(n) \times \mathsf{O}(n) \to \mathsf{O}(n) \qquad S \cdot Q = QS.$$

The action is proper because $\mathsf{SD}(n)$ is a compact space.

Example 2 [A proper action of $\mathsf{GL}(n, \mathbb{R})$] Consider the smooth action of the Lie group $\mathsf{GL}(n, \mathbb{R})$ on the smooth manifold

$$\mathsf{F}(m, n, \mathbb{R}) = \{X \in \mathsf{M}(m, n, \mathbb{R}) \, : \, \mathsf{rank}\, X = n\}$$

given by

$$\mathsf{GL}(n, \mathbb{R}) \times \mathsf{F}(m, n, \mathbb{R}) \to \mathsf{F}(m, n, \mathbb{R}) \qquad A \cdot X = XA^T.$$

The action is proper.

Theorem [Quotient Manifold Theorem] Let the Lie group G act smoothly, freely and properly on the smooth manifold M. Then, the orbit space M/G is a topological manifold of dimension equal to $\mathsf{dim}\, M/G = \mathsf{dim}\, M - \mathsf{dim}\, G$, and has an unique smooth structure making the projection map $\pi : M \to M/G$ a smooth submersion.

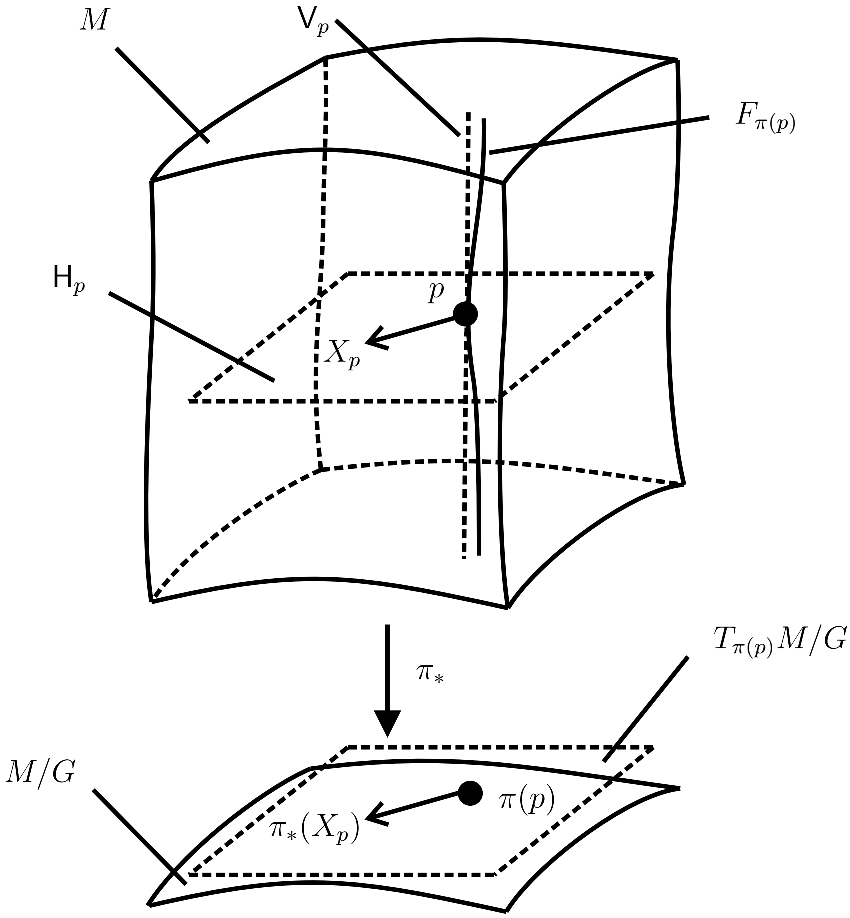
Corollary [Quotient Manifold Theorem] Let $\mathsf{V}_p = \mathsf{Ker}\,\pi_* \subset T_pM$ denote the vertical space at $p \in M$, where

$$\pi_* : T_pM \to T_{\pi(p)}M/G.$$

Let H_p denote any complementary subspace of V_p in T_pM. Then, the restriction

$$\pi_* : \mathsf{H}_p \to T_{\pi(p)}M/G$$

is an isomorphism.



Example 1 [Grassmann spaces] Consider the action of the Lie group $\mathsf{GL}(n, \mathbb{R})$ on the smooth manifold

$$\mathsf{F}(m, n, \mathbb{R}) = \{X \in \mathsf{M}(m, n, \mathbb{R}) \,:\, \mathsf{rank}\, X = n\}$$

given by

$$\mathsf{GL}(n, \mathbb{R}) \times \mathsf{F}(m, n, \mathbb{R}) \to \mathsf{F}(m, n, \mathbb{R}) \qquad A \cdot X = XA^T.$$

The action is smooth, free and proper.

Thus, the orbit space $\mathsf{G}(m, n) = \mathsf{F}(m, n, \mathbb{R})/\mathsf{GL}(n, \mathbb{R})$, also called the Grassmann space of n-dimensional subspaces of $\mathbb{R}^m$, is as smooth manifold of dimension $\mathsf{dim}\, \mathsf{G}(m, n) = n(m - n)$ and

$$\pi \,:\, \mathsf{F}(m, n, \mathbb{R}) \to \mathsf{G}(m, n)$$

is a smooth submersion.

Note that the case $n = 1$ corresponds to the real projective space $\mathbb{RP}^{m-1}$.

Example 2 [Eigenvalue decomposition (EVD) is a diffeomorphism] Let

$$\mathsf{S}_\lambda(n, \mathbb{R}) = \{S \in \mathsf{S}(n, \mathbb{R}) \,:\, \text{ all eigenvalues of } S \text{ are distinct } \}.$$

Note that $\mathsf{S}_\lambda(n, \mathbb{R})$ is a smooth manifold: it is an open subset of the vector space (smooth manifold) $\mathsf{S}(n, \mathbb{R})$ of symmetric matrices.

Let

$$\mathsf{D}_\lambda(n, \mathbb{R}) = \left\{ \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \,:\, \lambda_1 > \lambda_2 > \cdots > \lambda_n \right\}.$$

Note that $\mathsf{D}_\lambda(n, \mathbb{R})$ is a smooth manifold: it is an open subset of the vector space (smooth manifold) of diagonal matrices.

The map

$$F\,:\,\mathsf{O}(n)/\mathsf{SD}(n)\times\mathsf{D}_\lambda(n,\mathbb{R})\to\mathsf{S}_\lambda(n,\mathbb{R})\qquad F([Q],D)=QDQ^T$$

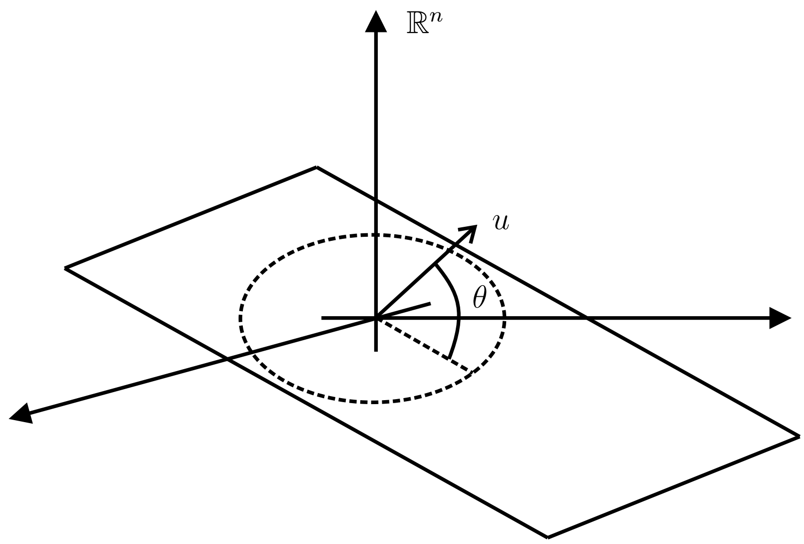
is bijective and smooth.

Its inverse map

$$\mathsf{EVD}\,:\,\mathsf{S}_\lambda(n,\mathbb{R})\to\mathsf{O}(n)/\mathsf{SD}(n)\times\mathsf{D}_\lambda(n,\mathbb{R})$$

is smooth.

Example 3 [**Principal angle**] Let $u\in\mathsf{S}^{n-1}(\mathbb{R})$ be a fixed unit-norm vector in $\mathbb{R}^n$. Let $W\subset\mathsf{G}(n,n-1)$ denote the subset of hyperplanes which do not contain the straight line spanned by u.



W is an open subset of $\mathsf{G}(n,n-1)$. Thus, W is a smooth manifold.

Consider the principal angle function

$$\theta\,:\,W\to\mathbb{R}\qquad\cos(\theta(\mathcal{S}))=\max_{v\in\mathcal{S}\,:\,\|v\|=1}u^Tv.$$

The function θ is smooth.

Definition [Embedding] A smooth map $F : N \to M$ between smooth manifolds is said to be an embedding if it is an immersion and a topological embedding (a homeomorphism of N onto its image $\widetilde{N} = F(N)$, viewed as a subspace of M).

Lemma [Useful criterion for detecting embeddings] Let the smooth map $F : M \to N$ be an injective immersion. If either

(a) M is compact, or,

(b) F is proper,

then F is an embedding.

Example 1 [Orthogonal projectors] Consider the subset of $n \times n$ orthogonal projectors of rank p:

$$\begin{aligned} \mathsf{Proj}(n,p,\mathbb{R}) &= \left\{X \in \mathsf{M}(n,\mathbb{R}) : X^2 = X,\ X^T = X,\ \mathsf{rank}\, X = p\right\}. \\ &= \{X \in \mathsf{S}(n,\mathbb{R}) : \lambda_1(X) = \cdots = \lambda_p(X) = 1, \\ &\qquad \lambda_{p+1}(X) = \cdots = \lambda_n(X) = 0\}. \end{aligned}$$

$\mathsf{Proj}(n,p,\mathbb{R})$ is an embedded submanifold of $\mathsf{M}(n,\mathbb{R})$.

Example 2 [Scaled orthogonal projectors] Consider the subset of $n \times n$ scaled orthogonal projectors of rank p:

$$\begin{aligned} \mathsf{SProj}(n,p,\mathbb{R}) &= \{X \in \mathsf{S}(n,\mathbb{R}) : \lambda_1(X) = \cdots = \lambda_p(X) = \lambda > 0, \\ &\qquad \lambda_{p+1}(X) = \cdots = \lambda_n(X) = 0\}. \end{aligned}$$

$\mathsf{SProj}(n,p,\mathbb{R})$ is an embedded submanifold of $\mathsf{M}(n,\mathbb{R})$.

References

[1] J. Lee, *Introduction to Smooth Manifolds*, Springer-Verlag, 2000.