Nonlinear Signal Processing (2004/2005)

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The Tangent Bundle

Definition [**Tangent vector**] Let M be a smooth manifold and let p be a point of M. A tangent vector to M at p is a map

$$X_p : C^\infty(M) \to \mathbb{R} \qquad f \mapsto X_p f$$

which satisfies

$$\begin{aligned} X_p(af + bg) &= a(X_p f) + b(X_p g) & \text{[linearity]} \\ X_p(fg) &= (X_p f)g(p) + f(p)(X_p g) & \text{[Leibniz rule]}, \end{aligned}$$

for all $a, b \in \mathbb{R}$ and $f, g \in C^\infty(M)$.

Example 1 [**Euclidean space**] Let $M = \mathbb{R}^n$ and $a \in M$. Choose a vector $v = (v^1, v^2, \ldots, v^n) \in \mathbb{R}^n$. The map

$$X_a : C^\infty(\mathbb{R}^n) \to \mathbb{R} \qquad f \mapsto \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i} f(a)$$

is a tangent vector to $\mathbb{R}^n$ at a.

The tangent vector X_a is also written as

$$v^i \frac{\partial}{\partial x^i}\bigg|_a .$$

Definition [**Tangent space**] Let M be a smooth manifold and let p be a point of M. The tangent space to M at p is denoted by T_pM and is a vector space under the operations

$$\begin{aligned} (X_p + Y_p)f &= X_p f + Y_p f \\ (aX_p)f &= a(X_p f) \end{aligned}$$

for all $a \in \mathbb{R}$ and $X_p, Y_p \in T_pM$.

Example 1 [**Euclidean space**] The tangent space $T_a\mathbb{R}^n$ is finite-dimensional (dimension n). The map

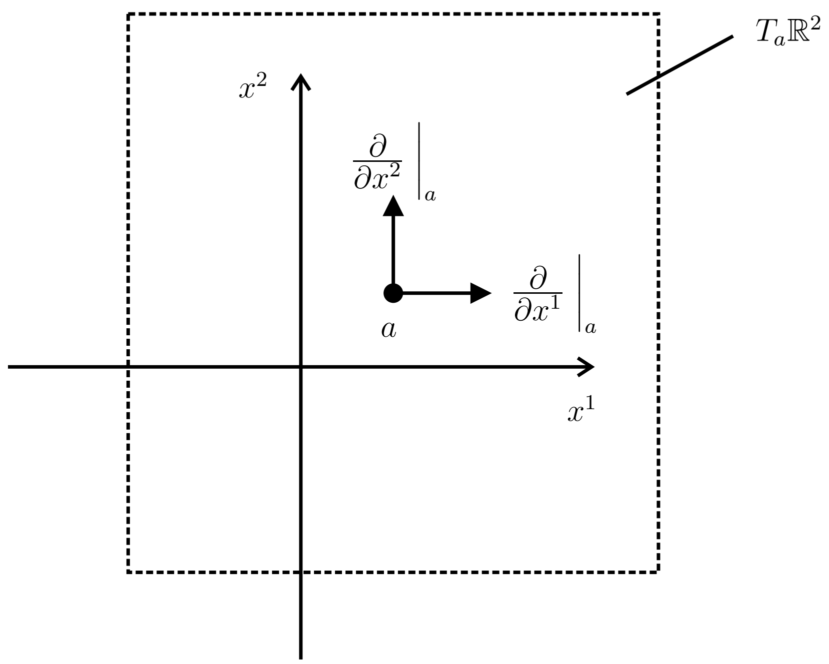
$$\mathbb{R}^n \to T_a\mathbb{R}^n \qquad v = (v^1, v^2, \ldots, v^n) \mapsto v^i \frac{\partial}{\partial x^i}\bigg|_a$$

provides an isomorphism.

The ordered set

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_a, \frac{\partial}{\partial x^2}\bigg|_a, \ldots, \frac{\partial}{\partial x^n}\bigg|_a \right\}$$

is called the canonical basis of $T_a\mathbb{R}^n$.



Example 2 [**Tangent space to a vector space**] Let V be a finite-dimensional vector space and $p \in V$. The tangent space T_pV is finite-dimensional (n-dimensional). The map

$$V \to T_pV \qquad v \mapsto \frac{d}{dt} f(p + tv)\bigg|_{t=0}$$

provides an isomorphism. This identification is written as $V \simeq T_pV$.

- This identification is used in practice as follows. For example, consider $\mathsf{S}(n,\mathbb{R})$, the set of $n \times n$ symmetric matrices with real entries. Note that $\mathsf{S}(n,\mathbb{R})$ is an $n(n+1)/2$-dimensional vector space, hence a smooth manifold with the same dimension.

  Let $S_0 \in \mathsf{S}(n,\mathbb{R})$ and consider a tangent vector $\Delta_{S_0} \in T_{S_0}\mathsf{S}(n,\mathbb{R})$. Note that, by definition, Δ_{S_0} is a certain kind of differential operator on $\mathsf{S}(n,\mathbb{R})$.

  However, the identification

  $$T_{S_0}\mathsf{S}(n,\mathbb{R}) \simeq \mathsf{S}(n,\mathbb{R})$$

  allows you to represent Δ_{S_0} by a much nicer object: a symmetric matrix ! Within this representation, if you have to compute later the action of Δ_{S_0} on a given smooth function $f : \mathsf{S}(n,\mathbb{R}) \to \mathbb{R}$ you calculate

  $$\Delta_{S_0} f = \frac{d}{dt} f(S_0 + t\Delta_{S_0}) \Big|_{t=0}.$$

  Note that in the above equation, the symbol Δ_{S_0} is given two different meanings (which come from the identification): in the left hand-side it represents an element of $T_{S_0}\mathsf{S}(n,\mathbb{R})$ (a certain differential operator) whereas in the right hand-side it represents an element of $\mathsf{S}(n,\mathbb{R})$ (an $n \times n$ symmetric matrix).

  To illustrate: consider the smooth function

  $$f : \mathsf{S}(n,\mathbb{R}) \to \mathbb{R} \qquad f(S) = \mathsf{tr}(S^2) - \mathsf{tr}(S).$$

  Let $\Delta_{S_0} \in T_{S_0}\mathsf{S}(n,\mathbb{R}) \simeq \mathsf{S}(n,\mathbb{R})$. Then,

  $$\begin{aligned} \Delta_{S_0} f &= \frac{d}{dt} f(S_0 + t\Delta_{S_0}) \Big|_{t=0} \\ &= \frac{d}{dt} \mathsf{tr}\left((S_0 + t\Delta_{S_0})^2\right) - \mathsf{tr}(S_0 + t\Delta_{S_0}) \Big|_{t=0} \\ &= \frac{d}{dt} \mathsf{tr}(S_0^2) - \mathsf{tr}(S_0) + t\,\mathsf{tr}(S_0\Delta_{S_0} + \Delta_{S_0}S_0 - \Delta_{S_0}) + t^2\Delta_{S_0}^2 \Big|_{t=0} \\ &= \mathsf{tr}(S_0\Delta_{S_0} + \Delta_{S_0}S_0 - \Delta_{S_0}). \end{aligned}$$

- For another example, consider

  $$\mathsf{L}(n,\mathbb{R}) = \{X \in \mathsf{M}(n,\mathbb{R}) : X_{ij} = 0 \text{ for } i < j\}$$

the set of $n \times n$ lower-triangular matrices with real entries. Note that $\mathsf{L}(n, \mathbb{R})$ is an $n(n+1)/2$-dimensional vector space, hence a smooth manifold with the same dimension.

Let $L_0 \in \mathsf{S}(n, \mathbb{R})$ and consider a tangent vector $\Delta_{L_0} \in T_{L_0}\mathsf{L}(n, \mathbb{R})$. The identification

$$T_{L_0}\mathsf{L}(n, \mathbb{R}) \simeq \mathsf{L}(n, \mathbb{R})$$

allows you to represent Δ_{L_0} by a nice object: a lower-triangular matrix ! Again, with respect to this representation, if you have to compute later the action of Δ_{L_0} on a given smooth function $f : \mathsf{L}(n, \mathbb{R}) \to \mathbb{R}$ you calculate

$$\Delta_{L_0} f = \frac{d}{dt} f(L_0 + t\Delta_{L_0}) \bigg|_{t=0}.$$

Definition [Push-forward] Let $F : M \to N$ be a smooth map between smooth manifolds. For each $p \in M$, we define the push-forward map

$$F_* : T_pM \to T_{F(p)}N \qquad (F_*X_p)f = X_p(F^*f)$$

where $F^*f = f \circ F$ is the pull-back of $f \in C^\infty(N)$ by F (note that $F^*f \in C^\infty(M)$).

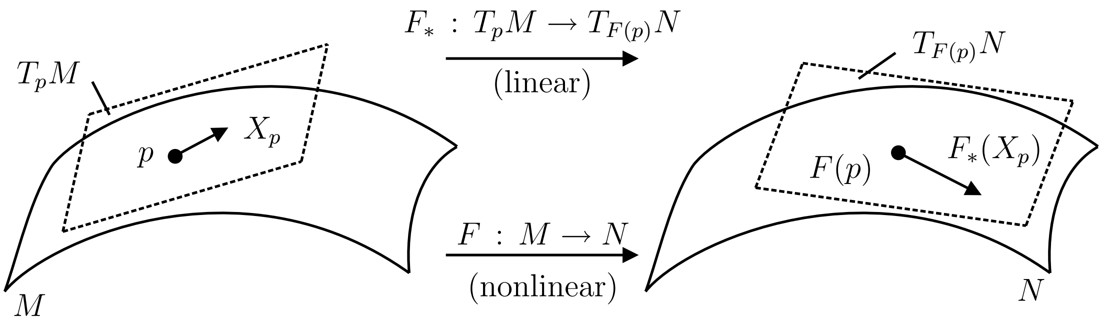
Note that F_* is a linear map:

$$F_*(aX_p + bY_p) = aF_*(X_p) + bF_*(Y_p)$$

for all $a, b \in \mathbb{R}$ and $X_p, Y_p \in T_pM$.

Example 1 [Interpretation of push-forwards in Euclidean spaces] Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. Let $p \in \mathbb{R}^n$ and $q = F(p) \in \mathbb{R}^m$. Consider the canonical basis for the tangent spaces $T_p\mathbb{R}^n$ and $T_q\mathbb{R}^m$ given by

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p \right\} \quad \text{and} \quad \left\{ \frac{\partial}{\partial y^1}\bigg|_q, \ldots, \frac{\partial}{\partial y^m}\bigg|_q \right\},$$



respectively.

With respect to those bases, the linear operator $F_* : T_p\mathbb{R}^n \to T_q\mathbb{R}^m$ is given by the derivative matrix of $F = (F^1, \ldots, F^m)$ at p :

$$DF(p) = \begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \frac{\partial F^1}{\partial x^2}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \frac{\partial F^2}{\partial x^1}(p) & \frac{\partial F^2}{\partial x^2}(p) & \cdots & \frac{\partial F^2}{\partial x^n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \frac{\partial F^m}{\partial x^2}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}.$$

Equivalently, this means that, under the canonical identifications

$$T_p\mathbb{R}^n \simeq \mathbb{R}^n \quad \text{and} \quad T_{F(p)}\mathbb{R}^m \simeq \mathbb{R}^m$$

corresponding to

$$a^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto (a^1, a^2, \ldots, a^n) \quad \text{and} \quad b^j \frac{\partial}{\partial y^j}\bigg|_{F(p)} \mapsto (b^1, b^2, \ldots, b^m),$$

respectively, the push-forward $F_* : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ is represented by the linear map

$$F_* : \mathbb{R}^n \to \mathbb{R}^m \qquad F_*(\Delta) = DF(p)\Delta.$$

Lemma [Properties of push-forwards] Let $F : M \to N$ and $G : N \to P$ be smooth maps between smooth manifolds.

(a) $(G \circ F)_* = G_* \circ F_* : T_pM \to T_{G(F(p))}P$

(b) $\mathsf{id}_{M*} = \mathsf{id}_{T_pM} : T_pM \to T_pM$

(c) If F is a diffeomorphism, then $F_* : T_pM \to T_{F(p)}N$ is an isomorphism.

Proposition [Tangent vectors are "local" objects] Let M be a smooth manifold and $p \in M$. If $f, g \in C^\infty(M)$ coincide on some open subset containing p, then $X_pf = X_pg$.

Proposition [Tangent spaces to open submanifolds] Let M be a smooth manifold, $U \subset M$ an open submanifold and $\iota : U \to M$ the inclusion map. For any $p \in U$, the map $\iota_* : T_pU \to T_pM$ is an isomorphism. In particular, $T_pM = \iota_*(T_pU)$. This identification is written as $T_pU \simeq T_pM$ and $\iota_* X_p \simeq X_p$.

Example 1 [Tangent spaces to open submanifolds of vector spaces] Let V be a finite-dimensional vector space, hence a smooth manifold. Let U be an open submanifold of V and $p \in U$. The last proposition offers the isomorphism

$$T_pU \simeq T_pV.$$

On the other hand, we also have the isomorphism

$$T_pV \simeq V.$$

Concatenating the two isomorphisms we have

$$T_pU \simeq V.$$

That is, any tangent vector $\Delta_p \in T_pU$ can be represented by an element of the vector space V. Within this representation, the action of Δ_p on a smooth function $f : U \to \mathbb{R}$ is given by

$$\Delta_p f = \frac{d}{dt} f(p + t\Delta_p) \bigg|_{t=0}.$$

- As an example, consider the set

$$\mathsf{P}(n,\mathbb{R}) = \{X \in \mathsf{S}(n,\mathbb{R}) \,:\, X \text{ is positive-definite}\}.$$

  Note that $\mathsf{P}(n,\mathbb{R})$ is an open submanifold of the smooth manifold $\mathsf{S}(n,\mathbb{R})$. Let $P_0 \in \mathsf{P}(n,\mathbb{R})$. We have the isomorphism

$$T_{P_0}\mathsf{P}(n,\mathbb{R}) \simeq \mathsf{S}(n,\mathbb{R})$$

  which means that any tangent vector $\Delta_{P_0} \in T_{P_0}\mathsf{P}(n,\mathbb{R})$ can be represented by a symmetric matrix ! Again, within this representation, if you have to compute the action of Δ_{P_0} on a given smooth function $f \,:\, \mathsf{P}(n,\mathbb{R}) \to \mathbb{R}$ you calculate

$$\Delta_{P_0} f = \frac{d}{dt} f(P_0 + t\Delta_{P_0})\bigg|_{t=0}.$$

- For another example, consider

$$\mathsf{L}^+(n,\mathbb{R}) = \{X \in \mathsf{L}(n,\mathbb{R}) \,:\, L_{ii} > 0 \text{ for all } i\}.$$

  Note that $\mathsf{L}^+(n,\mathbb{R})$ is an open submanifold of the smooth manifold $\mathsf{L}(n,\mathbb{R})$. For $L_0 \in \mathsf{L}^+(n,\mathbb{R})$ we have the isomorphism

$$T_{L_0}\mathsf{L}^+(n,\mathbb{R}) \simeq \mathsf{L}(n,\mathbb{R})$$

  meaning that any tangent vector $\Delta_{L_0} \in T_{L_0}\mathsf{L}^+(n,\mathbb{R})$ can be represented by a lower-triangular matrix !

Example 2 [Cholesky decomposition] The Cholesky decomposition asserts that for any $P \in \mathsf{P}(n,\mathbb{R})$ there is an unique $L \in \mathsf{L}^+(n,\mathbb{R})$ such that

$$P = LL^T.$$

In this example, we consider the map

$$F \,:\, \mathsf{L}^+(n,\mathbb{R}) \to \mathsf{P}(n,\mathbb{R}) \qquad F(L) = LL^T.$$

We already know that F is bijective. The map F is also smooth (why?). The goal of this example is to compute the push-forward map

$$F_* \,:\, T_{L_0}\mathsf{L}^+(n,\mathbb{R}) \to T_{F(L_0)}\mathsf{P}(n,\mathbb{R})$$

at a given point $L_0 \in \mathsf{L}^+(n,\mathbb{R})$. Thanks to the isomorphisms

$$T_{L_0}\mathsf{L}^+(n,\mathbb{R}) \simeq \mathsf{L}(n,\mathbb{R}) \quad \text{and} \quad T_{F(L_0)}\mathsf{P}(n,\mathbb{R}) \simeq \mathsf{S}(n,\mathbb{R})$$

the push-forward can be represented by a linear map

$$F_* : \mathsf{L}(n,\mathbb{R}) \to \mathsf{S}(n,\mathbb{R}).$$

We want to find out this linear map.
Let $\Delta_{L_0} \in T_{L_0}\mathsf{L}^+(n,\mathbb{R}) \simeq \mathsf{L}(n,\mathbb{R})$. Note that

$$F_*(\Delta_{L_0}) \in T_{F(L_0)}\mathsf{P}(n,\mathbb{R}) \simeq \mathsf{S}(n,\mathbb{R})$$

is characterized by its action on smooth functions $f : \mathsf{P}(n,\mathbb{R}) \to \mathbb{R}$ as follows:

$$F_*(L_0)f = \frac{d}{dt} f(F(L_0) + tF_*(\Delta_{L_0})) \Big|_{t=0}.$$

Now, take a smooth function $f : \mathsf{P}(n,\mathbb{R}) \to \mathbb{R}$. We have

$$\begin{aligned}
F_*(\Delta_{L_0})f &= \Delta_{L_0}(F^*f) \\
&= \Delta_{L_0}(f \circ F) \\
&= \frac{d}{dt}(f \circ F)(L_0 + t\Delta_{L_0}) \Big|_{t=0} \\
&= \frac{d}{dt} f\left((L_0 + t\Delta_{L_0})(L_0 + t\Delta_{L_0})^T\right) \Big|_{t=0} \\
&= \frac{d}{dt} f\left(L_0L_0^T + t(L_0\Delta_{L_0}^T + \Delta_{L_0}L_0^T) + t^2\Delta_{L_0}^2\right) \Big|_{t=0} \\
&= \frac{d}{dt} f\left(L_0L_0^T + t(L_0\Delta_{L_0}^T + \Delta_{L_0}L_0^T)\right) \Big|_{t=0}.
\end{aligned}$$

We conclude that

$$F_*(\Delta_{L_0}) = L_0\Delta_{L_0}^T + \Delta_{L_0}L_0^T.$$

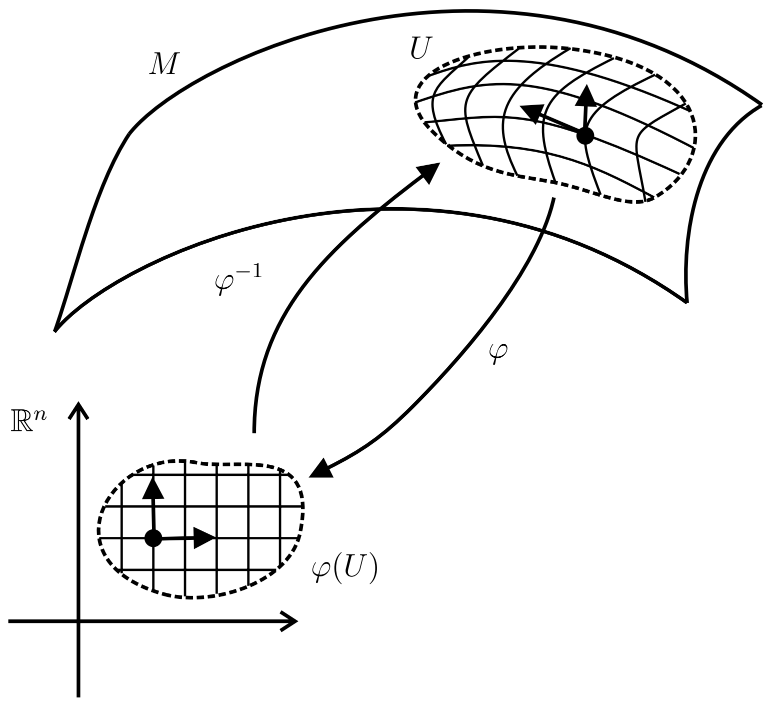
Example 3 [Smooth charts provide local bases for tangent spaces]
Let M be an n-dimensional smooth manifold and (U,φ) a smooth chart on M. Since $\varphi : U \subset M \to \varphi(U) \subset \mathbb{R}^n$ is a diffeomorphism, the map

$$\varphi_* : T_pU \simeq T_pM \to T_{\varphi(p)}\varphi(U) \simeq T_{\varphi(p)}\mathbb{R}^n$$

is a linear isomorphism. Thus,

$$\left\{ \varphi_*^{-1}\left(\frac{\partial}{\partial x^1}\bigg|_{\varphi(p)}\right), \varphi_*^{-1}\left(\frac{\partial}{\partial x^2}\bigg|_{\varphi(p)}\right), \ldots, \varphi_*^{-1}\left(\frac{\partial}{\partial x^n}\bigg|_{\varphi(p)}\right) \right\}$$

is a basis for $T_pU \simeq T_pM$.



For example, consider the unit-circle

$$\mathsf{S}^1(\mathbb{R}) = \{(x^1, x^2) \in \mathbb{R}^2 \,:\, (x^1)^2 + (x^2)^2 = 1\}$$

and the point $p = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

The point p is contained in the smooth chart (U, φ), where

$$U = \{(x^1, x^2) \in \mathsf{S}^1(\mathbb{R}) \,:\, x^2 > 0\}$$

and

$$\varphi \,:\, U \to (-1, 1) \qquad \varphi(x^1, x^2) = x^1.$$

The one-dimensional tangent space $T_{\varphi(p)}\mathbb{R}$ is spanned by

$$\left.\frac{d}{dt}\right|_{t=\frac{1}{2}}.$$

Thus, the one-dimensional tangent space $T_p\mathsf{S}^1(\mathbb{R})$ is spanned by

$$E_{1p} = \varphi_*^{-1}\left(\left.\frac{d}{dt}\right|_{t=\frac{1}{2}}\right).$$

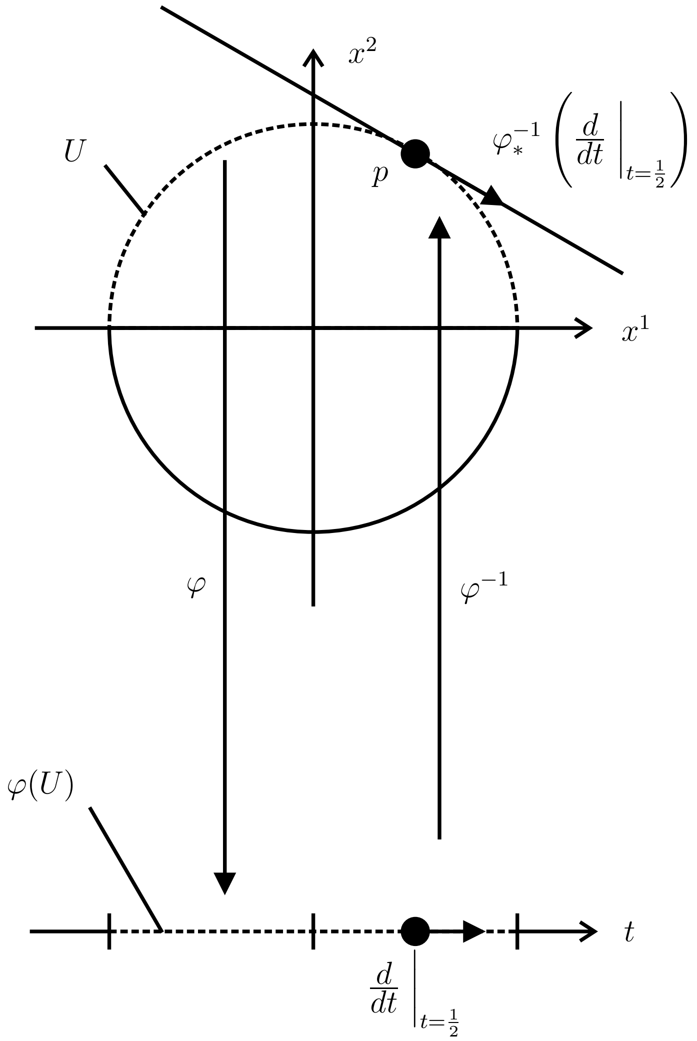
Example 4 [Computation of $X_p f$ in coordinates] Let M be an n-dimensional smooth manifold and (U, φ) containing $p \in M$. Let $X_p \in T_pM$ be given by

$$X_p = a^i E_{ip} \quad \text{where} \quad E_{ip} = \varphi_*^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right).$$

For a smooth function $f : M \to \mathbb{R}$, we have

$$\begin{aligned} X_p f &= a^i E_{ip} f \\ &= a^i \varphi_*^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) f \\ &= a^i \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} f \circ \varphi^{-1} \\ &= a^i \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} \widehat{f} \\ &= a^i \frac{\partial \widehat{f}}{\partial x^i}(\varphi(p)). \end{aligned}$$

Example 5 [Computation of F_* in coordinates] Let $F : M \to N$ be a smooth map of smooth manifolds ($\dim M = m$ and $\dim N = n$). Let $p \in M$ and $q = F(p) \in N$. Let (U, φ) and (V, ψ) denote smooth charts around p and q, respectively, with $F(U) \subset V$.



Let $\{E_{1p}, \dots, E_{mp}\}$ and $\{F_{1q}, \dots, F_{nq}\}$ be basis for T_pM and T_qN, respectively, where

$$E_{ip} = \varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\right) \quad \text{and} \quad F_{jq} = \psi_*^{-1}\left(\frac{\partial}{\partial y^j}\bigg|_{\psi(q)}\right).$$

With respect to those bases, the linear operator $F_* : T_pM \to T_qN$ is given by the derivative matrix of the map $\widehat{F} : \varphi(U) \to \psi(V)$, $\widehat{F} =$

$\psi \circ F \circ \varphi^{-1} = \left(\widehat{F}^1, \ldots, \widehat{F}^m\right)$ which represents F in those coordinates :

$$D\widehat{F}(\varphi(p)) = \begin{pmatrix} \dfrac{\partial \widehat{F}^1}{\partial x^1}(\varphi(p)) & \dfrac{\partial \widehat{F}^1}{\partial x^2}(\varphi(p)) & \cdots & \dfrac{\partial \widehat{F}^1}{\partial x^n}(\varphi(p)) \\ \dfrac{\partial \widehat{F}^2}{\partial x^1}(\varphi(p)) & \dfrac{\partial \widehat{F}^2}{\partial x^2}(\varphi(p)) & \cdots & \dfrac{\partial \widehat{F}^2}{\partial x^n}(\varphi(p)) \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial \widehat{F}^m}{\partial x^1}(\varphi(p)) & \dfrac{\partial \widehat{F}^m}{\partial x^2}(\varphi(p)) & \cdots & \dfrac{\partial \widehat{F}^m}{\partial x^n}(\varphi(p)) \end{pmatrix}.$$

Definition [Smooth curves] A smooth curve in a smooth manifold M is a smooth map

$$\gamma \,:\, (a,b) \subset \mathbb{R} \to M \qquad t \mapsto \gamma(t).$$

The tangent vector to γ at $t_0 \in (a,b)$ is the vector

$$\dot{\gamma}(t_0) = \gamma_* \left(\frac{d}{dt}\bigg|_{t_0} \right) \in T_{\gamma(t_0)}M.$$

Example 1 [Smooth curves in Euclidean spaces] Let

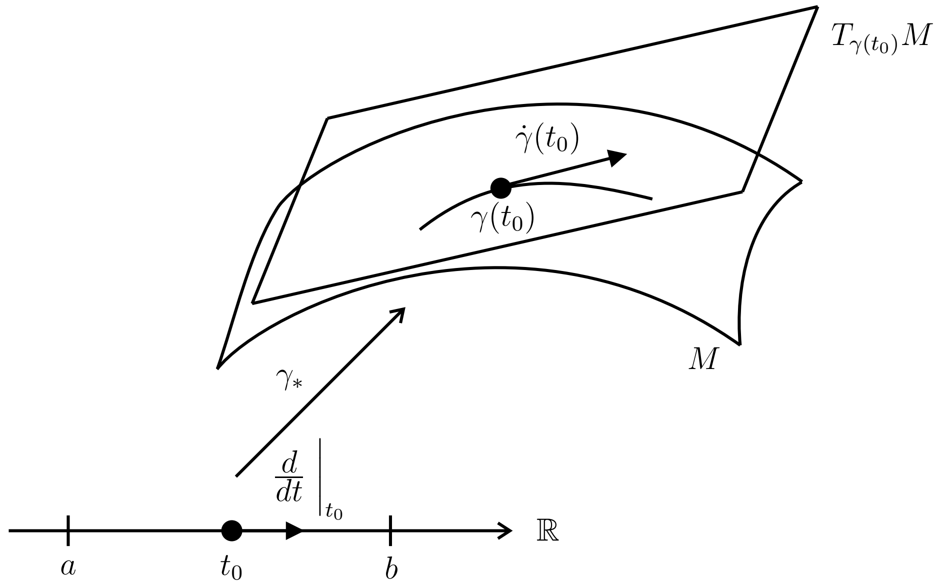
$$\gamma \,:\, (a,b) \to \mathbb{R}^n \qquad \gamma(t) = \left(\gamma^1(t), \gamma^2(t), \ldots, \gamma^n(t)\right)$$

be a smooth curve in $\mathbb{R}^n$.

Thus, $\dot{\gamma}(t_0)$ is an element of $T_{\gamma(t_0)}\mathbb{R}^n$ and, therefore, it can be expressed as

$$\dot{\gamma}(t_0) = c^i \frac{\partial}{\partial x^i}\bigg|_{\gamma(t_0)}$$

for some constants $c^i \in \mathbb{R}$. Each constant c^i can be found by the usual



trick: $c^i = \dot{\gamma}(t_0)x^i$. That is,

$$\begin{aligned} c^i &= \dot{\gamma}(t_0)x^i \\ &= \gamma_*\left(\left.\frac{d}{dt}\right|_{t_0}\right)x^i \\ &= \left.\frac{d}{dt}\right|_{t_0}(\gamma^* x^i) \\ &= \left.\frac{d}{dt}\right|_{t_0}(x^i \circ \gamma) \\ &= \left.\frac{d}{dt}\right|_{t_0}\gamma^i(t) \\ &= \frac{d\gamma^i}{dt}(t_0). \end{aligned}$$

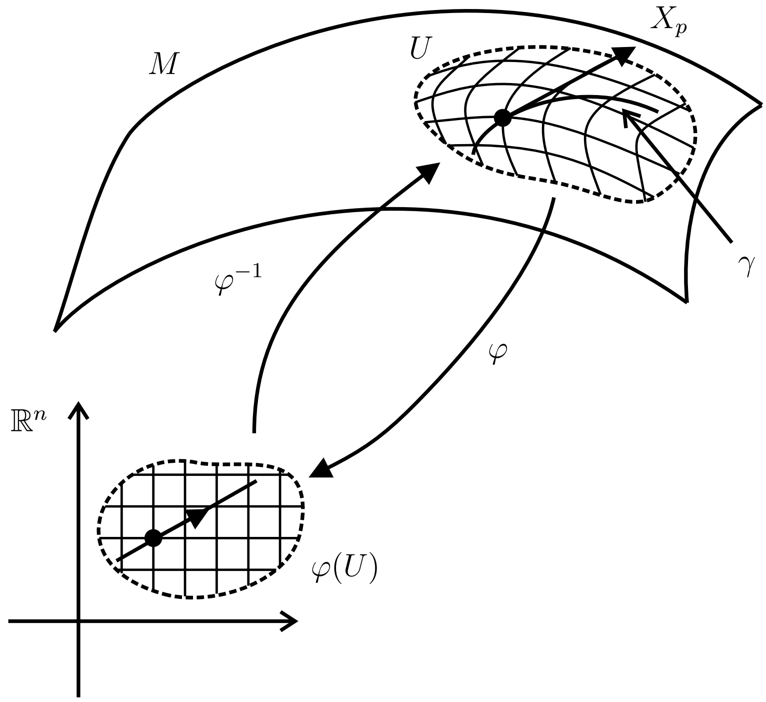
The conclusion is

$$\dot{\gamma}(t_0) = \frac{d\gamma^i}{dt}(t_0)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t_0)}.$$

For a specific example, let $\gamma(t) = \left(t^2 - 2, \sin\left(\frac{t}{2}\right), e^t\right)$. Then,

$$\dot{\gamma}(\pi) = 2\pi\left.\frac{\partial}{\partial x^1}\right|_{(\pi^2-2,1,e^\pi)} + e^\pi\left.\frac{\partial}{\partial x^3}\right|_{(\pi^2-2,1,e^\pi)}.$$

Lemma [Geometrical interpretation of tangent spaces] Let M be a smooth manifold and $p \in M$. Every $X_p \in T_pM$ is the tangent vector to some smooth curve in M.



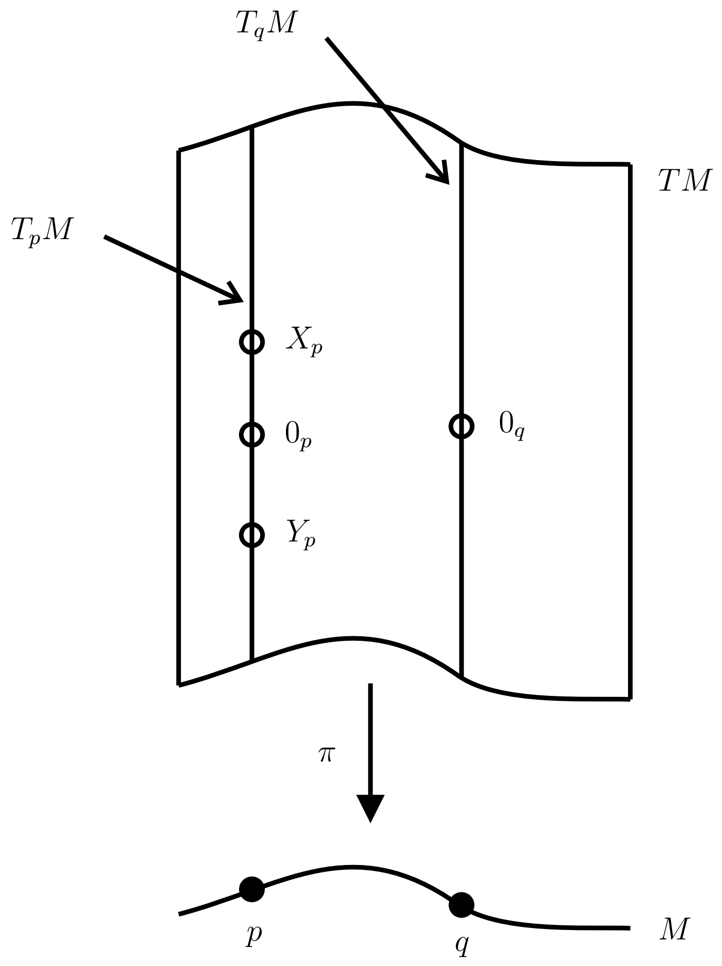
Definition [Tangent bundle] The tangent bundle of a smooth manifold M is the disjoint union of all the tangent spaces of M:

$$TM = \coprod_{p \in M} T_pM.$$

An element of TM is denoted by (p, X) where $p \in M$ and $X \in T_pM$, or, simply by X_p.

The canonical projection is the map

$$\pi : TM \to M \qquad X_p \mapsto p.$$

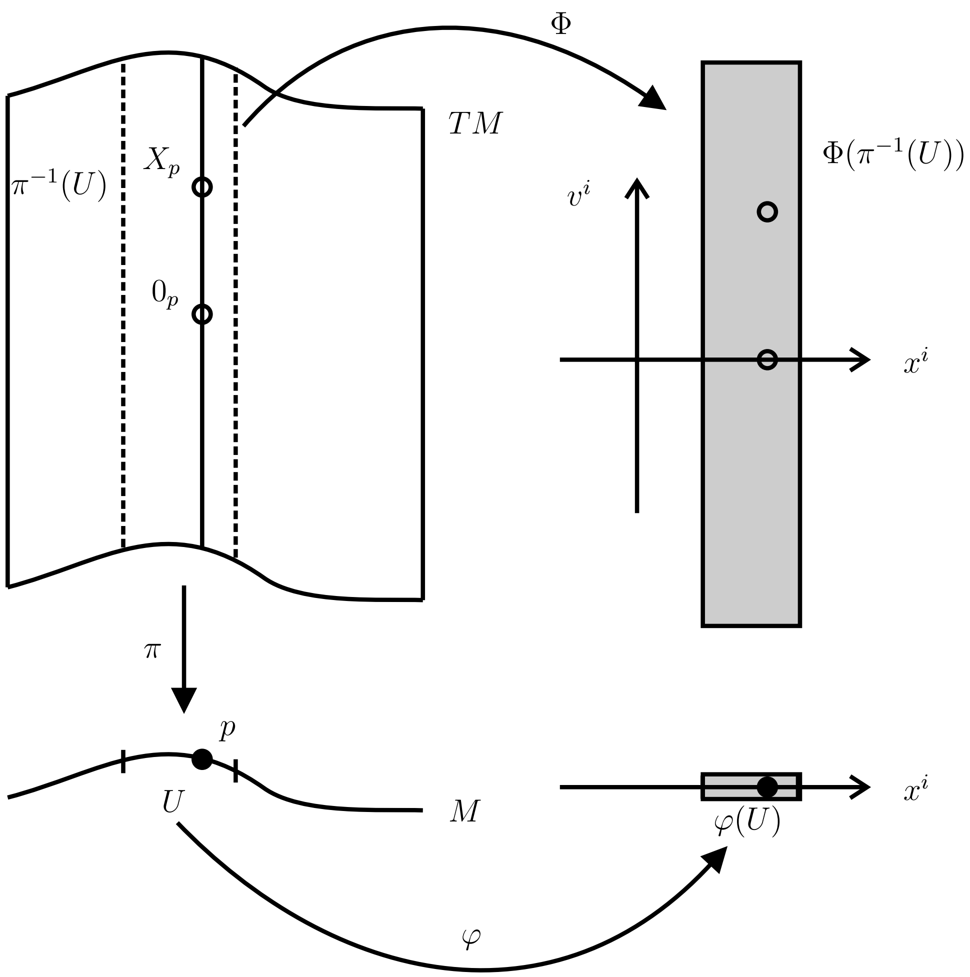


Lemma [Tangent bundle is a smooth manifold] Let M be an n-dimensional manifold. Then, TM can be given a smooth structure making it a $2n$-dimensional smooth manifold such that $\pi : TM \to M$ is a smooth map and, for each smooth chart (U, φ) in M, $(\pi^{-1}(U), \Phi)$ is a smooth chart in TM where

$$\Phi : \pi^{-1}(U) \to \mathbb{R}^{2n} \qquad v^i E_{ip} \mapsto (\varphi^1(p), \ldots, \varphi^n(p); v^1, \ldots, v^n)$$

and

$$E_{ip} = \varphi_*^{-1} \left(\frac{\partial}{\partial x^i} \bigg|_{\varphi(p)} \right).$$


Φ
TM
$\pi^{-1}(U)$
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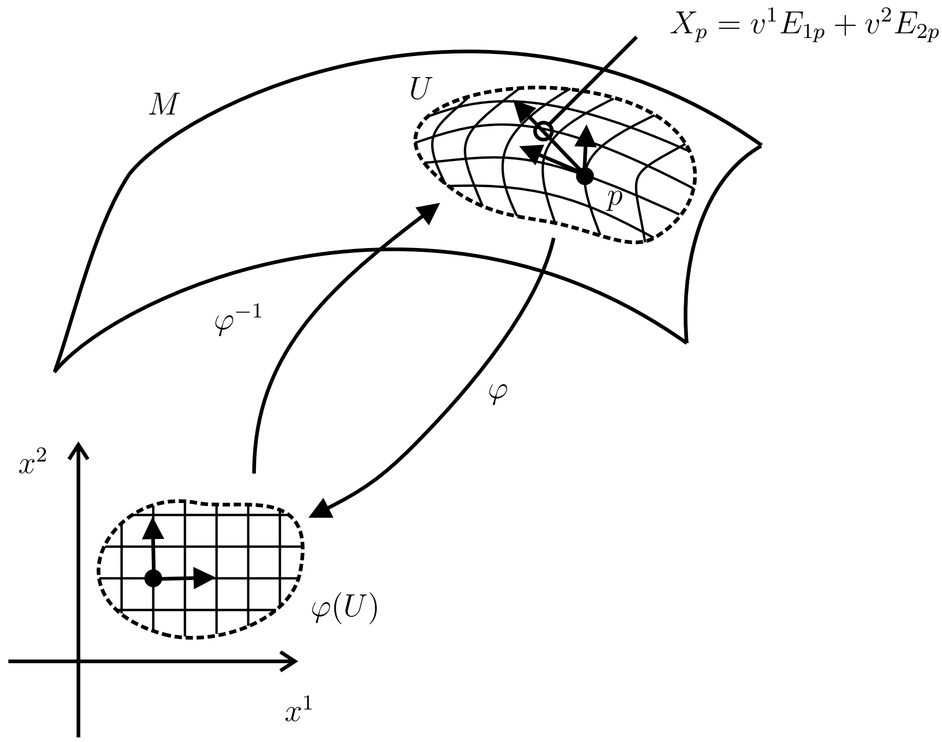
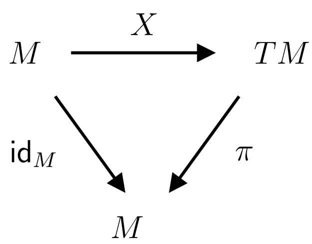


Definition [Smooth vector field] Let M be a smooth manifold. A smooth vector field is a smooth map

$$X \,:\, M \to TM \qquad p \mapsto X_p$$

such that the following diagram commutes



▷ *Intuition: the vector field X picks a representative in each tangent space*

Lemma [Characterization of smooth vector fields] Let M be a smooth manifold and X a vector field on M, that is, a map $X : M \to TM$ such that $\pi \circ X = \mathsf{id}_M$. The following are equivalent:

(a) X is a <u>smooth</u> vector field

(b) The components of X with respect to any chart on M are smooth

(c) The function

$$Xf : M \to \mathbb{R} \qquad p \mapsto X_p f$$

is smooth, for each $f \in C^\infty(M)$.

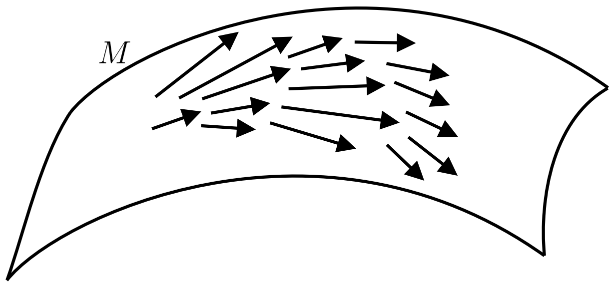
Example 1 [Euclidean space] The vector field

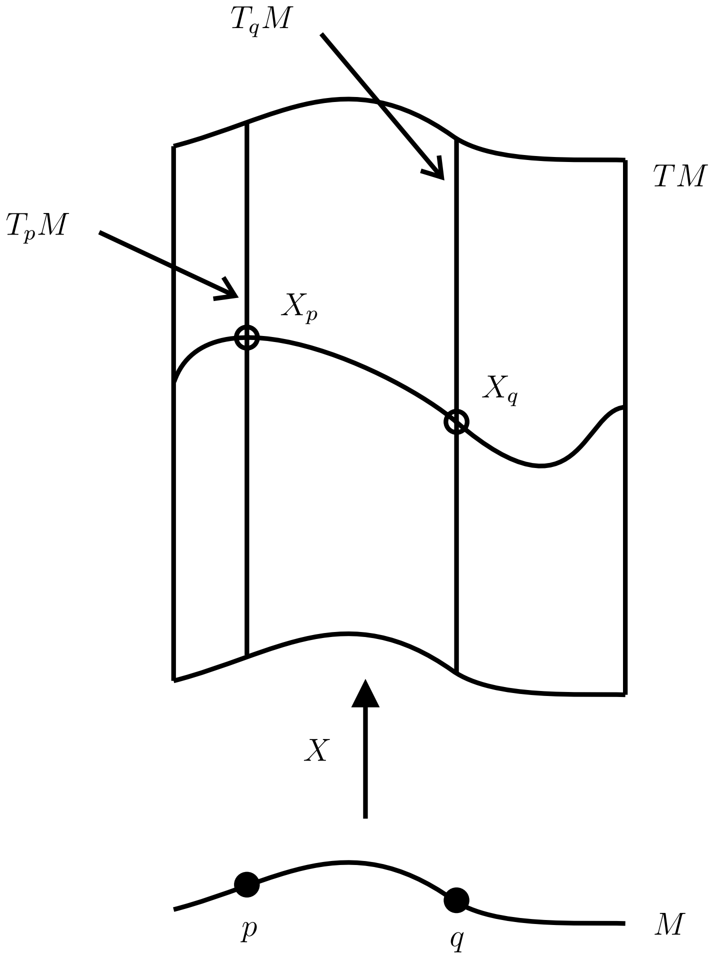
$$X = \underbrace{-4x^1 \cos(x^1 + 3x^2)}_{\text{smooth}} \frac{\partial}{\partial x^1} + \underbrace{\left(e^{x^1 - x^2} + 3x^2\right)}_{\text{smooth}} \frac{\partial}{\partial x^2}$$

is smooth.

References

[1] J. Lee, *Introduction to Smooth Manifolds*, Springer-Verlag, 2000.


M


T_qM
TM
T_pM
X_p
X_q
X
p
q
M