Nonlinear Signal Processing (2004/2005)

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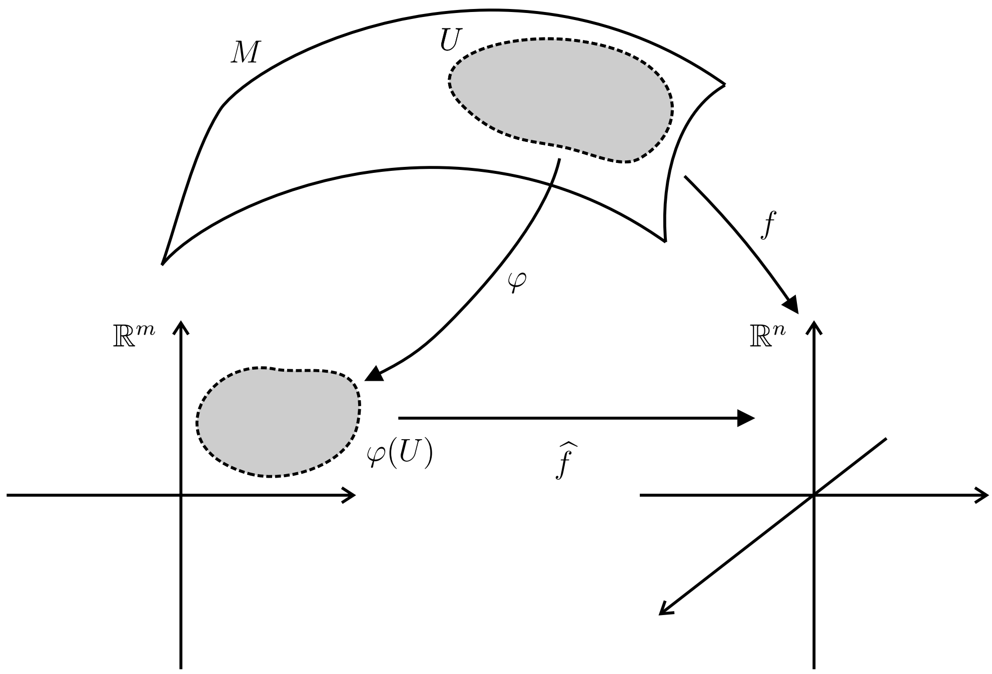
Smooth Maps

Definition [**Smooth function**] Let M be an m-dimensional smooth manifold. A function $f : M \to \mathbb{R}^n$ is said to be smooth if, for every smooth chart (U, φ), the function

$$\widehat{f} : \varphi(U) \subset \mathbb{R}^m \to \mathbb{R}^n \qquad \widehat{f} = f \circ \varphi^{-1}$$

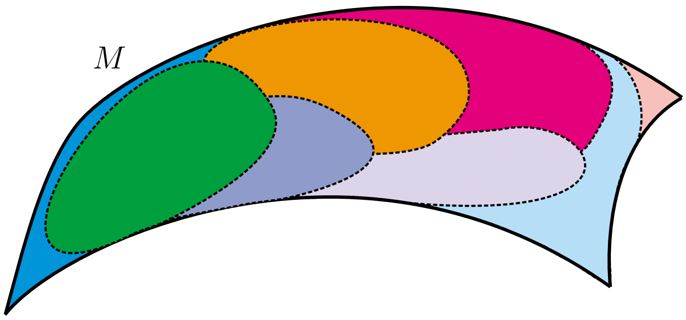
is smooth. The map $\widehat{f}$ is called the coordinate representation of f.



Definition [$C^\infty(M)$] The set of all smooth real valued functions $f : M \to \mathbb{R}$ is denoted by $C^\infty(M)$. Note that $C^\infty(M)$ is a vector space over $\mathbb{R}$ and a ring under pointwise multiplication:

$$f, g \in C^\infty(M) \Rightarrow af + bg \in C^\infty(M) \text{ for all } a, b, \in \mathbb{R} \text{ and } fg \in C^\infty(M).$$

Lemma [It is sufficient to check smoothness on a smooth atlas] Let $\mathcal{A} = \{(U_i, \varphi_i)\}$ be a smooth atlas for M. Then, $f : M \to \mathbb{R}^n$ is smooth if and only if $\widehat{f}_i = f \circ \varphi_i^{-1}$ is smooth for each i.



Example 1 [Map out of the unit-sphere] The inclusion map

$$\iota : \mathsf{S}^{n-1}(\mathbb{R}) \to \mathbb{R}^n \qquad \iota(x) = x$$

is smooth.

Example 2 [Unit-sphere (again)] The function

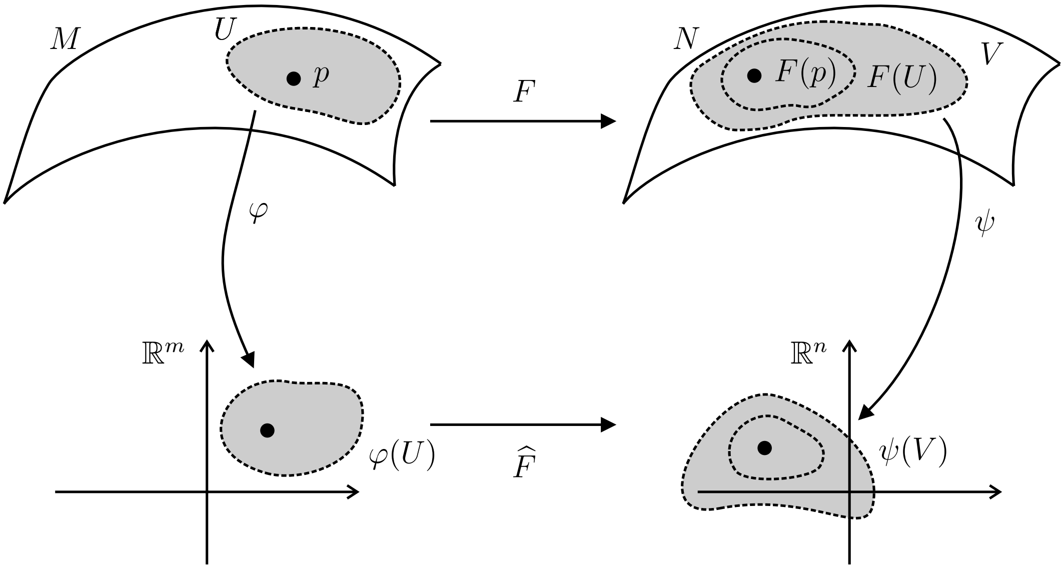
$$f : \mathsf{S}^{n-1}(\mathbb{R}) \to \mathsf{M}(n, \mathbb{R}) \qquad f(u) = uu^T$$

is smooth.

Definition [Smooth map] Let $F : M \to N$ be a map between smooth manifolds. The map F is said to be smooth if, for every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and

$$\widehat{F} : \varphi(U) \to \psi(V) \qquad \widehat{F} = \psi \circ F \circ \varphi^{-1}$$

is smooth.



Example 1 [Map into the unit-sphere] The map

$$F : \mathbb{R}^n - \{0\} \to \mathsf{S}^{n-1}(\mathbb{R}) \qquad F(x) = \frac{x}{\|x\|}$$

is smooth.

Example 2 [Map in and out of the unit-circle] The map

$$F : \mathsf{S}^1(\mathbb{R}) \to \mathsf{S}^1(\mathbb{R}) \qquad F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}$$

is smooth.

Example 3 **[Product manifolds]** Let M and N be smooth manifolds.

The projection map

$$\pi_M : M \times N \to M \qquad \pi_M(p,q) = p$$

is smooth.

For fixed $q \in N$, the inclusion map

$$\iota_q : M \to M \times N \qquad \iota(p) = (p,q)$$

is smooth.

Example 4 **[Inclusion map]** Let M be an n-dimensional smooth manifold and W be an open submanifold of M. The inclusion map

$$\iota : W \to M \qquad \iota(p) = p$$

is smooth.

Proof: Let $p \in W$ and choose a smooth chart in M containing p, say (U, φ). Then, $(V, \varphi|_V)$ is a smooth chart in W, where $V = W \cap U$. Note that $\iota(V) = V \subset U$. Also, the coordinate representation of ι with respect to the smooth charts $(V, \varphi|_V)$ in W and (U, φ) in M is given by

$$\widehat{\iota}(x^1, \dots, x^n) = \varphi \circ \iota \circ \varphi|_V^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^n)$$

which is smooth.

Lemma **[Smoothness implies continuity]** A smooth map between smooth manifolds is continuous.

Proof: Let $F : M \to N$ be a smooth map. Choose any $p \in M$. We will show that there exists an open subset U of M containing p such that $F|_U : U \to N$ is continuous (recall the local criterion for continuity from past lectures). Since F is smooth, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and $\widehat{F} = \psi \circ F \circ \varphi^{-1}$

is smooth. In particular, $\widehat{F}$ is continuous. Thus, $F|_U = \psi^{-1} \circ \widehat{F} \circ \varphi$ is continuous (composition of continuous maps)$\square$

Lemma [Composition of smooth maps is smooth] If $F : M \to N$ and $G : N \to P$ are smooth maps, then

$$G \circ F : M \to P$$

is smooth.

Lemma [Restriction of a smooth map to an open submanifold is smooth] Let $F : M \to N$ be a smooth map between smooth manifolds. If W is an open subset of M, then $F|_W : W \to N$ is smooth.

Proof: $F|_W = F \circ \iota$ where $\iota : W \to M$ denotes the inclusion map$\square$

Lemma [Local criterion for smoothness] Let $F : M \to N$ be a map between smooth manifolds. The map F is smooth if and only if each point $p \in M$ has an open neighborhood W such that $F|_W : W \to N$ is smooth.

$\triangleright$ *($\Rightarrow$) Take $W = M$. ($\Leftarrow$) Let $p \in M$. By hypothesis, there exist an open subset W containing p such that $F|_W : W \to N$ is smooth. This means that there exist smooth charts (U, φ) in W containing p and (V, ψ) in N containing $F(p)$ such that $F|_W(U) \subset V$ and $\widehat{F}|_W = \psi \circ F|_W \circ \varphi^{-1}$ is smooth. By the definition of open submanifolds, the chart (U, φ) is also a smooth chart in M and $F|_W = F$. Thus, $F(U) \subset V$ and $\widehat{F} = \psi \circ F \circ \varphi^{-1} = \widehat{F}|_W$ is smooth*$\square$

Lemma [Product manifolds] Let N be a smooth manifold and $M_1 \times \cdots \times M_n$ be a product smooth manifold. The map

$$F : N \to M_1 \times \cdots \times M_n$$

is smooth if and only if each map

$$F_i : N \to M_i \qquad F_i = \pi_{M_i} \circ F$$

is smooth.

$\triangleright$ *Intuition: analysis of F can be decoupled in simpler maps F_i*

$$\begin{array}{ccc} N & & \\ {\scriptstyle F}\downarrow & \searrow^{F_i} & \\ M_1 \times \cdots \times M_n & \xrightarrow[\pi_{M_i}]{} & M_i \end{array}$$

Example 1 [**Decomposing a vector in amplitude and direction**] The map

$$F \,:\, \mathbb{R}^n - \{0\} \to \mathbb{R}^+ \times \mathsf{S}^{n-1}(\mathbb{R}), \qquad F(x) = \left(\|x\|\,, \frac{x}{\|x\|}\right)$$

is smooth.

Definition [**Diffeomorphism**] Let $F \,:\, M \to N$ be a map between smooth manifolds. The map F is said to be a diffeomorphism if F is bijective, smooth and its inverse map $F^{-1} \,:\, N \to M$ is smooth.

Example 1 [**Unit-sphere**] The map

$$F \,:\, \mathsf{S}^1(\mathbb{R}) \to \mathsf{S}^1(\mathbb{R}) \qquad f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}$$

is a diffeomorphism.

Example 2 [**Decomposing a vector in amplitude and direction**] The map

$$F \,:\, \mathbb{R}^n - \{0\} \to \mathbb{R}^+ \times \mathsf{S}^{n-1}(\mathbb{R}), \qquad F(x) = \left(\|x\|\,, \frac{x}{\|x\|}\right)$$

is a diffeomorphism.

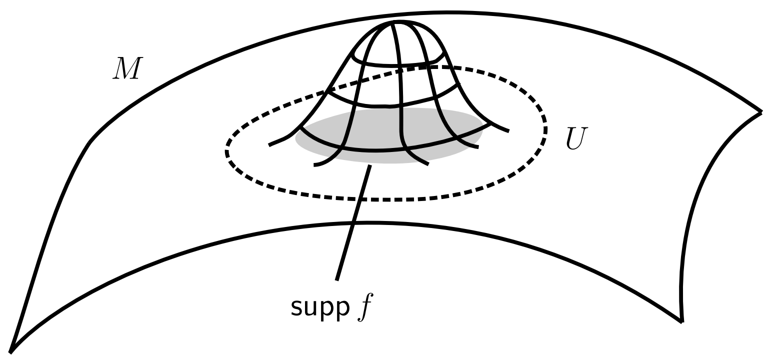
Definition [**Lie group**] Let G be a group which is at the same time a smooth manifold. Then, G is said to be a Lie group if the maps $m \,:\, G \times G \to G$, $m(x, y) = xy$ and $\iota \,:\, G \to G$, $\iota(x) = x^{-1}$ are smooth. (Note: a Lie group is, in particular, a topological group)

Example 1 [**Famous Lie group**] $\mathsf{GL}(n, \mathbb{R})$ is a Lie group.

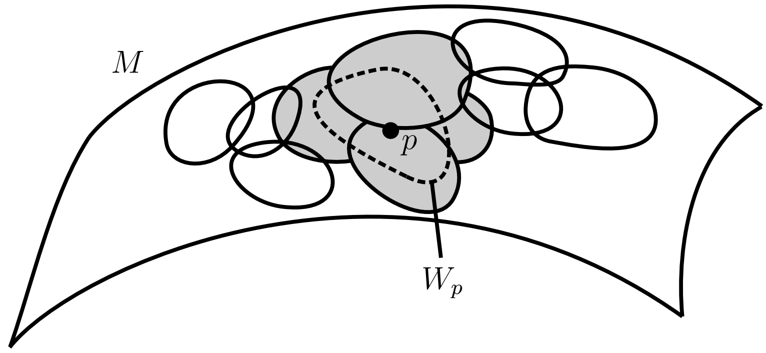
Definition [**Support of functions**] Let M be a smooth manifold and $f : M \to \mathbb{R}^n$ any function. The support of f is defined as

$$\operatorname{supp} f = \overline{\{p \in M \,:\, f(p) \neq 0\}}.$$

If $\operatorname{supp} f \subset U$, we say that f is supported in U.
If $\operatorname{supp} f$ is compact, we say that f is compactly supported.



Definition [**Locally finite collection of subsets**] Let X be a topological space. A collection of subsets $\mathcal{U} = \{U_i\}$ of X is said to be locally finite if each point $p \in X$ has a neighborhood W_p that intersects at most finitely many of the sets U_i.



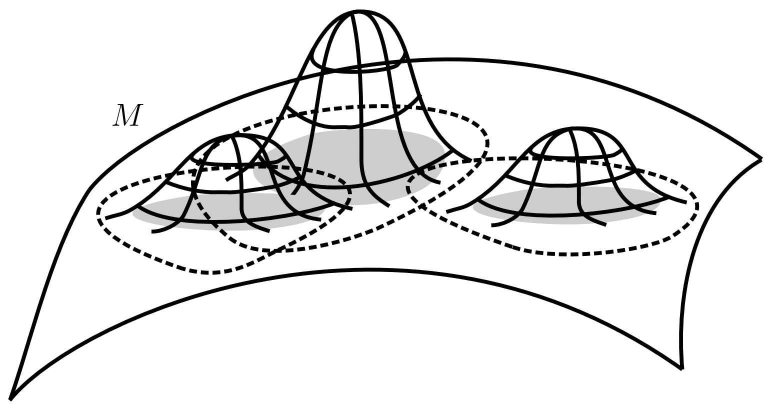
Definition [Partition of unity] Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of a smooth manifold M. A partition of unity subordinate to $\mathcal{U}$ is a collection $\{\varphi_i : M \to \mathbb{R}\}_{i\in I}$ of smooth functions such that

(i) $0 \le \varphi_i(x) \le 1$ for all $i \in I$ and $x \in M$

(ii) $\mathsf{supp}\,\varphi_i \subset U_i$

(iii) the collection $\{\mathsf{supp}\,\varphi_i\}_{i\in I}$ is locally finite

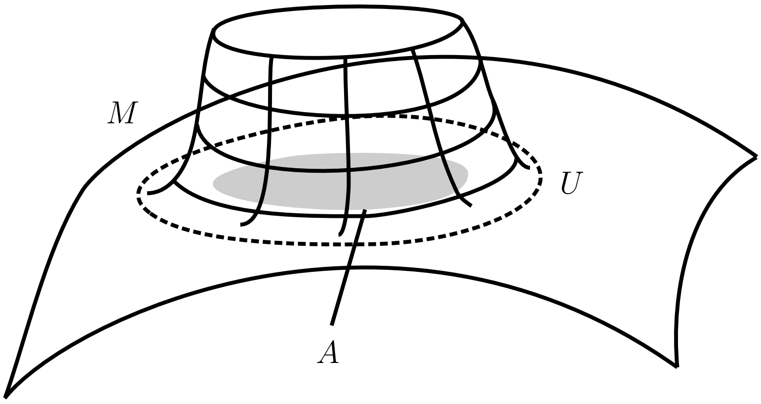
(iv) $\sum_{i\in I} \varphi_i(x) = 1$ for all $x \in M$.



Theorem [Existence of partitions of unity] Let $\mathcal{U} = \{U_i\}_{i\in I}$ be any open cover of M. There exists a partition of unity subordinate to $\mathcal{U}$.

Corollary [Bump functions] Let M be a smooth manifold. Let A be a closed subset of M and U an open subset containing A. There exists a smooth function $\varphi : M \to \mathbb{R}$ such that $\varphi \equiv 1$ on A and $\mathsf{supp}\,\varphi \subset U$.

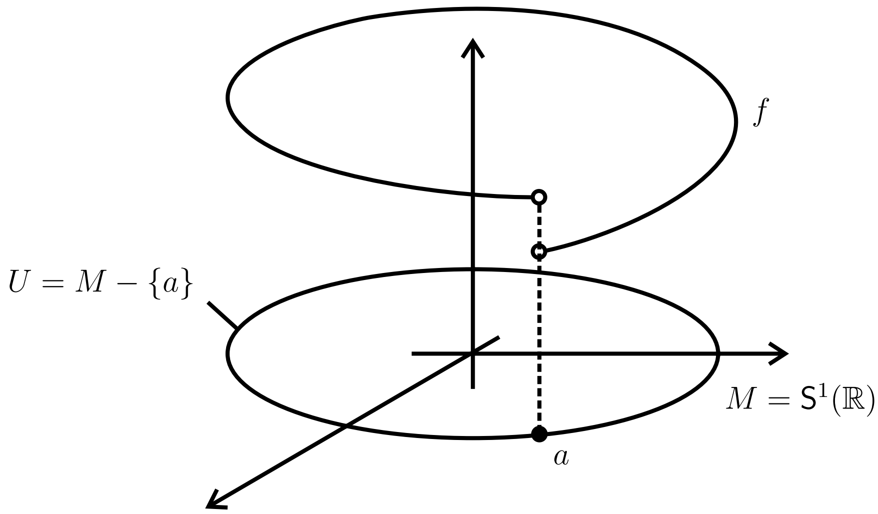
Such φ is called a bump function for A supported in U.



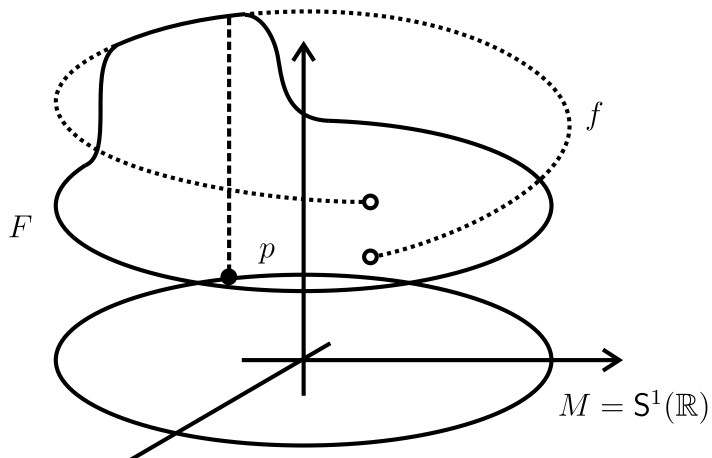
Example 1 [Globalizing local objects] Let M be a smooth manifold, U an open subset of M and $f : U \to \mathbb{R}$ a smooth function. For any $p \in U$, there exists a smooth function $F : M \to \mathbb{R}$ such that $F = f$ on an open neighborhood V of p.

Proof: Let V be an open neighborhood of p such that $\overline{V} \subset U$ (note: such V exists because M is locally compact and Hausdorff. In fact, one may even take $\overline{V}$ to be compact). Let φ be a bump function for $\overline{V}$ supported in U. We define $F : M \to \mathbb{R}$ as $F(x) = \varphi(x)f(x)$ for $x \in U$ and $F(x) = 0$ for $x \notin \mathsf{supp}\,\varphi$. The map F is smooth thanks to the local criterion for smoothness.

Note that, in general, we cannot hope to extend f from U to the whole manifold M as the next sketch shows (in that example, $M = \mathsf{S}^1(\mathbb{R})$ is the circle and U is the open set $M - \{a\}$).



However, given any point $p \in U$, the map $f : U \to \mathbb{R}$ can be extended to a smooth map $F : M \to \mathbb{R}$ which agrees with f on a neighborhood of p.



References

[1] J. Lee, *Introduction to Smooth Manifolds*, Springer-Verlag, 2000.