Nonlinear Signal Processing (2004/2005)

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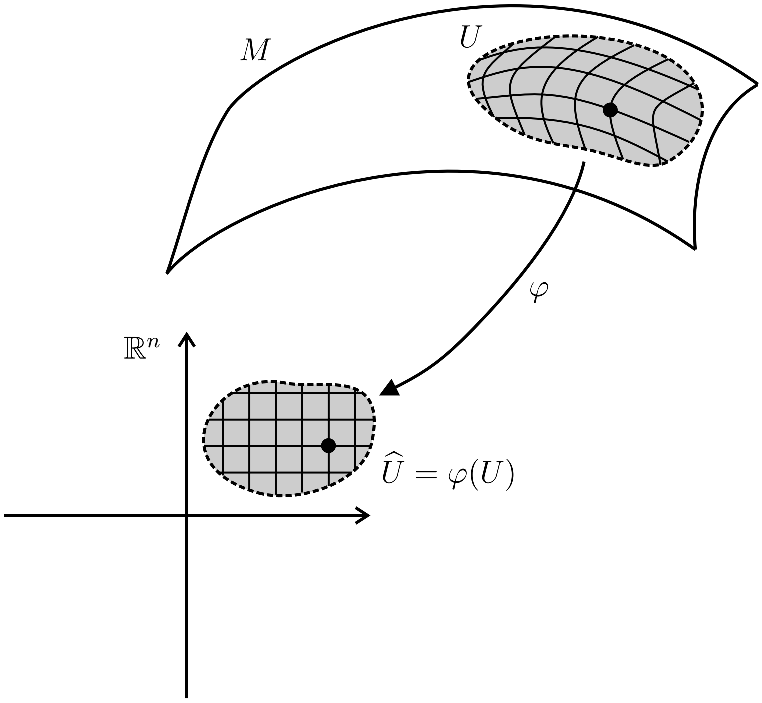
Smooth Manifolds

Definition [Coordinate chart] Let M be a topological n-manifold. A coordinate chart on M is a pair (U, φ) where U is an open subset of M and $\varphi : U \to \widehat{U}$ is a homeomorphism from U to an open subset $\widehat{U} = \varphi(U) \subset \mathbb{R}^n$.

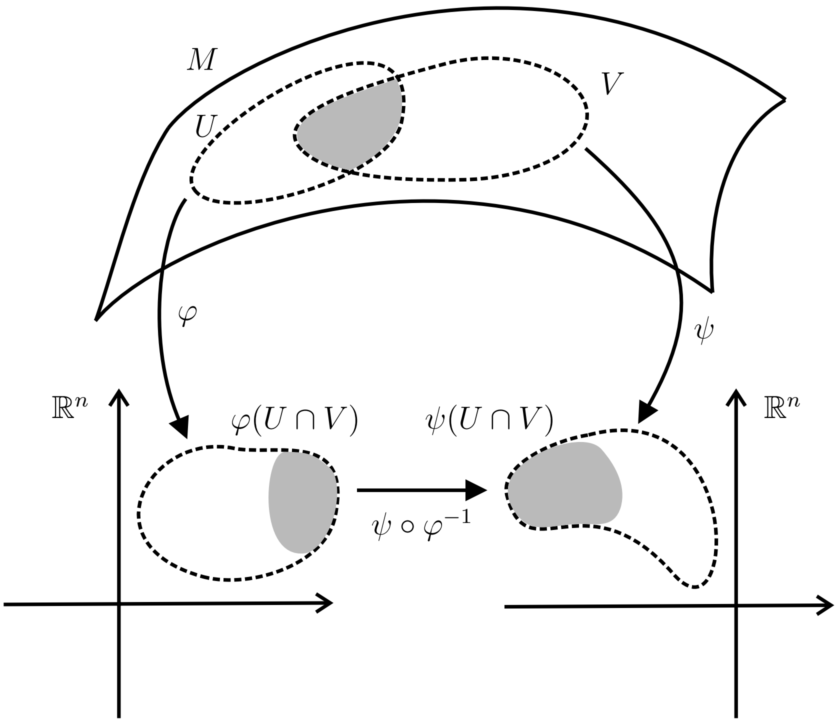
If (U, φ) is a coordinate chart, U is called the coordinate domain and φ is the coordinate map. The components x^i of $\varphi = (x^1, x^2, \ldots, x^n)$ are called local coordinates on U.



Definition [Smooth maps in Euclidean spaces] Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a map defined on an open subset $U \subset \mathbb{R}^n$. The map F is said to be smooth if all the component functions of $F = (F^1, F^2, \ldots, F^m)$ have continuous partial derivatives of all orders.

Definition [Diffeomorphisms in Euclidean spaces] Let $F : U \to V$ be a map defined on open subsets U, V of $\mathbb{R}^n$. The map F is said to be a diffeomorphism if it is bijective, smooth and its inverse map $F^{-1} : V \to U$ is smooth.

Definition [Smooth compatibility of charts] Let M be a topological manifold. Two charts (U, φ) and (V, ψ) on M are said to be smoothly compatible if either $U \cap V = \emptyset$ or $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism.



Definition [Smooth atlas, maximal smooth atlas] Let M be a topological manifold. A smooth atlas $\mathcal{A}$ on M is a collection of smoothly compatible charts whose domains cover M.

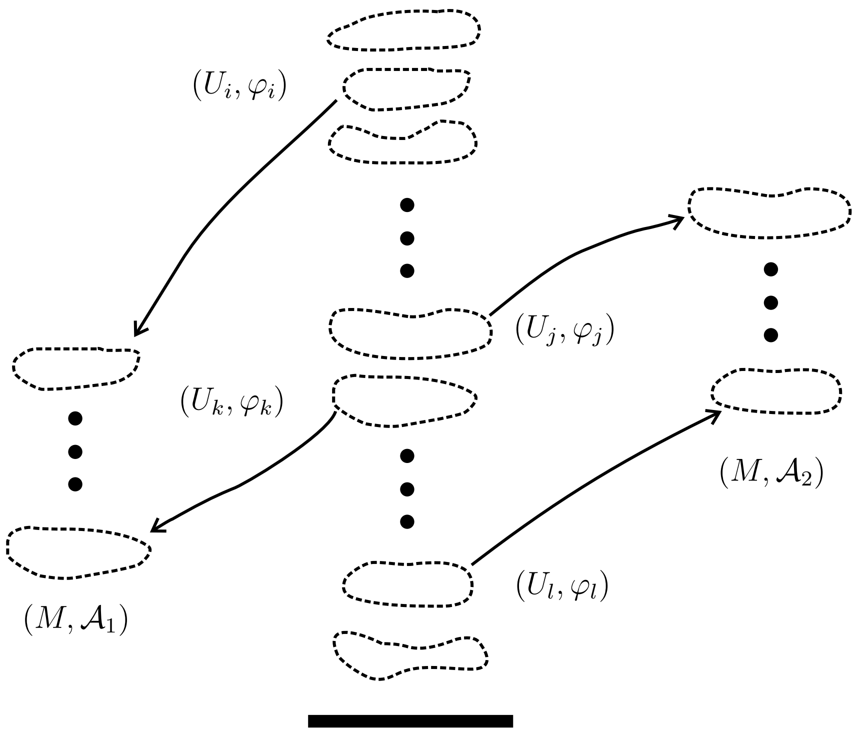
A smooth atlas $\mathcal{A}$ is said to be maximal if it does not exist a smooth atlas $\widetilde{\mathcal{A}}$ strictly larger that $\mathcal{A}$ ($\mathcal{A} \subset \widetilde{\mathcal{A}}$ and $\mathcal{A} \neq \widetilde{\mathcal{A}}$).

Definition [Smooth manifold] A smooth manifold is a pair $(M, \mathcal{A})$ where M is a topological manifold and $\mathcal{A}$ is a maximal smooth atlas on M.

▷ *Intuition: A smooth manifold is a topological manifold which can be covered by coordinate charts which fit nicely together.*

Theorem [Maximal smooth atlas] Let M be a topological manifold. If $\mathcal{A}$ is a smooth atlas on M, then there exists an unique maximal smooth atlas on M which contains $\mathcal{A}$.

▷ *Analogy: a straight-line through the origin can be specified by a single non-zero point (you don't need to furnish <u>all</u> points).*



Stack of all coordinate charts of M

Example 1 **[Obvious example]** $\mathsf{M}(n, m, \mathbb{R}) \simeq \mathbb{R}^{nm}$ is a smooth manifold of dimension nm.

Example 2 **[Vector space]** A vector space of dimension n is a smooth manifold of dimension n. Examples: any linear subspace in $\mathbb{R}^n$ and $\mathsf{S}(n, \mathbb{R})$.

Example 3 **[Unit-sphere]** The unit-sphere

$$\mathsf{S}^{n-1}(\mathbb{R}) = \{x \in \mathbb{R}^n \, : \, \|x\| = 1\}$$

is a smooth manifold of dimension $n - 1$.

Definition **[Smooth chart]** Let $(M, \mathcal{A})$ be a smooth manifold. Any coordinate chart (U, φ) contained in the given maximal smooth atlas $\mathcal{A}$ is called a smooth chart.

Lemma **[Existence of special smooth charts]** Let $(M, \mathcal{A})$ be a smooth manifold of dimension m. If (U, φ) is a smooth chart and $V \subset U$ is an open subset, then $(V, \varphi|_V)$ is a smooth chart.

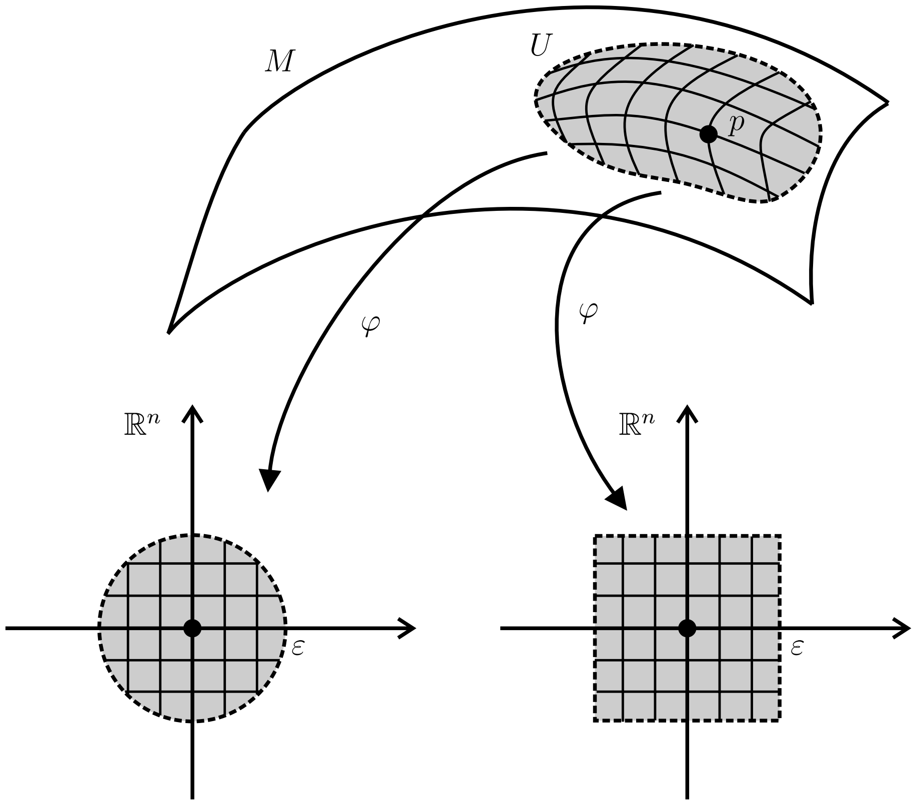
Let $p \in M$ and let $\varepsilon > 0$ be given. Then, there exists a smooth chart (U, φ) containing p such that $\varphi(U) = B^m_\varepsilon(0)$ or $\varphi(U) = C^m_\varepsilon(0)$, where

$$B^m_\varepsilon(0) = \left\{ (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \, : \, \sum_{i=1}^{m} x_i^2 < 1 \right\}$$

$$C^m_\varepsilon(0) = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \, : \, |x_i| < \varepsilon, \text{ for all } i\}\,.$$

(Note: a smooth chart (U, φ) is said to be centered at $p \in U$ if $\varphi(p) = (0, \ldots, 0)$).

▷ *Intuition: A smooth manifold contains a lot of smooth charts, many of them nice.*



Lemma [Open submanifold] Let $(M, \mathcal{A})$ be a smooth n-manifold and $W \subset M$ an open subset. Then, $\mathcal{A}_W = \{(U, \varphi) \in \mathcal{A} : U \subset W\}$ is a smooth atlas on the topological n-manifold W (W is given the subspace topology).

Theorem [Product manifolds] Let $(M, \mathcal{A}_M)$ and $(N, \mathcal{A}_N)$ be smooth manifolds of dimension m and n. Then,

$$\mathcal{A}_{M \times N} = \{(U \times V, \varphi \times \psi) : (U, \varphi) \in \mathcal{A}_M,\, (V, \psi) \in \mathcal{A}_N\}$$

is a smooth atlas on $M \times N$ and it is called the smooth (canonical) product atlas on the topological $(m+n)$-manifold $M \times N$.

References

[1] J. Lee, *Introduction to Smooth Manifolds*, Springer-Verlag, 2000.