Nonlinear Signal Processing (2004/2005)

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Connectedness and compactness

Definition [Connected space] Let X be a topological space. A separation of X is a pair of nonempty, disjoint, open subsets $U, V \subset X$ such that $X = U \cup V$. X is said to be disconnected if there exists a separation of X, and connected otherwise.

Definition [Connected subset] Let X be a topological space. A subset $A \subset X$ is said to be connected if the subspace A is connected.

In equivalent terms, the subset A is disconnected if there exist open sets U, V in X such that

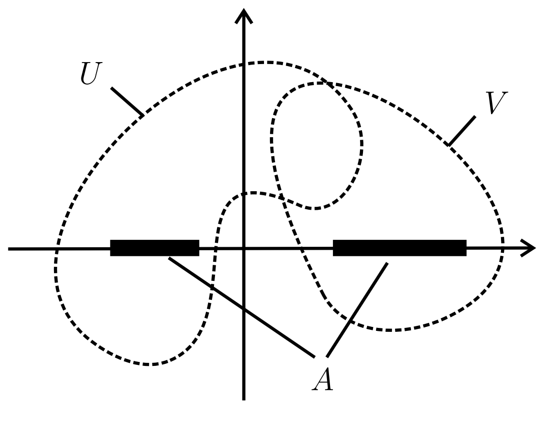
$$A \cap U \neq \emptyset, \quad A \cap V \neq \emptyset, \quad (A \cap U) \cap (A \cap V) = \emptyset, \quad A \subset U \cup V.$$

The sets U, V above are also considered a separation of A.

Example 1 [A simple disconnected subset] The subset

$$A = \{(x, y) \in \mathbb{R}^2 \,:\, x \in [-3, 1) \cup (2, 5],\ y = 0\}$$

of $\mathbb{R}^2$ is disconnected. Equivalently, the topological space A (endowed with the subspace topology) is disconnected.



Example 2 **[A more interesting disconnected subset]** The subset

$$\mathsf{O}(n) = \{X \in \mathsf{M}(n, \mathbb{R}) \, : \, X^T X = I_n\}$$

of $\mathsf{M}(n, \mathbb{R})$ is disconnected. Equivalently, the topological space $\mathsf{O}(n)$ (endowed with the subspace topology) is disconnected.

Note that $\mathsf{O}(n) \subset \{X \in \mathsf{M}(n, \mathbb{R}) \, : \, \det X = \pm 1\}$. The open sets

$$U = \{X \in \mathsf{M}(n, \mathbb{R}) \, : \, \det X < 0\} \quad V = \{X \in \mathsf{M}(n, \mathbb{R}) \, : \, \det X > 0\}$$

provide a separation of $\mathsf{O}(n)$.

(Remark that $\mathsf{O}(n) \cap U \neq \emptyset$ and $\mathsf{O}(n) \cap V \neq \emptyset$; why ?)

Proposition **[Characterization of connectedness]** A topological space X is connected if and only if the only subsets of X that are both open and closed are $\emptyset$ and X.

Example 1 **[Application of connectedness]** Let X be a connected topological space and $A : X \to \mathsf{S}(n, \mathbb{R})$ a continuous map. Thus, the map $x \mapsto A(x)$ assigns (continuously) a symmetric matrix to each point in X. Suppose the polynomial equation

$$\sum_{k=0}^{n} c_k A(x)^k = 0$$

is satisfied for all $x \in X$, where $c_k \in \mathbb{R}$ are fixed, real coefficients. Then, the spectrum (set of eigenvalues, including multiplicities) of $A(x)$ is constant over $x \in X$.

Proof: Pick a $x_0 \in X$, let $A_0 = A(x_0)$ and let

$$\sigma_0 = \{\lambda_1(A_0), \lambda_2(A_0), \ldots, \lambda_n(A_0)\}$$

denote its spectrum. We assume that the eigenvalues are ordered in non-increasing order:

$$\lambda_1(A_0) \geq \lambda_2(A_0) \geq \cdots \geq \lambda_n(A_0).$$

Define the subset

$$S = \{x \in X \, : \, \sigma(A(x)) = \sigma(A_0)\}.$$

Note that $S \neq \emptyset$ *because* $x_0 \in S$. *We will show that* S *is both open and closed in* X. *Since* X *is connected, this establishes that* $S = X$ *by the previous proposition. To show that* S *is closed, let* $\eta_i : X \to \mathbb{R}$, $\eta_i(x) = \lambda_i(A(x))$, *for* $i = 1, 2, \ldots, n$. *That is,* $\eta_i(x)$ *computes the ith ordered eigenvalue of* $A(x)$. *Note that each* η_i *is a continuous function (composition of continuous maps). Thus, each subset* $S_i = \eta_i^{-1}(\lambda_i(A_0))$ *is closed in* X. *Since* $S = S_1 \cap S_2 \cap \cdots \cap S_n$, *it follows that* S *is closed in* X. *To show that* S *is open, we reason as follows. Let* $z_1, z_2, \ldots, z_m \in \mathbb{C}$ *be the distinct roots of the polynomial equation*

$$p(z) = \sum_{k=0}^{n} c_k z^k = 0.$$

Note that, since $p(A(x)) = 0$, *we have* $\lambda_i(A(x)) \in \{z_1, z_2, \ldots, z_m\}$ *for all* i *and* $x \in X$. *Let*

$$\delta = \min_{k \neq l} |z_k - z_l|$$

be the minimum distance between the distinct roots. Thus, if $z \in \{z_1, z_2, \ldots, z_m\}$ *and* $|z - z_i| < \delta$, *then* $z = z_i$. *The subset*

$$U_i = \eta_i^{-1}\left((\lambda_i(A_{x_0}) - \delta, \lambda_i(A_{x_0}) + \delta)\right)$$

is open in X *(thanks to the continuity of* η_i*). By the previous argument,* $x \in U_i$ *implies* $\lambda_i(A(x)) = \lambda_i(A(x_0))$. *Thus, the open subset* $U = \bigcap_{i=1}^{n} U_i$ *is contained in* S. *But, also trivially,* $S \subset U$. *Thus,* $S = U$.

Proposition [Characterization of connected subsets of $\mathbb{R}$**]** A nonempty subset of $\mathbb{R}$ is connected if and only if it is an interval.

Definition [Path connected space] Let X be a topological space and $p, q \in X$. A path in X from p to q is a continuous map $f : [0, 1] \to X$ such that $f(0) = p$ and $f(1) = q$.

We say that X is path connected if for every $p, q \in X$ there is a path in X from p to q.

Theorem [Easy sufficient criterion for connectedness] If X is a path connected topological space, then X is connected.

Example 1 [Obvious example] $\mathsf{M}(n, m, \mathbb{R}) \simeq \mathbb{R}^{nm}$ is connected

Example 2 [Convex sets are connected]

$$\mathsf{S}(n, \mathbb{R}) = \{X \in \mathsf{M}(n, \mathbb{R}) \,:\, X = X^T\} \text{ is connected}$$

$$U^+(n, \mathbb{R}) = \{X \in \mathsf{M}(n, \mathbb{R}) : X \text{ upper-triangular and } X_{ii} > 0\} \text{ is connected}$$

Example 3 [Special orthogonal matrices]

$$\mathsf{SO}(n) = \{X \in \mathsf{O}(n) \,:\, \det(X) = 1\} \text{ is connected}$$

because there is a path in $\mathsf{SO}(n)$ from I_n to any $X \in \mathsf{SO}(n)$.

Illustrative example: suppose $X \in \mathsf{SO}(5)$ has the eigenvalue decomposition

$$X = Q \begin{bmatrix} \cos\theta & -\sin\theta & & & \\ \sin\theta & \cos\theta & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & 1 \end{bmatrix} Q^T, \qquad Q \in \mathsf{O}(n).$$

(Note: if $X \in \mathsf{SO}(n)$ the multiplicity of the eigenvalue -1 is even.)

Then, $f \,:\, [0, 1] \to \mathsf{SO}(5)$,

$$f(t) = Q \begin{bmatrix} \cos(\theta t) & -\sin(\theta t) & & & \\ \sin(\theta t) & \cos(\theta t) & & & \\ & & \cos(\pi t) & -\sin(\pi t) & \\ & & \sin(\pi t) & \cos(\pi t) & \\ & & & & 1 \end{bmatrix} Q^T,$$

is a path in $\mathsf{SO}(5)$ from I_5 to X.

Example 4 **[Non-singular matrices with positive determinant]**

$$\mathsf{GL}^+(n,\mathbb{R}) = \{X \in \mathsf{M}(n,\mathbb{R}) \,:\, \det(X) > 0\} \text{ is connected}$$

because there is a path in $\mathsf{GL}^+(n,\mathbb{R})$ from I_n to any $X \in \mathsf{GL}^+(n,\mathbb{R})$.

Proof: let $X \in \mathsf{GL}^+(n,\mathbb{R})$. Invoking the QR decomposition of X (and noting that $\det X > 0$), we see that there exist $Q \in \mathsf{SO}(n)$ and $U \in \mathsf{U}^+(n,\mathbb{R})$ such that

$$X = QU.$$

Since both $\mathsf{SO}(n)$ and $\mathsf{U}^+(n,\mathbb{R})$ are connected, let $Q(t)$ and $U(t)$ be paths in $\mathsf{SO}(n)$ and in $\mathsf{U}^+(n,\mathbb{R})$ from I_n to Q and U, respectively. Then, $X(t) = Q(t)U(t)$ is a path in $\mathsf{GL}^+(n,\mathbb{R})$ from I_n to X.

Example 5 **[Special Euclidean group]**

$$\mathsf{SE}(n) = \left\{ \begin{bmatrix} Q & \delta \\ 0 & 1 \end{bmatrix} \,:\, Q \in \mathsf{SO}(n), \delta \in \mathbb{R}^n \right\} \text{ is connected}$$

because there is a path in $\mathsf{SE}(n)$ from

$$\begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}$$

to any $X \in \mathsf{SE}(n)$.

Proof: let

$$X = \begin{bmatrix} Q & \delta \\ 0 & 1 \end{bmatrix} \in \mathsf{SE}(n).$$

Let $Q(t)$ be a path in $\mathsf{SO}(n)$ from I_n to Q, and $\delta(t)$ a path in $\mathbb{R}^n$ from 0 to δ. Then

$$f(t) = \begin{bmatrix} Q(t) & \delta(t) \\ 0 & 1 \end{bmatrix}$$

is the desired path.

Theorem **[Main theorem on connectedness]** Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map. If X is connected, then $f(X)$ (as a subspace of Y) is connected.

Example 1 **[Unit-sphere]**

$$\mathsf{S}^{n-1}(\mathbb{R}) = \{x \in \mathbb{R}^n \,:\, \|x\| = 1\}$$

is connected, because it is the image of the connected space $\mathbb{R}^{n+1} - \{0\}$ through the continuous map

$$f \,:\, \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^n \qquad f(x) = \frac{x}{\|x\|}.$$

Example 2 **[Ellipsoid]** Any non-flat ellipsoid in $\mathbb{R}^n$ can be described as

$$\mathsf{E} = \left\{Au + x_0 \,:\, u \in \mathsf{S}^{n-1}(\mathbb{R})\right\}$$

where $x_0 \in \mathbb{R}^n$ is the center of the ellipsoid and $A \in \mathsf{GL}(n, \mathbb{R})$ defines the shape and spatial orientation of E.

Thus E is connected because it is the image of the connected space $\mathsf{S}^{n-1}(\mathbb{R})$ through the continuous map

$$f \,:\, \mathsf{S}^{n-1}(\mathbb{R}) \to \mathbb{R}^n \qquad f(x) = Ax + x_0.$$

Example 3 **[Projective space $\mathbb{RP}^n$]** $\mathbb{RP}^n$ is connected because it is the image of the connected space $\mathbb{R}^{n+1} - \{0\}$ through the continuous projection map

$$\pi \,:\, \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n \qquad \pi(x) = [x].$$

Proposition **[Properties of connected spaces]**

(a) Suppose X is a topological space and U, V are disjoint open subsets of X. If A is a connected subset of X contained in $U \cup V$, then either $A \subset U$ or $A \subset V$.

(b) Suppose X is a topological space and $A \subset X$ is connected. Then $\overline{A}$ is connected.

(c) Let X be a topological space, and let $\{A_i\}$ be a collection of connected subsets with a point in common. Then $\bigcup_i A_i$ is connected.

(d) The Cartesian product of finitely many connected topological spaces is connected.

(e) Any quotient space of a connected topological space is connected.

Theorem **[Intermediate value theorem]** Let X be a connected topological space and f is a continuous real-valued function on X. If $p, q \in X$ then

f takes on all values between $f(p)$ and $f(q)$.

Example 1 [Antipodal points at the same temperature] Let

$$T : \mathsf{S}^1(\mathbb{R}) \subset \mathbb{R}^2 \to \mathbb{R}$$

be a continuous map on the unit-circle in $\mathbb{R}^2$. Then, there exist a point $p \in \mathsf{S}^1(\mathbb{R})$ such that $T(p) = T(-p)$.

Proof: The map

$$f : [0, 2\pi] \to \mathbb{R} \qquad f(\theta) = T(\cos\theta, \sin\theta) - T(-\cos\theta, -\sin\theta)$$

is continuous. If $f(0) = 0$, we can pick $p = (1, 0)$. Otherwise, $f(0)f(\pi) < 0$ and there exists $\theta_0 \in [0, \pi]$ such that $f(\theta_0) = 0$. Make $p = (\cos\theta_0, \sin\theta_0)$.

As a consequence, this shows that there two antipodal points in the Earth's equator line at the same temperature.

Definition [Components] Let X be a topological space. A component of X is a maximally connected subset of X, that is, a connected set that is not contained in any larger connected set.

▷ *Intuition: X consists of a union of disjoint "islands"/components.*

Example 1 [Orthogonal group] The orthogonal group

$$\mathsf{O}(n) = \{X \in \mathsf{M}(n, \mathbb{R}) : X^T X = I_n\}$$

has two components:

$$\mathsf{SO}(n) = \{X \in \mathsf{O}(n, \mathbb{R}) : \det X = 1\}$$
$$\mathsf{O}^-(n) = \{X \in \mathsf{O}(n, \mathbb{R}) : \det X = -1\}.$$

Proof: We have already seen that $\mathsf{SO}(n)$ is connected. Any attempt to enlarge $\mathsf{SO}(n)$ involves taking a point in $\mathsf{O}^-(n)$. But, then, the sets $U = \{X \in \mathsf{M}(n, \mathbb{R}) : \det X < 0\}$ and $V = \{X \in \mathsf{M}(n, \mathbb{R}) : \det X > 0\}$

would provide a separation of such set. Thus, $\mathsf{SO}(n)$ is a component of $\mathsf{O}(n)$. The set $\mathsf{O}^-(n)$ is connected because it is the image of $\mathsf{SO}(n)$ through the continuous map

$$f : \mathsf{SO}(n) \to \mathsf{M}(n, \mathbb{R}) \qquad f(X) = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} X.$$

Thus, $\mathsf{O}^-(n)$ is a component of $\mathsf{O}(n)$.

Proposition [Properties of components] Let X be any topological space.

(a) Each component of X is closed in X.

(b) Any connected subset of X is contained in a single component.

Definition [Compact space] A topological space X is said to be compact if every open cover of X has a finite subcover. That is, if $\mathcal{U}$ is any given open cover of X, then there are finitely many sets $U_1, \ldots, U_k \in \mathcal{U}$ such that $X = U_1 \cup \cdots \cup U_k$.

Definition [Compact subset] Let X be a topological space. A subset $A \subset X$ is said to be compact if the subspace A is compact.

In equivalent terms, the subset A is compact if and only if given any collection of open subsets of X covering A, there is a finite subcover.

Proposition [Characterization of compact sets in $\mathbb{R}^n$] A subset X in $\mathbb{R}^n$ is compact if and only if X is closed and bounded.

Example 1 [Stiefel manifold] The set

$$\mathsf{O}(n, m) = \{X \in \mathsf{M}(n, m, \mathbb{R}) \, : \, X^T X = I_m\}$$

is compact because it is closed and bounded.

It is closed because $\mathsf{O}(n,m) = f^{-1}(\{I_m\})$ and

$$f : \mathsf{M}(n,m,\mathbb{R}) \to \mathsf{M}(m,\mathbb{R}) \qquad f(X) = X^T X$$

is continuous.

It is bounded because, if $X \in \mathsf{O}(n,m)$ then

$$\|X\|^2 = \mathsf{tr}(X^T X) = \mathsf{tr}(I_m) = m.$$

Note that $\mathsf{O}(n,1) = \mathsf{S}^{n-1}(\mathbb{R})$ and $\mathsf{O}(n,n) = \mathsf{O}(n)$.

Theorem [Main theorem on compactness] Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map. If X is compact, then $f(X)$ (as a subspace of Y) is compact.

Example 1 [Projective space $\mathbb{RP}^n$] The projective space $\mathbb{RP}^n$ is compact because it is the image of the compact set $\mathsf{S}^n(\mathbb{R})$ through the continuous projection map $\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n$.

Proposition [Properties of compact spaces]

(a) Every closed subset of a compact space is compact.

(b) In a Hausdorff space X, compact sets can be separated by open sets. That is, if $A, B \subset X$ are disjoint compact subsets, there exist disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$.

(c) Every compact subset of a Hausdorff space is closed.

(d) The Cartesian product of finitely many compact topological spaces is compact.

(e) Any quotient space of a compact topological space is compact.

Example 1 [Special orthogonal matrices]

$$\mathsf{SO}(n) = \{X \in \mathsf{O}(n) : \det X = 1\}$$

is compact because it is a closed subset of the compact space $\mathsf{O}(n)$.

It is closed because $\mathsf{SO}(n) = f^{-1}(\{1\})$ and

$$f : \mathsf{O}(n) \to \mathbb{R} \qquad f(X) = \det X$$

is continuous.

Theorem [Extreme value theorem] If X is a compact space and $f : X \to \mathbb{R}$ is continuous, then f attains its maximum and minimum values on X.

Proposition [Characterization of compactness in 2nd countable Hausdorff spaces] Let X be a 2nd countable Hausdorff space. The following are equivalent:

(a) X is compact

(b) Every sequence of points in X has a subsequence that converges to a point in X.

Example 1 [Principal component analysis is a continuous map] Let X, Y be 2nd countable Hausdorff spaces. Furthermore, let Y be compact. Let $F : X \times Y \to \mathbb{R}$ be a continuous function. For each $x \in X$, we define the function $F_x : Y \to \mathbb{R}$, $F_x(y) = F(x, y)$.

Suppose that, for each $x \in X$, there exists only one global minimizer in Y of the function F_x. Let $\phi : X \to Y$ be the map which, given $x \in X$, returns the (unique) global minimizer in Y of the function F_x. The map ϕ is continuous.

Proof: In this topological setting, it suffices to prove that $x_n \to x_0$ implies $y_n = \phi(x_n) \to y_0 = \phi(x_0)$. Suppose $y_n \not\to y_0$ (we will reach a contradiction). Then, there exists an open set U in Y containing y_0 and a subsequence y_{n_k} such that $y_{n_k} \notin U$. Since Y is compact, the sequence y_{n_k} admits a convergent subsequence, say, $y_{n_{k_l}} \to z$. Now, we claim $z = y_0$. To show this, choose $y \in Y$ arbitrarily. We have

$$F(x_{n_{k_l}}, y_{n_{k_l}}) \leq F(x_{n_{k_l}}, y) \quad \text{for all } l.$$

Taking the limit $l \to \infty$ yields

$$F(x_0, z) \leq F(x_0, y).$$

Since y was chosen arbitrarily, this shows that z is a global minimizer of F_{x_0}. By uniqueness, $z = y_0$. Since the sequence $y_{n_{k_l}} \to z = y_0$ it has a point in U (contradiction!).

Let $P = [p_1\, p_2\, \ldots\, p_k] \in \mathsf{M}(n, k, \mathbb{R})$ denote a constellation of k points in $\mathbb{R}^n$. A one-dimensional principal component analysis (PCA) of P consists in extracting the "dominant" straight line in P. That is, the straight line spanned by a vector

$$\begin{aligned} \widehat{x}(P) \quad &\in \quad \underset{x \in \mathbb{R}^n - \{0\}}{\arg\min} \sum_{j=1}^{k} \left\| p_j - \frac{xx^T}{\|x\|^2} p_j \right\|^2 \\ &= \quad \underset{x \in \mathbb{R}^n - \{0\}}{\arg\max} \frac{x^T P P^T x}{\|x\|^2}. \end{aligned}$$

The straight line is unique if $\lambda_{\mathsf{max}}(PP^T)$ is simple. In equivalent terms, order the eigenvalues of the $n \times n$ symmetric matrix PP^T as

$$\underbrace{\lambda_n(PP^T)}_{\lambda_{\mathsf{min}}(PP^T)} \leq \lambda_{n-1}(PP^T) \leq \cdots \leq \lambda_2(PP^T) \leq \underbrace{\lambda_1(PP^T)}_{\lambda_{\mathsf{max}}(PP^T)}.$$

The dominant straight line is unique for those constellations P belonging to

$$\mathcal{P} = \left\{ P \in \mathsf{M}(n, k, \mathbb{R}) \,:\, \lambda_1(PP^T) > \lambda_2(PP^T) \right\}.$$

Since each $\lambda_j \,:\, \mathsf{S}(n, \mathbb{R}) \to \mathbb{R}$ is a continuous function on $\mathsf{S}(n, \mathbb{R})$, the set of $n \times n$ symmetric matrices with real entries (see your homework for this result!), the set $\mathcal{P}$ is open in $\mathsf{M}(n, k, \mathbb{R})$.

Thus, we have a map $\mathsf{PCA} \,:\, \mathcal{P} \to \mathbb{RP}^{n-1}$

$$P \in \mathcal{P} \longrightarrow \boxed{\mathsf{PCA}} \longrightarrow \pi(\widehat{x}(P)) \in \mathbb{RP}^{n-1}$$

The map PCA is continuous, because

Step 1: The map

$$F : \mathbb{RP}^{n-1} \times \mathsf{M}(n,k,\mathbb{R}) \to \mathbb{R} \qquad F([x],P) = \sum_{j=1}^{k} \left\| p_j - \frac{xx^T}{\|x\|^2} p_j \right\|^2$$

is continuous (as we have already seen in a previous example).

Step 2: Its restriction to the subspace $\mathbb{RP}^{n-1} \times \mathcal{P} \subset \mathbb{RP}^{n-1} \times \mathsf{M}(n,k,\mathbb{R})$ is also continuous (for brevity of notation, we keep the same symbol F):

$$F : \mathbb{RP}^{n-1} \times \mathcal{P} \to \mathbb{R} \qquad F([x],P) = \sum_{j=1}^{k} \left\| p_j - \frac{xx^T}{\|x\|^2} p_j \right\|^2 .$$

Step 3: $\mathsf{PCA} : \mathcal{P} \to \mathbb{RP}^{n-1}$ extracts, for each $P \in \mathcal{P}$, the (unique) minimizer of F_P in $\mathbb{RP}^{n-1}$. Since $\mathbb{RP}^{n-1}$ is compact, the previous result shows that PCA is continuous.

References

[1] J. Lee, *Introduction to Topological Manifolds*, Springer-Verlag, 2000.