Nonlinear Signal Processing (2004/2005)

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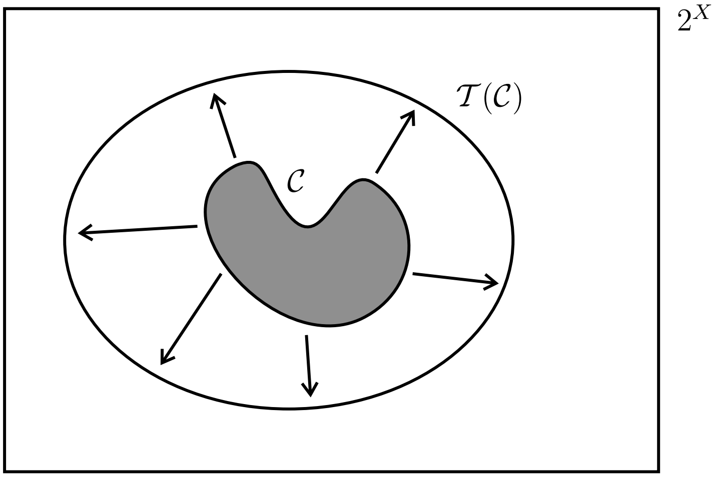
New Spaces from Old

$$\textit{Three basic mechanisms :} \begin{cases} \textit{subspaces} \\ \textit{Cartesian products} \\ \textit{quotients} \end{cases}$$

Definition [**Topology generated by a class of subsets**] Let X be a nonempty set and $\mathcal{C}$ a class of subsets of X. The topology generated by $\mathcal{C}$, written $\mathcal{T}(\mathcal{C})$, is defined as the smallest topology containing the class $\mathcal{C}$.



Lemma [**Characterization of generated topologies**] Let X be a nonempty set and $\mathcal{C}$ a class of subsets of X. Then $\mathcal{T}(\mathcal{C})$ is the class of all unions of finite intersections of sets in $\mathcal{C}$.

That is, $U \in \mathcal{T}(\mathcal{C})$ if and only if

$$U = \bigcup_{\alpha \in A} U_\alpha, \qquad U_\alpha = C_\alpha^1 \cap C_\alpha^2 \cap \cdots C_\alpha^n, \qquad C_\alpha^i \in \mathcal{C}.$$

Also, the collection $\{U_\alpha\}$ is a basis for $\mathcal{T}(\mathcal{C})$.

Definition [Subspace topology] Let X be a topological space and $A \subset X$ be any subset. The subspace topology $\mathcal{T}_A$ on A is defined as

$$\mathcal{T}_A = \{A \cap U \,:\, U \text{ open in } X\}\,.$$

Let $A \subset X$ be any subset. By the subspace A of X we mean the topological space $(A, \mathcal{T}_A)$ where $\mathcal{T}_A$ is the subspace topology on A.

Example 1 [Open ? Depends where...] Consider the subspace $A = [0,2)$ of $X = \mathbb{R}$.

The set $[0,1)$ is not open in X.

The set $[0,1)$ is open in A, because it can be written as

$$[0,1) = \underbrace{[0,2)}_{A} \cap \underbrace{(-1,1)}_{\text{open in } X}\,.$$

Example 2 [Unit sphere in $\mathbb{R}^n$]

$$\mathsf{S}^{n-1}(\mathbb{R}) = \{x \in \mathbb{R}^n \,:\, \|x\| = 1\}$$

is a subspace of $\mathbb{R}^n$.

The set $U_i^+ = \{x \in \mathsf{S}^{n-1}(\mathbb{R}) \,:\, x_i > 0\}$ is open in $\mathsf{S}^{n-1}(\mathbb{R})$:

$$U_i^+ = \mathsf{S}^{n-1}(\mathbb{R}) \cap \underbrace{\{x \in \mathbb{R}^n \,:\, x_i > 0\}}_{\text{open in } \mathbb{R}^n}.$$

Example 3 [Group of $n \times n$ orthogonal matrices]

$$\mathsf{O}(n) = \{X \in \mathsf{M}(n,\mathbb{R}) \,:\, X^T X = I_n\}$$

is a subspace of $\mathsf{M}(n,\mathbb{R})$.

The set $U = \{X \in \mathsf{O}(n) \,:\, \det(X) > 0\}$ is open in $\mathsf{O}(n)$:

$$U = \mathsf{O}(n) \cap \underbrace{\{X \in \mathsf{M}(n,\mathbb{R}) \,:\, \det(X) > 0\}}_{\text{open in } \mathsf{M}(n,\mathbb{R})}.$$

Definition [Topological embedding] Let X and Y be topological spaces. An injective, continuous map $f : X \to Y$ is said to be a topological embedding if it is a homeomorphism onto its image $f(X)$ (endowed with the subspace topology).

▷ *Intuition: we can interpret* $X \simeq f(X)$ *as a subspace of* Y *(*X *is simply another label for a subspace of* Y*)*

Theorem [Characteristic property of subspace topologies] Let X, Y be topological spaces. Let A be a subspace of X. Then, a map $f : Y \to A$ is continuous if and only if $\widehat{f} = \iota_A \circ f$ is continuous.

Here, $\iota_A : A \to X$, $\iota_A(x) = x$ denotes the inclusion map of A into X.

$$\begin{tikzcd} Y \arrow[d, "f"'] \arrow[dr, "\widehat{f}"] & \\ A \arrow[r, "\iota_A"'] & X \end{tikzcd}$$

▷ *Intuition: continuity of the "hard" map* f *can be investigated through the easier map* $\widehat{f}$

Example 1 [Map into the unit sphere] The map

$$f : \mathbb{R}^n - \{0\} \to \mathsf{S}^{n-1}(\mathbb{R}), \qquad f(x) = \frac{x}{\|x\|}$$

is continuous, because

$$\widehat{f} : \mathbb{R}^n - \{0\} \to \mathbb{R}^n, \qquad \widehat{f}(x) = \frac{x}{\|x\|}$$

is clearly continuous.

Lemma [Other properties of the subspace topology] Suppose A is a subspace of the topological space X.

(a) The inclusion map $\iota_A : A \to X$ is continuous (in fact, a topological embedding).

(b) If $\widehat{f} : X \to Y$ is continuous then $f = \widehat{f}|_A : A \to Y$ is continuous.

(c) If $B \subset A$ is a subspace of A, then B is a subspace of X; in other words, the subspace topologies that B inherits from A and from X agree.

(d) If $\mathcal{B}$ is a basis for the topology of X, then

$$\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$$

is a basis for the topology of A.

(e) If X is Hausdorff and second countable then A is also Hausdorff and second countable.

Example 1 [Map out of the unit sphere] The map

$$f : \mathsf{S}^{n-1}(\mathbb{R}) \to \mathsf{M}(n, \mathbb{R}), \qquad f(x) = xx^T$$

is continuous, because

$$\widehat{f} : \mathbb{R}^n \to \mathsf{M}(n, \mathbb{R}), \qquad \widehat{f}(x) = xx^T$$

is <u>clearly</u> continuous and $f = \widehat{f}|_{\mathsf{S}^{n-1}(\mathbb{R})}$.

Example 2 **[Concatenating the techniques...]** The map

$$f \,:\, \mathsf{O}(n) \to \mathsf{S}^{n-1}(\mathbb{R}), \qquad f(X) = f([\,x_1\,x_2\,\cdots\,x_n\,]) = x_1$$

is continuous because

Step 1:

$$\widehat{f} \,:\, \mathsf{M}(n,\mathbb{R}) \to \mathbb{R}^n, \qquad f(X) = x_1$$

is clearly continuous

Step 2:

$$\widehat{f}|_{\mathsf{O}(n)} \,:\, \mathsf{O}(n) \to \mathbb{R}^n, \qquad \widehat{f}|_{\mathsf{O}(n)}(X) = x_1$$

is continuous due to

$$\begin{tikzcd}
\mathsf{M}(n,\mathbb{R}) \arrow[r, "\widehat{f}"] & \mathbb{R}^n \\
\mathsf{O}(n) \arrow[u, "\iota_{\mathsf{O}(n)}"] \arrow[ur, "\widehat{f}|_{\mathsf{O}(n)}"'] &
\end{tikzcd}$$

Step 3:

$$f \,:\, \mathsf{O}(n) \to \mathsf{S}^{n-1}(\mathbb{R}), \qquad f(X) = x_1$$

is continuous due to

$$\begin{tikzcd}
& \mathbb{R}^n \\
\mathsf{O}(n) \arrow[ur, "\widehat{f}|_{\mathsf{O}(n)}"] \arrow[r, "f"'] & \mathsf{S}^{n-1}(\mathbb{R}) \arrow[u, "\iota_{\mathsf{S}^{n-1}(\mathbb{R})}"']
\end{tikzcd}$$

Example 3 **[A topological manifold]** $\mathsf{S}^{n-1}(\mathbb{R})$ is a topological manifold of dimension $n-1$.

Example 4 [**Another(?) topological manifold**] The set of 2×2 special orthogonal matrices

$$\mathsf{SO}(2) = \left\{ X \in \mathsf{M}(2, \mathbb{R}) \ : \ X^T X = I_2, \det(X) = 1 \right\}$$

is a topological manifold of dimension 1, because the map

$$f \, : \, \mathsf{S}^1(\mathbb{R}) \to \mathsf{SO}(2), \qquad f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

is a homeomorphism.

Definition [**Product topology**] Let $X_1, X_2, \ldots, X_n$ be topological spaces. The product topology on the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ is the topology generated by the collection of rectangles

$$\mathcal{C} = \{U_1 \times U_2 \times \cdots U_n \ : \ U_i \text{ is open in } X_i\}.$$

Note that $\mathcal{C}$ is a basis for the product topology.

The set $X_1 \times \cdots \times X_n$ equipped with the product topology is called a product space.

Theorem [**Characteristic property of product topologies**] Let $X_1 \times \cdots \times X_n$ be a product space and let Y be a topological space. Then, the map $f \, : \, Y \to X_1 \times \cdots \times X_n$ is continuous if and only if each map $f_i \, : \, Y \to X_i$, $f_i = \pi_i \circ f$ is continuous.

Here, $\pi_i \, : \, X_1 \times X_2 \times \cdots \times X_n \to X_i$, $\pi_i(x_1, x_2, \ldots, x_n) = x_i$ denotes the projection map onto the ith factor X_i.

$$\begin{tikzcd}
Y \arrow[d, "f"'] \arrow[dr, "f_i"] & \\
X_1 \times \cdots \times X_n \arrow[r, "\pi_i"'] & X_i
\end{tikzcd}$$

Example 1 [**Decomposing a vector in amplitude and direction**] The map

$$f : \mathbb{R}^n - \{0\} \to \mathbb{R}^+ \times \mathsf{S}^{n-1}(\mathbb{R}), \qquad f(x) = \left(\|x\|\,, \frac{x}{\|x\|} \right),$$

is continuous.

Lemma [**Other properties of the product topology**] Let $X_1, \ldots, X_n$ be topological spaces.

(a) The projection maps $\pi_i : X_1 \times \cdots \times X_n \to X_i$ are continuous and open.

(b) Let $x_j \in X_j$ be fixed for $j \neq i$. The map

$$f : X_i \to X_1 \times \cdots \times X_n, \qquad f(x) = (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)$$

is a topological embedding.

(c) If $\mathcal{B}_i$ is a basis for the topology of X_i, then the class

$$\mathcal{B} = \{B_1 \times \cdots \times B_n : B_i \in \mathcal{B}_i\}$$

is a basis for the topology of the product space $X_1 \times \cdots \times X_n$.

(d) If A_i is a subspace of X_i, for $i = 1, \ldots, n$, the product topology and the subspace topology on $A_1 \times \cdots \times A_n \subset X_1 \times \cdots \times X_n$ are identical.

(e) If each X_i is Hausdorff and second countable then the product space $X_1 \times \cdots \times X_n$ is also Hausdorff and second countable.

Definition [**Product map**] If $f_i : X_i \to Y_i$ are maps for $i = 1, \ldots, n$, their product map, written $f_1 \times \cdots \times f_n$, is defined as

$$f_1 \times \cdots \times f_n : X_1 \times \cdots \times X_n \to Y_1 \times \cdots \times Y_n,$$

$$(f_1 \times \cdots \times f_n)(x_1, \ldots, x_n) = (f_1(x_1), \ldots, f_n(x_n)).$$

Proposition [**Product map**] A product of continuous maps is continuous, and a product of homeomorphisms is a homeomorphism.

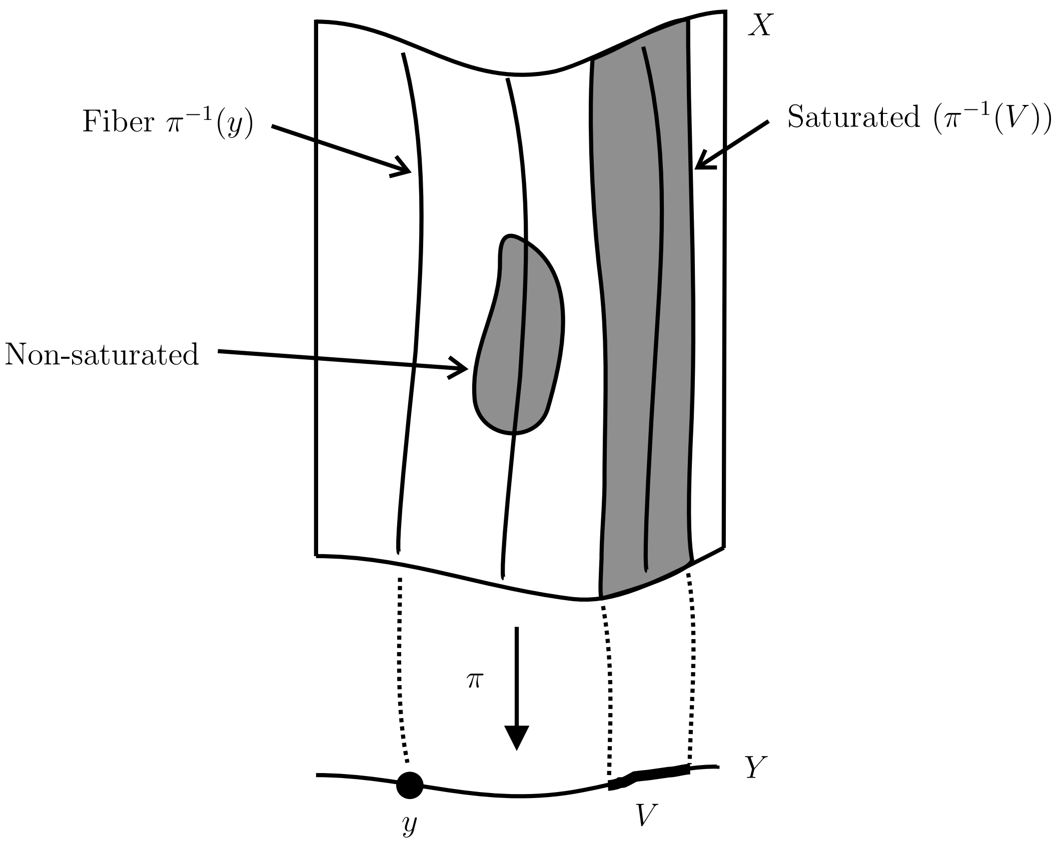
Proposition [Product manifolds] If $M_1, \ldots, M_k$ are topological manifolds of dimensions $n_1, \ldots, n_k$, respectively, the product space $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$.

▷ *Intuition: if each X_i has n_i "degrees of freedom", then $X_1 \times \cdots \times X_k$ has $n_1 + \cdots + n_k$ "degrees of freedom"*

Definition [Saturated sets, fibers] Let X and Y be sets and $\pi : X \to Y$ be a surjective map.

A subset $\pi^{-1}(y) \subset X$ for $y \in Y$ is called a fiber of π.

A subset $U \subset X$ is said to be saturated (with respect to π) if $U = \pi^{-1}(V)$ for some subset $V \subset Y$. Equivalent characterizations: $U = \pi^{-1}(\pi(U))$ or U is a union of fibers.



Example 1 Consider the surjective map

$$\pi : \mathbb{R}^2 \to \mathbb{R}_0^+, \qquad \pi(x) = \|x\| .$$

The fibers of π are the circles centered at the origin and the origin itself.

The annulus $U = \{x \in \mathbb{R}^2 : 1 < \|x\| < 2\}$ is a saturated set.

Each coordinate axis of $\mathbb{R}^2$ is non-saturated.

Definition [Quotient topology] Let X be a topological space, Y be any set, and $\pi : X \to Y$ be a surjective map. The quotient topology on Y induced by the map π is defined as

$$\mathcal{T}_\pi = \{U \subset Y : \pi^{-1}(U) \text{ is open in } X\}.$$

Example 1 [Real projective space $\mathbb{RP}^n$] Introduce the equivalence relation $\sim$ in $X = \mathbb{R}^{n+1} - \{0\}$:

$$x \sim y \qquad \text{if and only if} \qquad x \text{ and } y \text{ are colinear} .$$

Let $\mathbb{RP}^n = X/\sim$ denote the set of equivalence classes.

The map $\pi : X \to \mathbb{RP}^n$, $x \mapsto \pi(x) = [x]$ is surjective.

The projective space $\mathbb{RP}^n$ becomes a topological space by letting π induce the quotient topology.

The fibers of π are straight lines in $\mathbb{R}^{n+1} - \{0\}$.

Definition [Quotient map] Let X and Y be topological spaces. A surjective map $f : X \to Y$ is called a quotient map if the topology of Y coincides with $\mathcal{T}_f$ (the quotient topology induced by f). This is equivalent to saying that U is open in Y if and only if $f^{-1}(U)$ is open in X.

Lemma [Characterization of quotient maps] Let X and Y be topological spaces. A continuous surjective map $f : X \to Y$ is a quotient map if and only if it takes saturated open sets to open sets, or saturated closed sets to closed sets.

Lemma [Easy sufficient conditions for quotient maps] If $f : X \to Y$ is a surjective continuous map that is also an open or closed map, then it is a quotient map.

Lemma [Composition property of quotient maps] Suppose $\pi_1 : X \to Y$ and $\pi_2 : Y \to Z$ are quotient maps. Then their composition $\pi_2 \circ \pi_1 : X \to Z$ is also a quotient map.

Theorem [Characteristic property of quotient topologies] Let $\pi : X \to Y$ be a quotient map. For any topological space B, a map $f : Y \to B$ is continuous if and only if $\widehat{f} = f \circ \pi$ is continuous.

$$
\begin{tikzcd}
X \arrow[d, "\pi"'] \arrow[dr, "\widehat{f}"] & \\
Y \arrow[r, "f"'] & B
\end{tikzcd}
$$

$\triangleright$ *Intuition: continuity of the "hard" map f can be investigated through the easier map $\widehat{f}$*

Example 1 [Real projective space $\mathbb{RP}^n$] For $[x] \in \mathbb{RP}^n$, let $\mathsf{line}([x])$ be the straight line spanned by x. Let $x_0 \in \mathbb{R}^{n+1}$ be fixed.

The map

$$f : \mathbb{RP}^n \to \mathbb{R}, \qquad f([x]) = \mathsf{dist}(x_0, \mathsf{line}([x])),$$

is continuous, because

$$\widehat{f} : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}, \qquad \widehat{f}(x) = \left\| \left(I_n - x(x^T x)^{-1} x^T\right) x_0 \right\|$$

is clearly continuous.

Corollary [Passing to the quotient] Suppose $\pi : X \to Y$ is a quotient map, B is a topological space, and $\widehat{f} : X \to B$ is any continuous map that is constant on the fibers of π (that is, if $\pi(p) = \pi(q)$ then $\widehat{f}(p) = \widehat{f}(q)$). Then, there exists an unique continuous map $f : Y \to B$ such that $\widehat{f} = f \circ \pi$:

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\widehat{f}"] & \\ Y \arrow[r, "f"'] & B \end{tikzcd}$$

Example 1 [Elementary descent to $\mathbb{RP}^n$] Let $x_0 \in \mathbb{R}^{n+1} - \{0\}$ be fixed. The map

$$\widehat{f} : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}, \qquad \widehat{f}(x) = \mathsf{arccos}\left(\frac{|x_0^T x|}{\|x_0\|\,\|x\|}\right),$$

is continuous and descends to a continuous map in $\mathbb{RP}^n$.

Definition [Group] A group is an ordered pair $(G, *)$ consisting of a set G and a binary operation $* : G \times G \to G$ such that **i) [associativity]** for every $x, y, z \in G$ we have $(x * y) * z = x * (y * z)$, **ii) [identity]** there is $e \in G$ such that $e * x = x * e = e$ for all $x \in G$, and **iii) [inverse]** for each $x \in G$ there is a $y \in G$ such that $x * y = y * x = e$.

If the operation $*$ can be understood from the context, the group $(G, *)$ is simply denoted by G. Also, the symbol $*$ is usually dropped and we write xy instead of $x * y$.

Lemma [Elementary properties of groups] Let $(G, *)$ be a group.

(a) The identity element is unique (and is usually denoted by e), that is, if $e_1, e_2 \in G$ satisfy $e_i * x = x * e_i = x$ for all $x \in G$ and $i \in \{1, 2\}$, then $e_1 = e_2$

(b) The inverse is unique (and is usually denoted by x^{-1}), that is, if for a given $x \in G$, the elements $y_1, y_2 \in G$ satisfy $x * y_i = y_i * x = e$ for $i \in \{1, 2\}$, then $y_1 = y_2$.

Example 1 **[General linear group]** $\mathsf{GL}(n, \mathbb{R})$ is a group with matrix multiplication as the group operation. The identity element of the group is I_n. The inverse of A is A^{-1}.

Example 2 **[Group of orthogonal matrices]** $\mathsf{O}(n)$ is a group with matrix multiplication as the group operation.

Example 3 **[Group of special orthogonal matrices]**

$$\mathsf{SO}(n) = \{X \in \mathsf{O}(n) \, : \, \det(X) = 1\}$$

is a group with matrix multiplication as the group operation.

Example 4 **[Upper triangular matrices with positive diagonal entries]** The set

$$U^+(n, \mathbb{R}) = \{X \in \mathsf{M}(n, \mathbb{R}) : X \text{ is upper-triangular and } X_{ii} > 0 \text{ for all i } \}$$

is a group with matrix multiplication as the group operation.

Example 5 **[Group of rigid motions in $\mathbb{R}^n$]** The set

$$\mathsf{SE}(n) = \left\{ \begin{bmatrix} Q & \delta \\ 0 & 1 \end{bmatrix} \, : \, Q \in \mathsf{SO}(n), \delta \in \mathbb{R}^n \right\}$$

is a group with matrix multiplication as the group operation.

Definition [Subgroup, left translation, right translation, homomorphism, kernel of a homomorphism] Let $(G, *)$ be a group.

A subgroup of G is a set $H \subset G$ such that $e \in H$, $x * y \in H$ whenever $x, y \in H$, and $x^{-1} \in H$ whenever $x \in H$.

For each $g \in G$, we define the left translation map $L_g : G \to G$, $L_g(x) = g*x$. Similarly, we have the right translation map $R_g : G \to G$, $R_g(x) = x*g$.

Let $(H, \widetilde{*})$ denote a group with identity element $\widetilde{e}$. A map $F : G \to H$ is said to be a homomorphism if $F(x * y) = F(x)\widetilde{*}F(y)$ for all $x, y \in G$. The kernel of F is defined as

$$\mathsf{Ker}\, F = \{x \in G \,:\, F(x) = \widetilde{e}\}\,.$$

Note that $\mathsf{Ker}\, F$ is a subgroup of G.

Example 1 [Subgroups of the general linear group] $\mathsf{O}(n)$, $\mathsf{SO}(n)$ and $\mathsf{U}^+(n, \mathbb{R})$ are subgroups of $\mathsf{GL}(n, \mathbb{R})$.

$\mathsf{SE}(n)$ is a subgroup of $\mathsf{GL}(n+1, \mathbb{R})$.

Example 2 [Homomorphism] The map

$$F \,:\, \mathsf{GL}(n, \mathbb{R}) \to \mathsf{GL}(1, \mathbb{R}), \qquad F(X) = \det(X),$$

is a homomorphism.

Example 3 [Generalization of the previous result] The map

$$F \,:\, \mathsf{GL}(n, \mathbb{R}) \to \mathsf{GL}\left(\begin{pmatrix} n \\ k \end{pmatrix}, \mathbb{R}\right), \qquad f(X) = X^{[k]},$$

is a homomorphism (Cauchy-Binet formula).

Definition [Topological group] Let G be a group which is at the same time a topological space. Then, G is said to be a topological group if the maps $m : G \times G \to G$, $m(x, y) = xy$ and $\iota : G \to G$, $\iota(x) = x^{-1}$ are continuous.

Example 1 **[Famous topological groups]** $\mathsf{GL}(n,\mathbb{R})$, $\mathsf{O}(n)$, $\mathsf{SO}(n)$, $\mathsf{U}^+(n,\mathbb{R})$ and $\mathsf{SE}(n)$ are topological groups.

Definition **[Group action]** Let G be a group and X be a set. A left action of G on X is a map $\theta : G \times X \to X$ such that **i)** $\theta(e,x) = x$ for all $x \in X$ and **ii)** $\theta(g,\theta(h,x)) = \theta(gh,x)$ for all $g,h \in G$ and $x \in X$.

If the action θ is clear from the context, we write gx instead of $\theta(g,x)$. Thus, a left action satisfies $ex = x$ and $g(hx) = (gh)x$.

A right action of G on X is a map $\theta : X \times G \to X$ such that **i)** $\theta(x,e) = x$ for all $x \in X$ and **ii)** $\theta(\theta(g,x),h) = \theta(x,gh)$ for all $g,h \in G$ and $x \in X$.

If G is a topological group and X is a topological space, the action is said to be continuous if θ is continuous.

Example 1 **[$\mathsf{GL}(n,\mathbb{R})$ acts on $\mathbb{R}^n$]** The map

$$\theta : \mathsf{GL}(n,\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n, \qquad \theta(A,x) = Ax,$$

defines a continuous left action of $\mathsf{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$.

This is called the natural action of $\mathsf{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$.

Example 2 **[$\mathsf{O}(n)$ acts on $\mathsf{S}(n,\mathbb{R})$]** Let

$$\mathsf{S}(n,\mathbb{R}) = \{X \in \mathsf{M}(n,\mathbb{R}) : X = X^T\}$$

denote the set of $n \times n$ symmetric matrices with real entries.

The map

$$\theta : \mathsf{O}(n) \times \mathsf{S}(n,\mathbb{R}) \to \mathsf{S}(n,\mathbb{R}), \qquad \theta(Q,S) = QSQ^T,$$

defines a continuous left action of $\mathsf{O}(n)$ on $\mathsf{S}(n,\mathbb{R})$.

Lemma [Continuous left actions] Let $\theta : G \times X \to X$ be a continuous left action of G on X. For each $g \in G$, the map

$$\theta_g : X \to X, \qquad \theta_g(x) = \theta(g, x) = gx$$

is a homeomorphism.

▷ *Proof: The map θ_g is bijective because the map $\theta_{g^{-1}}$ is a left and right inverse for it, that is, $\theta_g \circ \theta_{g^{-1}} = \theta_{g^{-1}} \circ \theta_g = \mathsf{id}_X$. The map θ_g is continuous because it is the composition of two continuous maps: $\theta_g = \theta \circ \iota_g$, where $\iota_g : G \to G \times X$, $\iota_g(x) = (g, x)$. It is a homeomorphism because its inverse is given by $\theta_{g^{-1}}$, which is continuous*$\square$

Definition [Orbits,free/transitive actions,invariants,maximal invariants] Let $\theta : G \times X \to X$ denote a left action of the group G on a set X.

The orbit of $p \in X$ is the set $Gp = \{\theta(g, p) : g \in G\}$.

The action is said to be transitive if, for any given $p, q \in X$ there exists $g \in G$ such that $\theta(g, p) = q$.

The action is said to be free if $\theta(g, p) = p$ implies $g = e$.

An invariant of the action is a map $\phi : X \to Y$ (where Y denotes a set) which is constant on orbits, that is, $x, y \in Gp$ imply $\phi(x) = \phi(y)$.

A maximal invariant of the action is an invariant ϕ which differs from orbit to orbit, that is, $x \notin Gy$ implies $\phi(x) \neq \phi(y)$.

▷ *Intuition: when an action is transitive, there is only one orbit. If the action is free, each orbit is a "copy" of G. A maximal invariant permits to index the orbits.*

Example 1 The natural action of $\mathsf{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ is not transitive (it has two orbits, namely, $\{0\}$ and $\mathbb{R}^n - \{0\}$), it is not free and a maximal invariant is $\phi : \mathbb{R}^n \to \mathbb{R}$, $\phi(0) = 0$ and $\phi(x) = 1$ if $x \neq 0$.

Example 2 The action of $\mathsf{O}(n)$ on $\mathsf{S}(n, \mathbb{R})$ discussed above is not transitive, it is not free and a maximal invariant is $\phi : \mathsf{S}(n, \mathbb{R}) \to \mathbb{R}^n$,

$$\phi(S) = (\lambda_1(S), \lambda_2(S), \ldots, \lambda_n(S))^T,$$

where $\lambda_1(S) \geq \lambda_2(S) \geq \cdots \geq \lambda_n(S)$ denote the eigenvalues of S sorted in non-increasing order.

Definition [**Orbit space**] Let $\theta : G \times X \to X$ denote a continuous action of the topological group G on the topological space X.

Introduce an equivalence relation on X by declaring $x \sim y$ if they share the same orbit, that is, $x \sim y$ if and only if there exists $g \in G$ such that $y = \theta(g, x)$.

The set of equivalence classes is denoted by X/G and is called the orbit space of the action.

Lemma [**Orbit space**] Suppose the topological group G acts continuously on the left of the topological space X. Let X/G be given the quotient topology.

(a) The projection map $\pi : X \to X/G$ is open.

(b) If X is second countable, then X/G is second countable.

(c) X/G is Hausdorff if and only if the set

$$A = \{(p, q) \in X \times X \,:\, q = \theta(g, p) \text{ for some } g \in G\}$$

is closed in $X \times X$.

$\triangleright$ *Proof: (a) Let U be open in X. We must show that $\pi(U)$ is open in X/G, that is, $V = \pi^{-1}(\pi(U))$ is open in X. But*

$$V = \bigcup_{g \in G} \theta_g(U),$$

where $\theta_g : X \to X$, $\theta_g(x) = gx$. Since each θ_g is a homeomorphism, $\theta_g(U)$ is open in X. Thus, V is open in X. (b) If $\mathcal{B}$ is a countable basis for X, then $\pi(\mathcal{B}) = \{\pi(B) : B \in \mathcal{B}\}$ is a countable basis for X/G. (c) ($\Rightarrow$) Let $(x, y) \notin A$. Thus, x and y lie in distinct orbits, that is, $\pi(x) \neq \pi(y)$. Since X/G is Hausdorff, let U and V be disjoint neighborhoods of $\pi(x)$ and $\pi(y)$, respectively. Then, $\pi^{-1}(U) \times \pi^{-1}(V)$ is open in $X \times X$, contains (x, y) and does not intersect A (why?). Thus, the complement of A in $X \times X$ is open. ($\Leftarrow$) Let $\pi(x)$ and $\pi(y)$ be two distinct points in X/G. Then, $(x, y) \notin A$. Let U and V be neighborhoods of x and y, respectively, such that $U \times V$ does not intersect A. Then, $\pi(U)$ and $\pi(V)$ are disjoint neighborhoods of $\pi(x)$ and $\pi(y)$, respectively (why?) $\square$

Example 1 [**Projective space** $\mathbb{RP}^n$] Let $G = \mathsf{GL}(1,\mathbb{R})$ act continuously on $X = \mathbb{R}^{n+1} - \{0\}$ as $\theta : G \times X \to X$, $\theta(\lambda, x) = \lambda x$. Then, $\mathbb{RP}^n = X/G$.

$\mathbb{RP}^n$ is second countable.

$\mathbb{RP}^n$ is Hausdorff because

$$A = \{(x, y) \in X \times X \,:\, x \text{ and } y \text{ are in the same orbit}\}$$

is closed: it can be written as $A = f^{-1}(\{0\})$ where f if the continuous map

$$f \,:\, X \times X \to \mathbb{R}, \qquad f(x, y) = (x^T x)(y^T y) - (x^T y)^2.$$

Lemma [**Product of open maps is open**] Let A, B, X, Y be topological spaces. Let $f : A \to X$ and $g : B \to Y$ be open maps. Then, the product map

$$f \times g \,:\, A \times B \to X \times Y \qquad (f \times g)(a, b) = (f(a), g(b))$$

is open.

$\triangleright$ *Proof: Let W be an open set in $A \times B$. Then, W may be written as a union of rectangles*

$$W = \bigcup_i U_i \times V_i,$$

where each U_i in open in A and each V_i is open in B. We have

$$(f \times g)(W) = (f \times g)\left(\bigcup_i U_i \times V_i\right) = \bigcup_i (f \times g)(U_i \times V_i) = \bigcup_i f(U_i) \times g(V_i).$$

Since $f(U_i)$ is open in X and $g(V_i)$ is open in Y (by hypothesis), then $f(U_i) \times g(V_i)$ is open in $X \times Y$. Since W is an union of open sets, it is open$\square$

Lemma [**Hybrid spaces**] Let the topological group G act continuously on the left of the topological space X. Let the orbit space X/G be given the quotient topology and let $\pi : X \to X/G$ be the corresponding projection map. Let Y be any topological space. Then, the map

$$\pi \times \mathsf{id}_Y \,:\, X \times Y \to (X/G) \times Y \qquad (\pi \times \mathsf{id}_Y)(x, y) = (\pi(x), y)$$

is a quotient map.

▷ *Proof: To abbreviate notation, let* $f = \pi \times \mathsf{id}_Y$*. The map* f *is clearly surjective and continuous. Thus, if we show that* f *is an open map, we are done. Now, both* $\pi : X \to X/G$ *and* $\mathsf{id}_Y : Y \to Y$ *are open maps. Since* $f = \pi \times \mathsf{id}_Y$ *is the product of open maps, it is itself open*□

Corollary [Hybrid spaces] Let the topological group G act continuously on the left of the topological space X. Let the orbit space X/G be given the quotient topology and let $\pi : X \to X/G$ be the corresponding projection map. Let Y and B be any topological spaces. Then, the map $f : (X/G) \times Y \to B$ is continuous if and only if the map $\widehat{f} : X \times Y \to B$, $\widehat{f} = f \circ (\pi \times \mathsf{id}_Y)$ is continuous.

$$\begin{tikzcd}
X \times Y \arrow[d, "\pi \times \mathsf{id}_Y"'] \arrow[dr, "\widehat{f}"] & \\
(X/G) \times Y \arrow[r, "f"'] & B
\end{tikzcd}$$

▷ *Intuition: continuity of the "hard" map* f *can be investigated through the easier map* $\widehat{f}$

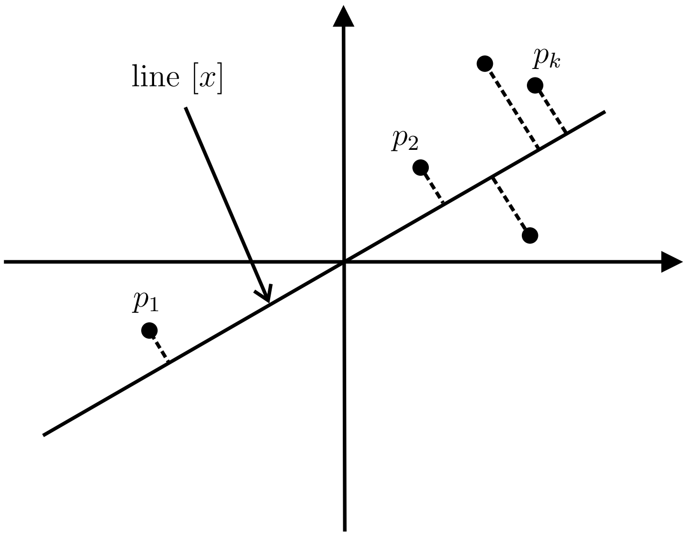
Example 1 [Projective space $\mathbb{RP}^{n-1}$] We write a matrix $P \in \mathsf{M}(n, k, \mathbb{R})$ is columns:

$$P = [\, p_1 \, p_2 \, \cdots \, p_k \,].$$

Consider the map

$$f : \mathbb{RP}^{n-1} \times \mathsf{M}(n, k, \mathbb{R}) \to \mathbb{R} \qquad f([x], P) = \sum_{j=1}^{k} \left\| p_j - \frac{xx^T}{\|x\|^2} p_j \right\|^2 .$$

In geometric terms, the map f computes the total squared distance from the constellation of points $\{p_1, p_2, \ldots, p_k\}$ to the straight line $[x]$.



The map f is continuous because

$$\widehat{f} : \mathbb{R}^n - \{0\} \times \mathsf{M}(n,k,\mathbb{R}) \to \mathbb{R} \qquad \widehat{f}(x,P) = \sum_{j=1}^{k} \left\| p_j - \frac{xx^T}{\|x\|^2} p_j \right\|^2$$

is clearly continuous.

References

[1] J. Lee, *Introduction to Topological Manifolds*, Springer-Verlag, 2000.