Nonlinear Signal Processing (2004/2005)

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Topological Spaces

Definition [**Topology**] A topology on a set X is a collection $\mathcal{T}$ of subsets of X (called *open sets* by definition) satisfying the following properties:

i) X and $\emptyset$ are elements of $\mathcal{T}$

ii) $\mathcal{T}$ is closed under finite intersections: if $U_1, \ldots, U_n \in \mathcal{T}$, then their intersection $U_1 \cap \cdots \cap U_n$ is in $\mathcal{T}$

iii) $\mathcal{T}$ is closed under arbitrary unions: if $\{U_\alpha\}_{\alpha \in A}$ is any (finite or infinite) collection of elements of $\mathcal{T}$, then their union $\bigcup_{\alpha \in A} U_\alpha$ is in $\mathcal{T}$.

$\triangleright$ *Remark: A set X can accept many topologies*

Definition [**Topological space**] A pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X is called a topological space. If the topology $\mathcal{T}$ is understood from the context, we simply say X is a topological space.

Example 1 [**Most familiar example: standard topology on $\mathbb{R}^n$**] $(\mathbb{R}^n, \mathcal{T})$ with $\mathcal{T}$ as the collection of all usual open sets in $\mathbb{R}^n$ (note: $\emptyset$ is in $\mathcal{T}$)

Same example (in disguised forms):

(a) [**Standard topology on $\mathsf{M}(m, n, \mathbb{R})$**]

$$\mathsf{M}(m, n, \mathbb{R}) = \{X \,:\, X \text{ is a } m \times n \text{ matrix with real entries}\}$$

We have $\mathsf{M}(m, n, \mathbb{R}) \simeq \mathbb{R}^{mn}$. Example:

$$\mathsf{M}(3, 2, \mathbb{R}) \simeq \mathbb{R}^6 \qquad \text{with} \qquad X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} \simeq \mathsf{vec}(X) = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{12} \\ x_{22} \\ x_{32} \end{bmatrix}.$$

Notation: $\mathsf{M}(n, \mathbb{R}) \equiv \mathsf{M}(n, n, \mathbb{R})$.

(b) [**Standard topology on** $\mathbb{C}^n$] We have $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Example:

$$\mathbb{C}^3 \simeq \mathbb{R}^6 \qquad \text{with} \qquad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ x_3 + iy_3 \end{bmatrix} \simeq \iota(z) = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}.$$

(c) [**Standard topology on** $\mathsf{M}(m, n, \mathbb{C})$]

$$\mathsf{M}(m, n, \mathbb{C}) = \{Z \,:\, Z \text{ is a } m \times n \text{ matrix with complex entries}\}$$

We have $\mathsf{M}(m, n, \mathbb{C}) \simeq \mathbb{C}^{mn} \simeq \mathbb{R}^{2mn}$. Example:

$$\mathsf{M}(3, 2, \mathbb{C}) \simeq \mathbb{C}^6 \simeq \mathbb{R}^{12}$$

with

$$Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \\ z_{31} & z_{32} \end{bmatrix} = \begin{bmatrix} x_{11} + iy_{11} & x_{12} + iy_{12} \\ x_{21} + iy_{21} & x_{22} + iy_{22} \\ x_{31} + iy_{31} & x_{32} + iy_{32} \end{bmatrix} \simeq \mathsf{vec}(X) = \begin{bmatrix} z_{11} \\ z_{21} \\ z_{31} \\ z_{12} \\ z_{22} \\ z_{32} \end{bmatrix} \simeq \iota(\mathsf{vec}(Z)) = \begin{bmatrix} x_{11} \\ y_{11} \\ x_{12} \\ y_{12} \\ x_{31} \\ y_{31} \\ x_{12} \\ y_{12} \\ x_{22} \\ y_{22} \\ x_{32} \\ y_{32} \end{bmatrix}.$$

Notation: $\mathsf{M}(n, \mathbb{C}) \equiv \mathsf{M}(n, n, \mathbb{C})$.

Example 2 [**Trivial space**] $(X, \mathcal{T})$ with X some set and $\mathcal{T} = \{\emptyset, X\}$

Example 3 [**Discrete space**] $(X, \mathcal{T})$ with X any set and $\mathcal{T} = 2^X$ (collection of *all* subsets of X)

Example 4 [**Toy example 1**] $(X, \mathcal{T})$ with $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{b\}, \{a, b\}, X\}$

Example 5 [**Toy example 2**] $(X, \mathcal{T})$ with $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a, b\}, X\}$

Example 6 [**A fake topological space**] $(X, \mathcal{T})$ with $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{c\}, X\}$ because $\{a\} \cup \{c\} = \{a, c\} \notin \mathcal{T}$

Example 7 [**Subspaces**] Let $(X, \mathcal{T})$ be a topological space and U an open set (that is, $U \in \mathcal{T}$). Then, $(U, \mathcal{T}_U)$, with

$$\mathcal{T}_U = \{V \in \mathcal{T} \,:\, V \subset U\} = \{W \cap U \,:\, W \in \mathcal{T}\}$$

is a topological space.

The topology $\mathcal{T}_U$ is called the subspace topology and, with this topology, U is called a subspace of X.

Definition [**Convergent sequence**] Let X be a topological space. A sequence $\{x_n : n = 1, 2, 3, \ldots\}$ of points in X is said to converge to $x \in X$ (notation: $x_n \to x$) if for every open set U containing x there exists N such that $x_i \in U$ for all $i \geq N$.

$\triangleright$ *Intuition: The tail of the sequence $\{x_n\}$ is arbitrarily close to x*

Example 1 [**In standard $\mathbb{R}^n$, life as usual**] The previous definition is our familiar one for $\mathbb{R}^n$ with the standard topology

Example 2 [**Funny things can happen**] In a trivial space X every sequence converges to every point of X !

Definition [**Continuous maps**] If X and Y are topological spaces, a map $f : X \to Y$ is said to be continuous if for every open set $U \subset Y$, $f^{-1}(U)$ is open in X.

$\triangleright$ *Intuition: f is continuous iff it pulls back open sets in Y to open sets in X*

Example 1 [**In standard $\mathbb{R}^n$, life as usual**] The previous definition is our familiar one for maps $f : \mathbb{R}^n \to \mathbb{R}^m$ with $\mathbb{R}^n$ and $\mathbb{R}^m$ with the standard topology.

Some examples:

(a) $f : \mathbb{R}^n \to \mathbb{R}$, $f(x, y, z) = 3x^2z - 2y^3 + 5xy^2z$

(b) $f : \mathsf{M}(n, \mathbb{R}) \to \mathbb{R}$, $f(X) = \det(X)$

Application: the general linear group

$$\mathsf{GL}(n, \mathbb{R}) = \{X \in \mathsf{M}(n, \mathbb{R}) : \det(X) \neq 0\}$$

is an open subset of $\mathsf{M}(n, \mathbb{R})$ because it equals

$$\{X : f(X) \neq 0\} = f^{-1}(\mathbb{R} - \{0\})$$

(c) $f : \mathsf{GL}(n, \mathbb{R}) \to \mathsf{M}(n, \mathbb{R})$, $f(X) = X^{-1}$

(d) $f : \mathsf{M}(n, m, \mathbb{R}) \to \mathsf{M}\left(\begin{pmatrix} n \\ k \end{pmatrix}, \begin{pmatrix} m \\ k \end{pmatrix}, \mathbb{R}\right)$, $f(X) = X^{[k]}$

$(1 \leq k \leq \min\{n, m\})$. Remark: $A^{[k]}$ is the kth compound matrix of A.

Example: for $n = 3, m = 4, k = 2$ and

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix},$$

we have

$$A^{[2]} = \begin{bmatrix} \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} & \det\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} & \det\begin{bmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{bmatrix} & \det\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} & \det\begin{bmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{bmatrix} & \det\begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \\ \det\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix} & \det\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} & \det\begin{bmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{bmatrix} & \det\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} & \det\begin{bmatrix} a_{12} & a_{14} \\ a_{32} & a_{34} \end{bmatrix} & \det\begin{bmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{bmatrix} \\ \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} & \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} & \det\begin{bmatrix} a_{21} & a_{24} \\ a_{31} & a_{34} \end{bmatrix} & \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} & \det\begin{bmatrix} a_{22} & a_{24} \\ a_{32} & a_{34} \end{bmatrix} & \det\begin{bmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{bmatrix} \end{bmatrix}.$$

Application: the set of $n \times m$ matrices with real entries and rank greater than or equal to k

$$\mathsf{Rank}_{\geq k}(n, m, \mathbb{R}) = \{X \in \mathsf{M}(n, m, \mathbb{R}) \,:\, \mathsf{rank}(X) \geq k\}$$

is an open subset of $\mathsf{M}(n, m, \mathbb{R})$ because it equals $\{X \,:\, f(X) \neq 0\}$. As a special case, we have that the subset of k-frames in $\mathbb{R}^n$,

$$\mathsf{F}(n, k, \mathbb{R}) = \{X \in \mathsf{M}(n, k, \mathbb{R}) \,:\, \mathsf{rank}(X) = k\},$$

is an open subset of $\mathsf{M}(n, k, \mathbb{R})$, because

$$\mathsf{F}(n, k, \mathbb{R}) = \mathsf{Rank}_{\geq k}(n, k, \mathbb{R}).$$

Example 2 [**The topologies really matter**] Let $X = \{a, b, c\}$ and $\mathcal{T}_i$ the topologies in the ith toy example.

The map $\mathsf{id} \,:\, (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous

The map $\mathsf{id} \,:\, (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is not continuous.

Lemma [**Elementary properties of continuous maps**] Let X, Y, and Z be topological spaces.

i) Any constant map $f \,:\, X \to Y$ is continuous

ii) The identity map $\mathsf{id} \,:\, X \to X$ is continuous

iii) If $f \,:\, X \to Y$ and $g \,:\, Y \to Z$ are continuous, so is their composition $g \circ f \,:\, X \to Z$

Lemma [**Local criterion for continuity**] A map $f \,:\, X \to Y$ between topological spaces is continuous if and only if each point of X has a neighborhood on which the restriction of f is continuous

Definition [**Homeomorphism**] If X and Y are topological spaces, a homeomorphism from X to Y is a continuous bijective map $f \,:\, X \to Y$ with continuous inverse. If there exists a homeomorphism between X and Y, the sets X and Y are said to be homeomorphic (topologically equivalent).

▷ *Intuition: you can consider X and Y the "same" topological space (Y is simply another label for X and vice-versa)*

Definition [Open map] If X and Y are topological spaces, a map $f : X \to Y$ is said to be an open map if for any open set $U \subset X$, the image set $f(U)$ is open in Y.

▷ *Intuition: f is continuous iff pushes forward open sets in X to open sets in Y*

Example 1 [Elementary projection] $\pi : \mathbb{R}^n \to \mathbb{R}^m$,

$$\pi(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) = (x_1, \ldots, x_m)$$

Example 2 [A fake open map] $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ because $f(-1, 1) = [0, 1)$.

Definition [Closed set] A subset F of a topological space X is said to be closed if its complement $X - F$ is open.

Example 1

$$\mathsf{Rank}_{\leq k}(m, n, \mathbb{R}) = \{X \in \mathsf{M}(n, m, \mathbb{R}) : \mathsf{rank}(X) \leq k\}$$

is a closed subset of $\mathsf{M}(n, m, \mathbb{R})$ because

$$\mathsf{Rank}_{\leq k}(m, n, \mathbb{R}) = \mathsf{M}(n, m, \mathbb{R}) - \underbrace{\mathsf{Rank}_{\geq k+1}(m, n, \mathbb{R})}_{\text{open}}.$$

Lemma [**Elementary properties of closed sets**] Let X be a topological space.

i) X and $\emptyset$ are closed

ii) if $F_1, \ldots, F_n$ are closed, then $F_1 \cup \cdots \cup F_n$ is closed

iii) if $\{F_\alpha\}_{\alpha \in A}$ is any (finite or infinite) collection of closed sets, then their intersection $\bigcap_{\alpha \in A} F_\alpha$ is closed.

Lemma [**Characterization of continuous maps through closed sets**] A map between topological spaces is continuous if and only if the inverse image of every closed set is closed.

▷ *Intuition: f is continuous iff it pulls back closed sets in Y to closed sets in X*

Definition [**Closed map**] If X and Y are topological spaces, a map $f : X \to Y$ is said to be a closed map if for any closed set $F \subset X$, the image set $f(F)$ is closed in Y.

Definition [**Elementary topological concepts**] Let X be a topological space. Let $A \subset X$. The closure of A in X, written $\overline{A}$ or $\mathsf{cl}(A)$, is the set

$$\overline{A} = \bigcap \{B : A \subset B \text{ and } B \text{ is closed in } X\}.$$

The interior of A, written $\mathsf{Int}\, A$, is

$$\mathsf{Int}\, A = \bigcup \{C : C \subset A \text{ and } C \text{ is open in } X\}.$$

The exterior of A, written $\mathsf{Ext}\, A$, is $\mathsf{Ext}\, A = X - \overline{A}$, and the boundary of A, written ∂A, is $\partial A = X - (\mathsf{Int}\, A \cup \mathsf{Ext}\, A)$.

▷ *Intuition: $\overline{A}$ is the "smallest" closed set containing A ; $\mathsf{Int}(A)$ is the "largest" open set contained in A*

Lemma [**Characterization of closure**] Let X be a topological space and

$A \subset X$. A point $x \in \overline{A}$ if and only if every open set containing x intersects A.

Example 1

$$\overline{\mathsf{GL}(n, \mathbb{R})} = \mathsf{M}(n, \mathbb{R})$$

Example 2

$$\overline{\mathsf{Rank}_{\geq k}(n, m, \mathbb{R})} = \mathsf{M}(n, m, \mathbb{R}) \qquad (1 \leq k \leq \min\{n, m\})$$

Lemma [**Characterization of boundary**] Let X be a topological space and $A \subset X$. A point x is in the boundary of A if and only if every open set containing x contains both a point of A and a point of $X - A$.

Definition [**Basis**] Let X be a topological space. A basis for X is a class $\mathcal{B}$ of open sets with the property that every non-empty open set in X is a union of sets in the class $\mathcal{B}$.

That is, if $U \subset X$ is open and non-empty, we can write

$$U = \bigcup_{\alpha \in A} B_\alpha \qquad \text{for some } B_\alpha \in \mathcal{B}.$$

The sets in a basis are called basic open sets.

▷ *Intuition: a basis is the "DNA" of a topology; the basic open sets are the building blocks of all open sets*

Example 1 A topology $\mathcal{T}$ is a basis for itself (useless remark in practice)

Example 2 $\mathcal{B} = \{(a, b) \subset \mathbb{R} \,:\, a < b\}$ is a basis for $\mathbb{R}$

Example 3 $\mathcal{B} = \{B^n_\epsilon(x_0) \subset \mathbb{R}^n \,:\, \epsilon > 0,\, x_0 \in \mathbb{R}^n\}$ is a basis for $\mathbb{R}^n$, where

$$B^n_\epsilon(x_0) = \{x \in \mathbb{R}^n \,:\, \|x - x_0\|_2 < \epsilon\}$$

Example 4 $\mathcal{B} = \{C^n_\epsilon(x_0) \subset \mathbb{R}^n \,:\, \epsilon > 0,\, x_0 \in \mathbb{R}^n\}$ is a basis for $\mathbb{R}^n$, where

$$C^n_\epsilon(x_0) = \{x \in \mathbb{R}^n \,:\, \|x - x_0\|_\infty < \epsilon\}$$

Lemma [Bases simplify the detection of continuous maps and open maps] Let X and Y be topological spaces. Let $\mathcal{B}_X$ be a basis for X and $\mathcal{B}_Y$ be a basis for Y.

A map $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open for every basic set $V \in \mathcal{B}_Y$.

A map $g : X \to Y$ is open if and only if the image $g(W)$ is open for every basic set $W \in \mathcal{B}_X$.

Example 1 [Pointwise maximum of continuous functions is continuous] Let X be a topological space. Let $f_i : X \to \mathbb{R}$ be continuous for $i = 1, 2, \ldots, n$. Then, $f : X \to \mathbb{R}$ given by

$$f(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

is continuous.

Definition [Locally Euclidean] A topological space X is said to be locally Euclidean of dimension n if every point of X has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

▷ *Intuition: around each point X looks like $\mathbb{R}^n$; but not globally*

Definition [Chart] Let X be locally Euclidean of dimension n. A chart on X is a pair (U, φ) where $U \subset X$ is open and $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a homeomorphism.

Definition [Hausdorff space] A topological space X is said to be a Hausdorff space if given any pair of distinct points $x, y \in X$ there exist open neighborhoods U of x and V of y such that $U \cap V = \emptyset$.

Example 1 [In standard $\mathbb{R}^n$, life as usual] The standard $\mathbb{R}^n$ is a Hausdorff space

Example 2 [Toy topology 2 is not Hausdorff] The topological space $(X, \mathcal{T})$ with $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a, b\}, X\}$ is not Hausdorff

Lemma [Elementary properties of Hausdorff spaces] Let X be a Hausdorff space. Then every singleton set $\{x\}$ is closed in X. Also, the limits of convergent sequences in X are unique.

Definition [Second countable space] A topological space X is said to be second countable if it admits a countable basis.

Example 1 [In standard $\mathbb{R}^n$, life as usual] The standard $\mathbb{R}^n$ is second countable. A countable basis:

$$\mathcal{B} = \{B^n_\epsilon(x_0) \,:\, \epsilon \in \mathbb{Q}^+, \text{ coordinates of } x_0 \text{ in } \mathbb{Q}\}.$$

Definition [Cover/Subcover] Let X be a topological space. A class $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of open sets is said to cover X if $X = \bigcup_{\alpha \in A} U_\alpha$. A subcover of $\mathcal{U}$ is a subclass $\mathcal{V} \subset \mathcal{U}$ which still covers X.

Lemma [Fundamental property of second countable spaces] Let X be a second countable space. Then every open cover of X admits a countable subcover.

Lemma [Second countable spaces allow simple characterization of closures] Let X be a second countable topological space. Let $A \subset X$. Then, $x_0 \in \overline{A}$ if and only if there exists a sequence $x_n \in A$ such that $x_n \to x_0$.

$\triangleright$ *Proof: ($\Rightarrow$) Let $x_0 \in \overline{A}$ and $\mathcal{V} = \{V_1, V_2, V_3, \ldots\}$ the collection of basic open sets containing x_0. Define the shrinking sequence: $U_1 = V_1$, $U_2 = V_1 \cap V_2$, $U_3 = V_1 \cap V_2 \cap V_3$, $\ldots$. Take a point $x_n \in A$ in each U_n (this can be done because each U_n is an open set containing $x_0 \in \overline{A}$). We have $x_n \to x_0$ (why?). ($\Leftarrow$) For the reverse direction, let U be an open set containing x_0. Since $x_n \to x_0$, there is a $x_N \in A$ in U. Thus, $A \cap U \neq \emptyset$* $\square$

Lemma [Second countable spaces simplify detection of continuous maps] Let $f : X \to Y$ be a map between topological spaces. Assume X is second countable. Then, f is continuous if and only if $x_n \to x_0$ implies $f(x_n) \to f(x_0)$.

$\triangleright$ *Proof: ($\Rightarrow$) Let U be an open set containing $f(x_0)$. By hypothesis, $V = f^{-1}(U)$ is open. Since $x_n \to x_0$, the tail of x_n is in V. Thus, the tail of $f(x_n)$ is in $f(V) \subset U$. ($\Leftarrow$) Let F be a closed set. Suppose $A = f^{-1}(F)$ is not closed. Then there exists $x_0 \in \overline{A}$ such that $x_0 \notin A$. Let $x_n \in A$ with $x_n \to x_0$. We have $f(x_n) \in f(A) \subset F$ for all n and $f(x_0) \in Y - F$ (open set). Thus, $f(x_n) \not\to f(x_0)$ (contradiction)*$\Box$

Definition [Topological manifold of dimension n] A topological manifold of dimension n is a second countable Hausdorff space that is locally Euclidean of dimension n.

Example 1 [The obvious example] $\mathbb{R}^n$ endowed with the standard topology is an n-dimensional topological manifold.

Lemma [Open subsets of manifolds are manifolds] If U is an open subset of an n-dimensional topological manifold, then U (endowed with the subspace topology) is an n-dimensional topological manifold.

Lemma [Topological manifolds] Let X and Y be homeomorphic topological spaces. Then, X is an n-dimensional topological manifold if and only if Y is an n-dimensional topological manifold.

References

[1] J. Lee, *Introduction to Topological Manifolds*, Springer-Verlag, 2000.